Static hole in a critical antiferromagnet:
field-theoretic renormalization group

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Abstract

We consider the quantum field theory of a single, immobile, spin $S$ hole coupled to a two-dimensional antiferromagnet at a bulk quantum critical point between phases with and without magnetic long-range order. We present an alternative derivation of its two-loop beta function; the results agree completely with earlier work (M. Vojta et al, Phys. Rev. B 61, 15152 (2000)), and also determine a new anomalous dimension of the hole creation operator.

Keywords: Kondo spin, critical antiferromagnet, field theory (subject index); high temperature superconductor (materials index).

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PACS numbers: 71.27.+a, 75.20.Hr, 75.10.Jm
Recent papers [1, 2] have introduced the following model Hamiltonian for a single non-magnetic (Zn or Li) impurity in a two-dimensional $d$-wave superconductor or spin-gap insulator (see [3] for a review and experimental motivation):

$$\mathcal{H} = \mathcal{H}_\phi - \gamma_0 \hat{S}_\alpha(x = 0)$$

$$\mathcal{H}_\phi = \int d^dx \left[ \frac{\pi^2_\alpha + \epsilon^2(\nabla \phi_\alpha)^2 + s\phi^2_\alpha}{2} + \frac{g_0}{4!}(\phi^2_\alpha) \right].$$  \hfill (1)

We have written the Hamiltonian in $d$ spatial dimensions, and $\hat{S}_\alpha (\alpha = 1, 2, 3)$ are spin $S$ operators of a magnetic moment that is postulated to be present near the impurity (the case of physical interest has $S = 1/2$); these operators obey the SU(2) commutation relations

$$[\hat{S}_\alpha, \hat{S}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{S}_\gamma$$  \hfill (2)

and $\hat{S}_\alpha \hat{S}_\alpha = S(S + 1)$. The field $\phi_\alpha(x, t)$ represents the local orientation of the antiferromagnetic order parameter at spatial position $x$ and time $t$; its canonically conjugate momentum is $\pi_\alpha(x, t)$, and hence

$$[\phi_\alpha(x, t), \pi_\beta(x', t)] = i\delta_{\alpha\beta}\delta^d(x - x')$$  \hfill (3)

This theory has a bulk quantum critical point at $s = s_c$ between a phase with magnetic order ($s < s_c, \langle \phi_\alpha \rangle \neq 0$), and a symmetric phase with a spin gap ($s > s_c, \langle \phi_\alpha \rangle = 0$). We are interested in the spin correlations of $\mathcal{H}$ for $s$ close to $s_c$, and in the vicinity of the impurity at $x = 0$. As discussed in [1, 2], universal aspects of these correlations are associated with a renormalized continuum theory of $\mathcal{H}$ defined in an expansion in $\epsilon = 3 - d$. This renormalization involves the familiar bulk renormalizations which are insensitive to the impurity degree of freedom

$$\phi_\alpha = \sqrt{Z}\phi_{R\alpha}; \quad g_0 = \frac{\mu^\epsilon Z_4}{Z^2S_{d+1}}g$$  \hfill (4)

and new ‘boundary’ renormalizations associated with the impurity spin

$$\hat{S}_\alpha = \sqrt{Z}\hat{S}_{R\alpha}; \quad \gamma_0 = \frac{\mu^{\epsilon/2}Z_\gamma}{\sqrt{ZZZ'}S_{d+1}}\gamma.$$  \hfill (5)
Here $\mu$ is a renormalization momentum scale (we set the velocity $c = 1$), $S_d = 2/[\Gamma(d/2)(4\pi)^{d/2}]$, and $\bar{S}_d = \Gamma(d/2 - 1)/[4\pi^{d/2}]$. The renormalization constants $Z, Z_4$ were computed long ago [4]; their values in the minimal subtraction scheme to order $g^2$ are

$$Z = 1 - \frac{5g^2}{144\epsilon} \quad ; \quad Z_4 = 1 + \frac{11g}{6\epsilon} + \left(\frac{121}{36\epsilon^2} - \frac{37}{36}\right) g^2.$$  \hfill (6)

The boundary renormalizations were computed to the same order in [1, 2]:

$$Z' = 1 - \frac{2\gamma^2}{\epsilon} + \frac{\gamma^4}{\epsilon} \quad ; \quad Z_\gamma = 1 + \frac{\pi^2[S(S + 1) - 1/3]}{6\epsilon} \gamma^2 g.$$ \hfill (7)

This paper will rederive the above results by a new method which also yields a renormalization constant for the hole creation operator. Furthermore, the present approach, unlike that of [2], has the advantage of being formulated entirely in terms of perturbation expansion which has a Wick theorem, and can thus be presented in conventional time-ordered Feynman diagrams.

We will identify the spin $\hat{S}_\alpha$ with that of a hole, with creation operator $\psi_a^\dagger$, that has been injected into the antiferromagnet. So

$$\hat{S}_\alpha = \psi_a^\dagger L^\alpha_{ab} \psi_b$$ \hfill (8)

where $a, b$ take the $2S + 1$ values $-S, \ldots, S$, and the $L^\alpha$ are the $(2S + 1) \times (2S + 1)$ angular momentum matrices associated with the spin $S$ representation. The hole operators obey the anticommutation relation

$$\psi_a^\dagger \psi_b + \psi_b \psi_a^\dagger = \delta_{ab}.$$ \hfill (9)

So the remainder of this paper will consider the Hamiltonian

$$\mathcal{H}_\psi = \lambda \psi_a^\dagger \psi_a + \mathcal{H}_\phi - \gamma_0 \psi_a^\dagger L^\alpha_{ab} \psi_b \phi_a(x = 0)$$ \hfill (10)

We will only look at the Hilbert space with a single hole, and $\lambda$, the energy of this hole is an arbitrary positive number.

We now consider the renormalization of $\mathcal{H}_\psi$. The standard procedure suggests the parameterization

$$\psi_a = \sqrt{Z_h \psi_{Ra}} \quad ; \quad \gamma_0 = \frac{\mu^{d/2} \bar{Z}_\gamma}{Z_h \sqrt{Z \bar{S}_d + 1}} \gamma.$$ \hfill (11)
It is important to note that despite the relation (8), the renormalization of the spin $\hat{S}_\alpha$ is not the square of the renormalization of $\psi_a$, $Z' \neq Z^2_{\gamma}$; bringing the two Fermi operators to the same spacetime point introduces a composite operator renormalization which invalidates such a relation. Instead, the relationship between the two renormalization schemes emerges by comparing the renormalization of $\gamma_0$ in (5) and (11); consistency of these relations demands

$$Z_h^2 Z_{\gamma}^2 = \tilde{Z}_{\gamma}^2 Z'$$  \hspace{1cm} (12)$$

We will now compute $Z_h$ and $\tilde{Z}_{\gamma}$ by completely standard field theoretic methods, and verify that their values and (7) satisfy (12).

The Feynman diagrams for the renormalization of two-point $\psi$ Green’s function are shown in Fig 1. As an explicit example, we display the computation of the simplest one-loop graph in Fig 1a:

$$(1a) = \gamma_0^2 S(S + 1) \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega'}{2\pi} \frac{1}{\omega'^2 + k^2} \frac{1}{-i(\omega + \omega') + \lambda}$$

$$= \gamma_0^2 S(S + 1) \frac{S_d}{2} \int_0^\infty \frac{k^{d-2} dk}{(-i\omega + k + \lambda)}$$

$$= A_\mu(-i\omega + \lambda) \gamma^2 S(S + 1) \left[ -\frac{1}{\epsilon} + \mathcal{R}/2 + \mathcal{O}(\epsilon) \right], \hspace{1cm} (13)$$

where $A_\mu \equiv \mu^\epsilon(-i\omega + \lambda)^{-\epsilon} \tilde{Z}_{\gamma}^2/(Z_h^2 Z)$. In the last step, the integral was evaluated in dimensional regularization. The constant $\mathcal{R} = -0.8455686701969\ldots$ is a consequence of phase space factors and will eventually cancel out of our final results. The remaining diagrams can be evaluated in a very similar manner: the frequency integrals are performed first, followed by integrals over the radial momenta. The results for the two-loop diagrams in Fig 1 are

$$(1b) = A_\mu^2(-i\omega + \lambda) \gamma^4 S^2(S + 1)^2 \left[ \frac{1}{2\epsilon^2} + \frac{1 - \mathcal{R}}{2\epsilon} + \mathcal{O}(\epsilon^0) \right]$$

$$(1c) = A_\mu^2(-i\omega + \lambda) \gamma^4 S(S + 1)(S^2 + S - 1) \left[ -\frac{1}{\epsilon^2} + \frac{-1 + 2\mathcal{R}}{2\epsilon} + \mathcal{O}(\epsilon^0) \right] \hspace{1cm} (14)$$

Turning to the renormalization of the vertex $\gamma_0$, the Feynman diagrams are shown in Fig 2. Evaluating these as above we obtain

$$(2a) = \gamma_0 A_\mu \gamma^2 (S^2 + S - 1) \left[ \frac{1}{\epsilon} - 1 - \mathcal{R}/2 + \mathcal{O}(\epsilon) \right]$$
\[(2b) = \gamma_0 A_\mu^2 \gamma^4 (S^2 + S - 1)^2 \left[ \frac{1}{2\epsilon^2} - \frac{3 + N}{2\epsilon} + O(\epsilon^0) \right] \]
\[(2c) = \gamma_0 A_\mu^2 \gamma^4 (S - 1)(S + 2)(S^2 + S - 1) \left[ \frac{1}{2\epsilon} + O(\epsilon^0) \right] \]
\[(2d) = \gamma_0 A_\mu^2 \gamma^4 (S^2 + S - 1)^2 \left[ \frac{1}{\epsilon^2} - \frac{2 + N}{2\epsilon} + O(\epsilon^0) \right] \]
\[(2e) = \gamma_0 A_\mu^2 \gamma^4 (S + 1)(S^2 + S - 1) \left[ -\frac{1}{\epsilon^2} + \frac{2 + N}{2\epsilon} + O(\epsilon^0) \right] \]
\[(2f) = -\gamma_0 \frac{A_\mu Z^2 H Z^4}{Z^2 Z} \gamma^2 g (S^2 + S - 1/3) \left[ \frac{\pi^2}{6\epsilon} + O(\epsilon^0) \right] \]

The two-loop expression for the boundary renormalization constants follows immediately from the results (13,14,15). Demanding cancellation of poles in \(\epsilon\) in the expressions for the renormalized vertex and \(\psi\) Green’s function at external frequency \(-i\omega + \lambda = \mu\) we obtain

\[
Z_h = 1 - \gamma^2 \frac{S(S + 1)}{\epsilon} + \gamma^4 \left[ \frac{(S - 1)S(S + 1)(S + 2)}{2\epsilon^2} + \frac{S(S + 1)}{2\epsilon} \right]
\]
\[
\tilde{Z}_\gamma = 1 - \gamma^2 \frac{(S^2 + S - 1)}{\epsilon} + \gamma^4 \left[ \frac{(S^2 + S - 3)(S^2 + S - 1)}{2\epsilon^2} + \frac{(S^2 + S - 1)}{2\epsilon} \right] + g \gamma^2 \frac{\pi^2 (S^2 + S - 1/3)}{6\epsilon} \]

(16)

It can be checked that (16) and (7) satisfy (12).

The validity of (12) implies that the beta function for the coupling \(\gamma\) is the same as that in [2]. Using either (5,6,7) or (11,6,16) we obtain

\[
\beta(\gamma) = -\frac{\epsilon\gamma^2}{2} + \gamma^3 - \gamma^5 + \frac{5g^2\gamma}{144} + \frac{g\gamma^3}{3} \frac{\pi^2}{3} (S^2 + S - 1/3).
\]

(17)

The anomalous dimension of the \(\psi_a\) field at the quantum critical point also follows from (16)

\[
\eta_h = \beta(\gamma) \frac{d\ln Z_h}{d\gamma} = S(S + 1)(\gamma^2 - \gamma^4),
\]

(18)

while, as in [2], the anomalous dimension of the spin field, \(\tilde{S}_\alpha\), follows from (5,7):

\[
\eta' = 2(\gamma^2 - \gamma^4).
\]

(19)

For completeness we also note the beta function for the coupling \(g\) which follows from (6)

\[
\beta(g) = -\epsilon g + \frac{11g^2}{6} - \frac{23g^3}{12}.
\]

(20)
The stable fixed point of the beta functions (17,20) has \( g \neq 0 \) and \( \gamma \neq 0 \) [2]. Evaluating (18) at the fixed point of the beta functions [2], we obtain
\[
\eta_h = S(S+1) \left[ \frac{\epsilon}{2} - \left( \frac{5}{484} + \frac{\pi^2(S^2 + S - 1/3)}{11} \right) \epsilon^2 + O(\epsilon^3) \right]
\]  
(21)
\( (\eta' = 2\eta_h/[S(S+1)] \) at this order). This anomalous dimension implies that the Green’s function \( G = \langle \psi_a \psi_a^{\dagger} \rangle \) obeys
\[
G(\omega) \sim (\lambda - \omega)^{-1+\eta_h}.
\]  
(22)

The equations (16,18,21) are the main new results of this paper. Unfortunately, the order \( \epsilon^2 \) corrections in (21) are rather large: this suggests that truncating the asymptotic series for \( \eta_h \) at order \( \epsilon \) probably gives the most reasonable estimate for its numerical value.

There is also an unstable fixed point at which the bulk interactions vanish \((g = 0)\). As shown in [2], \( \eta' = \epsilon \) exactly at this fixed point, and here we find that \( \eta_h = S(S+1)\epsilon/2 + O(\epsilon^3) \). There appears to be no general reason for the higher order terms in \( \eta_h \) to vanish. The \( g = 0 \) fixed point can also be studied in a large \( N \) theory [2], and the \( N = \infty \) results are \( \eta' = 1 \) and \( \eta_h = 1/2 \).

The physical motivations and implications of the above results are discussed in a separate paper [5]: there we argue that the anomalous dimension \( \eta_h \) characterizes photoemission spectra of mobile holes in two-dimensional antiferromagnets and superconductors in the vicinity of points in the Brillouin zone where their dispersion spectra are quadratic \((i.e.\) near energy minima, maxima, and van Hove singularities). The intensively studied \((\pi,0), (0,\pi)\) points \((the \ anti-\ nodal \ points)\) in the high temperature superconductors are prominent examples.

**ACKNOWLEDGMENTS**

This research was supported by US NSF Grant No DMR 96–23181.
FIG. 1: Diagrams contributing to the $\psi$ fermion self energy. The full line is the fermion propagator, while the dashed line is the $\phi_\alpha$ propagator.

REFERENCES

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FIG. 2: Diagrams contributing to the renormalization of the coupling $\gamma$. The full circle is the interaction $g$. 