The Sachdev-Ye-Kitaev model and matter without quasiparticles

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Abstract

This thesis is devoted to the study of Sachdev-Ye-Kitaev (SYK) models, which describe matter without quasiparticles. The SYK model is an exactly solvable model that is hoped to bring insights into the understanding of strongly correlated materials.

The main part discusses different aspects of SYK models. Chapter 2 presents numerical studies of the SYK models, shows its non-Fermi liquid nature and verifies the analytical predictions on two-point function and thermal entropies. Chapter 3 considers the thermodynamic and transport properties of the complex SYK models in higher dimensional generalizations, we show some universal relations obtained from SYK models which are consistent with holographic theories. Chapter 4 discusses a supersymmetric generalization of the SYK model. We show a different a scaling from the normal SYK model both analytically and numerically. We also discuss a different origin of the extensive zero temperature entropy in $\mathcal{N} = 2$ theory. Chapter 5 considers a continuous set of boson-fermion SYK models with a tunable low-energy scaling dimension. From the explicit computation of out of time ordered correlators, we find the theory is less chaotic in the free fermion limit. Chapter 6 describes the phases of a solvable $t$-$J$ model of electrons with an SYK-like form. We find a $\mathbb{Z}_2$ fractionalization picture where the deconfined phase is similar to the pseudo gap regime of underdoped cuprates, while the ‘quasi-Higgs’ phase is similar to the metallic phase of overdoped cuprates.
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Citations to Previously Published Work

Most of this thesis has appeared in print elsewhere. Details for particular chapters are given below.

Chapter 2:


Chapter 3:

- Wenbo Fu, Yingfei Gu, Subir Sachdev and Grigory Tarnopolsky, “A study on complex SYK models”, in preparation

Chapter 4:


Chapter 5:

- Wenbo Fu, Chao-Ming Jian, Zhen Bi and Cenke Xu, “A continuous set of boson-fermion SYK models”, in preparation

Chapter 6:


Electronic preprints (shown in typewriter font) are available on the Internet at the following URL:

http://arXiv.org
To my parents.
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Introduction

1.1 Preliminaries and motivations

The quasiparticle idea is an important concept from Landau’s Fermi-liquid theory. Except for Fermi-liquid, it also plays an important role in understanding many of condensed matter systems including BCS superconductivity, Luttinger liquid, fractional quantum hall effect etc\textsuperscript{1}. The idea is that in a typical quantum many-body system, it is hard to understand the whole spectrum of the theory with exponential many degrees of freedom. However, since most of the time we are interested in the low energy physics, it will be enough for us to concentrate on the low-energy sub-space of the whole spectrum, while these states have a particle-like description with a relatively long lifetime.

There are several definitions about quasiparticle description, the most general one without the needs for translation symmetry and well-defined momentum space is that, the low-energy quasiparticles we refer are not restricted to fermionic, collective modes are also counted as quasiparticles.

\textsuperscript{1}Here the quasiparticles we refer are not restricted to fermionic, collective modes are also counted as quasiparticles.
many-body spectrum can be written as [128]:

\[
E = \sum_\alpha \delta n_\alpha \epsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha,\beta} \delta n_\alpha \delta n_\beta + \cdots 
\]  

(1.1)

where \( \epsilon_\alpha \) is single-particle energy, \( F_{\alpha,\beta} \) is called Landau-parameter, and \( \delta n_\alpha \) is the change of single particle occupation compared to the ground state, \( \cdots \) denotes higher order interaction terms. To test whether quasiparticle is long lived, one can compute the lifetime of quasiparticles. For example, in the case of Fermi liquid, quasiparticle-quasiparticle scattering event can happen. And one can use Fermi’s golden rule to compute the lifetime. At low temperature, a typical computation [37] shows that

\[
\frac{1}{\tau} \sim \frac{E_F}{T^2}
\]

(1.2)

so the quasiparticle has extremely long lifetime at low temperature. Notice that the above computation relies on the fact that the spectral density near Fermi level is a constant. If there is some singularity in the spectral density, we would have a different scaling behavior as we will see later in the SYK models.

There will be several predictions from this quasiparticle picture, for example, there will be a \( T^2 \) temperature dependence of resistivity at low temperatures. The heat capacity will have a linear in \( T \) temperature dependence. The quasiparticle picture will also predict a \( B^2 \) magnetic field dependence of magnetoresistance.

However there are also many exotic phenomena (non-Fermi liquid properties) cannot be described by quasiparticle picture in strongly correlated materials, like cuprates, pnictides and heavy-fermion materials [38, 178, 184]. A typical phase diagram is shown in Figure 1.1. An interesting feature in the strange metal phase is an anomalous linear in \( T \) resistivity for a wide range of temperature. Another feature is that the specific heat scales as \( T \ln T \) at low temperatures.
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Figure 1.1: Figure, adapted from Ref [87], A schematic phase diagram of hole-doped cuprates, as a function of hole-doping $p$ and temperature $T$. Here AF is anti-ferromagnetic, PG is pseudogap, DW is density wave, dSC is d-wave superconductor, SM is strange metal, $T^*$ denotes the temperature scale of the appearance of the anti-nodal spectral gap, $T_c$ stands for the superconductivity transition temperature.

On the theory side, the phenomenological “marginal Fermi liquid” theory [187] can illustrate these experimental scalings with the assumption of critical bosonic degrees of freedom coupled with fermions. However, it is not clear how to derive such a bosonic spectrum from a microscopic model. Furthermore, the strange metal behavior persists to relatively high temperature, it is not clear whether one needs to lie near a quantum critical point to include the bosonic critical fluctuations.

On the other hand, the Sachdev-Ye-Kitaev models are microscopic models that can help understand these experimental features. Some evidence are:

- The spectral function of SYK model indicates a non-Fermi liquid state (will be illustrated in the next section);
- The recent experiment on charge susceptibility [143] agrees with the form predicted by
SYK models;

- The linear-$T$ resistivity and $T \ln T$ specific heat can be obtained from SYK models generalized in higher dimensions [36, 156, 179];

- Just like in the cuprate phase diagram, the SYK model also has a pairing instability at low temperature [18];

- There are recent experiments showing a linear-in-$B$ anomalous magnetoresistance [76, 96] which can also be obtained in a generalized SYK model [156].

- The drop in the Hall conductivity [8] indicates a topological transition and a topological transition can also be realized in SYK models (We will discuss it in Chapter 6).

These evidence suggest the importance of the study of SYK model to help understand "strange metal" phases.

Another interesting topic related to SYK model is quantum chaos. And SYK model is shown to be maximally chaotic. Before defining quantum chaos, let’s consider the classical set up. Classically, a chaotic system is characterized by the fact that small change in the initial condition will result in a large deviation in the future:

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t}$$

(1.3)

where $\lambda_L$ is called the classical Lyapunov exponent. In classical mechanics, the variation ratio $\frac{\delta x(t)}{\delta x(0)}$ can be represented by the Poisson bracket:

$$\{x(t),p(0)\}_{P.B.}$$

(1.4)

The quantum analog is just to replace the Poisson bracket by the quantum commutator (or anti-commutator) and evaluate its size in a thermal ensemble. More generally, one can use any operator instead of simply $x$ or $p$. Then the quantum chaos is defined by the growth of
the commutator square:

\[
C(t) := \langle [W(t), V(0)]^2 \rangle_\beta
\]  

(1.5)

where \(W(t)\) and \(V(0)\) are generic operators in the Heisenberg picture. The subscript \(\beta\) means it is measuring the thermal expectation. We can further expand the above expression:

\[
\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle_\beta = \langle (V(0)^\dagger W(t)^\dagger W(t)V(0))_\beta + (W(t)^\dagger V(0)^\dagger V(0)W(t))_\beta \\
- \langle V(0)^\dagger W(t)^\dagger V(0)W(t) \rangle_\beta - \langle W(t)^\dagger V(0)^\dagger W(t)V(0) \rangle_\beta
\]

(1.6)

The first two terms are called time ordered correlators (TOC) while the last two terms are called out of time ordered correlators (OTOC). TOC will quickly saturate to the product of two-point functions after a "collision time", and there will be exponential increase behavior of \(C(t)\) from this time to the "scrambling time" when \(C(t)\) becomes significant. The exponent will be called the quantum Lyapunov exponent. This sharp increase in \(C(t)\) can also be understood from the sharp decrease in the OTOC. And in literature, people usually compute OTOC in a chaotic system, under some renormalization, it will have the form:

\[
f(t) = 1 - \frac{1}{N} e^{\lambda_L t}
\]

(1.7)

with \(\lambda_L\) the quantum Lyapunov exponent. There is a proof that there is a bound on quantum chaos \(\lambda_L \leq \frac{2\pi}{\beta} [136]\), and the bound is saturated in the Einstein gravity theory [177]. On the other hand, the SYK model also saturates the bound at low temperature, where the model exhibit near conformal symmetry [116, 135]. This indicates that SYK is a simple model that is holographic. And it is argued to dual to an Einstein-dilation gravity theory in an approximate AdS\(_2\) geometry [137]. Thus the study of SYK models also motivates from the understanding of 2d quantum gravity.

The OTOC has also been studied in other condensed matter systems [35, 154, 155]. And
the inverse of the Lyapunov exponent gives a time scale called Lyapunov time \( \tau_L = \frac{1}{\lambda_L} \). The upper bound on \( \lambda_L \) gives a lower bound on \( \tau_L \geq \frac{1}{2xT} \). There is a similar time scale in traditional condensed matter literature called dephasing time \([170]\). It is related to the scattering rate of quasiparticles\(^2\). And this time is found to be similar in many materials \([24]\). Whether the dephasing time and Lyapunov time are related is still an open question.

### 1.2 SYK 101

In this section, we will give a general introduction to SYK models. First, let’s look at some of its interesting features:

1. Solvable in large \( N \), which describes a non-Fermi liquid state;
2. Finite zero temperature entropy;
3. Emergent conformal symmetry at strong coupling;
4. Maximally chaotic: the Lyapunov exponent extracted from out-of-time-ordered correlation function in this model saturates the bound of chaos \( \lambda_L = \frac{2\pi}{\beta} \) (At strong coupling).

We will comment on these features later in the section. Before doing that, let’s first introduce the model:

\[
H = \frac{1}{4!} \sum_{j,k,l,m=1}^{N} J_{jklm} \chi_j \chi_k \chi_l \chi_m
\]

where \( \{\chi_j\} \) are Majorana fermion operators satisfying commutation relation: \( \{\chi_j, \chi_k\} = \delta_{jk} \). \( N \) is the number of sites. \( \{ J_{jklm} \} \) are independent random couplings satisfying the statistics:

\[
\overline{J_{jklm}} = 0, \quad \frac{1}{3!} N^3 \overline{J^2_{jklm}} = J^2
\]

the dimensionless coupling in this model is \( \lambda = \beta J \).

\(^2\)However, there is no unique definition of dephasing time.
The model can be understood in this way: we can rewrite the Hamiltonian as

$$H = \sum_j \chi_j \mathcal{O}_j$$  \hspace{1cm} (1.10)$$

where $\mathcal{O}_j = \frac{1}{4!} \sum_{k,l,m=1}^{N} J_{jklm} \chi_k \chi_l \chi_m$ is the heat-bath field for fermion $\chi_j$. In the large $N$ limit, the heat bath field $\mathcal{O}_j$ is a sum of many matrices with random coefficients, therefore individually behaves like a Gaussian random variable.

We can estimate the strength of this heat bath by averaging $\mathcal{O}_j^2$:

$$\overline{\mathcal{O}_j^2} \sim N^3 \mathcal{J}^2 |\chi_j|^3 \sim J^2$$  \hspace{1cm} (1.11)$$

therefore the physical meaning of coupling constant $J$ is the measures of the strength of the heat bath.

The model can be generalized to a $q$–body interaction Hamiltonian ($q \ll N$)

$$H_q = \frac{1}{q!} \sum_{j_1, \ldots, j_q}^{N} J_{j_1j_2\cdots j_q} \chi_{j_1} \chi_{j_2} \cdots \chi_{j_q}$$  \hspace{1cm} (1.12)$$

And require the random couplings $\{J_{j_1j_2\cdots j_q}\}$ have suitable scaling w.r.t. $N$:

$$\overline{J_{j_1j_2\cdots j_q}} = 0, \quad \frac{1}{(q-1)!} N^{q-1} \overline{J_{j_1j_2\cdots j_q}^2} = J^2$$  \hspace{1cm} (1.13)$$

one can further define a $q$–dependent coupling constant $\mathcal{J}_q = \frac{J^2}{2^{q-1}}$. We redefine $\mathcal{J}$ this way such that we will have a well-defined perturbative large $q$ limit.

This model is solvable in the large $N$ limit. We will first compute the two-point function in this limit using diagrammatic summation. The two-point function is defined as

$$G(\tau_1, \tau_2) = \langle \mathcal{T} \chi_j(\tau_1) \chi_j(\tau_2) \rangle$$  \hspace{1cm} (1.14)$$
The free propagator is given by

\[ G_0(\tau_1, \tau_2) = \frac{1}{2} \text{sgn}(\tau_1 - \tau_2), \quad G_0(i\omega_n) = -\frac{1}{i\omega_n} \] (1.15)

Since the interaction \( J_{jklm} \) are statistically independent random variables, only propagators connecting same sites will survive under average. Physically, this is because different heat bath fields are independent. We can start from weak coupling limit and consider the diagrams order by order:

\[ \sim N^3 J^2 \sim 1 \]

\[ \sim N^6 J^2 \sim 1 \]

\[ \sim N^4 J^2 \sim 1/N^2 \] (1.16)

where we use solid lines to represent the free propagator and dashed line to represent the pairing of random couplings. (Remember that dashed line connects two vertices with the same set of indices.) Then one can count the number of terms that contribute to each diagram with fixed external legs. We show the results of the diagrams above.

As we can see from the above diagrams, only rainbow (watermelon) diagrams survive in the large \( N \) limit, this results in the Schwinger-Dyson equation:

\[ \begin{align*}
\cdot & = \cdot + \cdot \\
\cdot & = \cdot
\end{align*} \] (1.17) (1.18)

where thick lines represent the dressed green function \( G(\tau_1, \tau_2) \), thin lines represents the
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free green function $G_0(\tau_1, \tau_2)$ and gray disk represents the self energy $\Sigma(\tau_1, \tau_2)$. In terms of functions, the SD equations read

$$G(i\omega_n)^{-1} = -i\omega_n - \Sigma(i\omega_n)$$

(1.19)

$$\Sigma(\tau) = J^2 G(\tau)^3$$

(1.20)

This can be generalized to the $q$-body interaction case:

$$G(i\omega_n)^{-1} = -i\omega_n - \Sigma(i\omega_n)$$

(1.21)

$$\Sigma(\tau) = J^2 G(\tau)^q$$

(1.22)

The above Schwinger-Dyson are exact in the large $N$ limit, which exhibits the solvability of the model. One can numerically iterate these two equations in frequency and time domain using Fourier transform. On the other hand, at low temperature limit $\beta J \gg 1$, we can drop the $i\omega_n$ term and assume a low-lying scaling form $G(\tau) \sim \frac{1}{\sqrt{\pi \tau}}$. By simply power matching, we find that

$$\Delta = \frac{1}{q}$$

(1.23)

This means that the spectral density $\rho(\omega) \sim \omega^{\frac{2}{q} - 1}$. So there is a singularity in the spectral density near zero energy for $q \geq 4$.

Now we want to compare the $q = 2$ and $q = 4$ cases to illustrate the non-Fermi liquid nature of the SYK model. Actually the $q = 2$ model, which can be viewed as random hopping model, is well studied in the random matrix theory. The Schwinger-Dyson equation can be solved exactly and the spectral density is the famous semicircle low:

$$\rho(\omega) = \frac{\sqrt{4J^2 - \omega^2}}{2J^2}, \quad |\omega| \leq 2J$$

(1.24)
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We see that at small $\omega$, $\rho(\omega) = \rho_0 = \frac{1}{T}$ will be a constant. If we include some weak interaction, then quasiparticle can scatter with each other. The standard scattering rate computation using Fermi’s Golden rule shows that:

$$\Gamma \propto \rho_0^2 T^2$$

(1.25)

This means that the quasiparticle lifetime is quite long, which is the typical Fermi-liquid behavior although we don’t have a Fermi surface. On the other hand, if we do the same computation for the $q = 4$ SYK case, with the divergent spectral density $\rho(\omega) \sim \frac{1}{\sqrt{\omega}}$, we find that:

$$\Gamma \propto T$$

(1.26)

with a relatively short lifetime. This justifies the non-Fermi liquid nature of SYK models.

Another way to justify this is to check Eq. (1.1) in the many-body spectrum. Here we show an exact diagonalization result in Figure. 1.2. A visible difference is that the spectrum is much sparser at the two ends in the random hopping model. The reason is that the level spacing in the low-energy subspace of the random hopping model is proportional to $\frac{1}{N}$, which is the same as the level spacing in the single particle energy $\epsilon$. So the low-energy many-body spectrum can be described by Eq. (1.1) with only a few non-zero $\delta n_a$. However, this picture does not apply to the $q = 4$ SYK model. There are theoretical studies [74] showing that in the $N \to \infty$ and then $T \to 0$ limit, there is an extensive zero temperature entropy density $s_0 = 0.464848$. This does not mean that we have a ground state degeneracy but means that the level spacing in the low-energy subspace is also exponential small $\delta E \propto e^{-s_0 N}$ (denser spectrum at the end shown in the right figure of Figure. 1.2). This means that there are exponential many low-energy states. If we have a quasiparticle description, the many-body spectrum is described by Eq. (1.1). The low-energy spectrum is described by changes in a few $n_a$. This will indicate that we can use polynomial in $N$ parameters $(\epsilon, F_{a,\beta}, \ldots)$ to describe exponential in $N$ states. This contradiction indicates that the SYK model does not
have a quasiparticle picture.

Figure 1.2: Spectrum obtained from $N = 12$ exact diagonalization for $q = 2$ random hopping model (left) and $q = 4$ SYK model (right).

Now let us switch to the emergent conformal symmetry at strong coupling. At strong coupling $\beta J \gg 1$, we can drop the $i\omega_n$ term in the Schwinger-Dyson equations (1.22). Then by plugging the transformation:

$$
\tau \rightarrow f(\tau), \quad G(\tau_1, \tau_2) \rightarrow G_f(\tau_1, \tau_2) = (f'(\tau_1)f'(\tau_2))^\Delta G(f(\tau_1), f(\tau_2))
$$

with $\Delta = \frac{1}{q}$ in Eq. (1.22), we find that $G_f$ still satisfies the same equations. So this is a symmetry of the theory. We can treat this $i\omega_n$ term as the explicit symmetry breaking term like a small magnetic field in the ferromagnetic model that will help pin down the real ground state. The saddle point found by numerics actually gives the scaling form: $G(\tau) \sim \frac{1}{|\tau|^\Delta}$. This solution does not have a full conformal symmetry but a smaller $\text{PSL}_2(\mathbb{R})$ symmetry$^3$:

$$
f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R})
$$

This is like a spontaneous symmetry breaking picture, and there will be infinite many Goldstone mode denoted by the coset $\text{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$. With the inclusion of the explicit

$^3$which is also the isometry of $\text{AdS}_2$
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symmetry breaking term $i\omega_n$, these modes will be lifted from zero modes to soft modes. The model is called a nearly conformal quantum mechanics (NCFT$_1$) and is expected to dual to a nearly AdS$_2$ gravity [135, 137].

The procedure to get to the maximal chaos is a bit sophisticated [116, 119, 135] (we will show an explicit computation in Chapter 5). The basic idea is to use the representation of PSL$_2(\mathbb{R})$ to diagnose the four-point function, which is found to be a series of ladders that can be exactly summed over in the large $N$ limit. Then one finds that the soft modes we mentioned above will generate this maximal Lyapunov exponent. Actually, it is not that surprising that we get the maximal chaos since the model is holographic, and indeed the same behaviour can also be obtained in the gravity theory [137].

1.3 More about SYK

In this section, we will talk about some concerns about the SYK model. And will also list different aspects of the model that people are studying in the field.

There are common concerns about the study of SYK models: the first one is whether these models are experimental realizable or these are just toy models played with by theorists; the other one is that the model is defined in the large $N$ limit, but the real material is usually described by finite $N$ theories. To the first issue, there are several experimental proposals for SYK models [31, 34, 43, 125, 159] in atom-molecule systems, Fu-Kane superconductor, disordered quantum dots, or graphene flake. And in Ref. [133], the SYK model is experimentally investigated using a four-qubit NMR simulator. To the second concern, there is a recent work [191] considering a conventional lattice model. By mapping the scale $N$ to the range of the interaction, the lattice model can be viewed as a randomness-free rank–3 tensor model [120], which has the same physics as the normal disordered SYK model. The work has presented a finite $N$ ED simulation and the result is qualitatively consistent with the large $N$ properties: there is still a linear–$T$ resistivity and has an instability towards pairing state. So it is possible that the large $N$ properties of the SYK model are still applicable to
realistic models.

Now let’s introduce different aspects of the study of SYK models. The basic formalism is introduced in Ref. [108, 116, 119, 135, 162]. Since the model can be viewed as a lattice spin model by the Jordan-Wigner transformation, exact diagonalization (ED) studies will be straightforward to apply. In Chapter 2, we discuss some numerics from ED. There is also numerics on discussions of ETH in SYKs [88, 180]. Another quantity people are interested in and can be computed by ED is the spectral form factor [39, 66, 75, 104, 150] and can be used to discuss its relation to the random matrix theory. Analytical spectral density is obtained in Ref. [66, 67, 69]. When we consider generalizations to SYK models, a natural way is to add symmetries or generalize to higher dimensions. In Chapter 3, we have considered complex SYK models with $U(1)$ symmetry, more examples about adding SYK models with symmetries are discussed in Ref. [78, 195]. We can also generalize the model to higher dimensions, these include lattice generalizations and field theory generalizations. In Chapter 3, we consider a lattice generalization of complex SYK models. Other lattice generalizations are discussed in Ref. [13, 15, 27, 36, 42, 81, 83, 85, 109–111, 114, 156, 179, 198, 199]. Field theory generalizations are discussed in Ref. [16, 186]. The dynamical properties of the model are explored in Ref. [55, 82, 105, 122, 192]. Entanglement structures are discussed in Ref. [82, 99, 105, 129, 192]. In Chapter 5, we discuss the chaotic properties, and there are other papers talking about chaos with different set up [10, 17, 32, 33, 68]. In Chapter 6, we consider phase transitions in SYK models, while other works discuss the transition or crossover to normal Fermi liquid, many-body localization state, pairing state, Mott insulator etc [13, 18, 36, 42, 65, 84, 109–111, 156, 179]. In Chapter 4, we discuss a supersymmetric generalization of SYK model, and many aspects we mentioned above are also explored in the SUSY model [26, 70, 70, 104, 145, 145, 147, 149, 157, 194].

In the high energy theory community, there are theoretical studies on the Schwarzian theory [124, 141, 142, 164, 183]. There are also tensor models proposed, which do not have disorder but share similar properties as SYK models [120, 190], properties of these tensor
Chapter 1. Introduction

models and comparison with SYK models can be found in Ref. [49] and the references within. There are also many discussions on the possible dual gravity theory [44, 106, 137, 139, 148].

The SYK model also draws interests from mathematicians since the model is algebraically well-defined, there are mathematical papers discussing the spectrum of the model [62].

1.4 Organization of the Thesis

In the following, we will discuss some properties of the SYK models mentioned in the previous sections and some generalizations. The main results are summarized below.

In Chapter 2, we present numerical studies of the SYK models. The high temperature expansion and exact diagonalization of the $N$-site fermion model are used to compute the entropy density: our results are consistent with the numerical solution of $N = \infty$ saddle point equations, and the presence of a non-zero entropy density in the limit of vanishing temperature. The exact diagonalization results for the fermion Green’s function also appear to converge well to the $N = \infty$ solution. For the hard-core boson model, the exact diagonalization study indicates spin glass order. Some results on the entanglement entropy and the out-of-time-order correlators are also presented.

In Chapter 3, we compute the thermodynamic properties of the SYK models of fermions with a conserved fermion number, $Q$. We extend a previously proposed Schwarzian effective action to include a phase field, and this describes the low temperature energy and $Q$ fluctuations. We obtain higher-dimensional generalizations of the SYK models which display disordered metallic states without quasiparticle excitations, and we deduce their thermoelectric transport coefficients. We find a precise match between low temperature transport and thermodynamics of the SYK and holographic models. In both models, the Seebeck transport coefficient is exactly equal to the $Q$-derivative of the entropy. For the SYK models, quantum chaos, as characterized by the butterfly velocity and the Lyapunov rate, universally determines the thermal diffusivity, but not the charge diffusivity.

In Chapter 4, we discuss a supersymmetric generalization of the Sachdev-Ye-Kitaev model.
These are quantum mechanical models involving $N$ Majorana fermions. The supercharge is given by a polynomial expression in terms of the Majorana fermions with random coefficients. The Hamiltonian is the square of the supercharge. The $\mathcal{N} = 1$ model with a single supercharge has unbroken supersymmetry at large $N$, but non-perturbatively spontaneously broken supersymmetry in the exact theory. We analyze the model by looking at the large $N$ equation, and also by performing numerical computations for small values of $N$. We also compute the large $N$ spectrum of “singlet” operators, where we find a structure qualitatively similar to the ordinary SYK model. We also discuss an $\mathcal{N} = 2$ version. In this case, the model preserves supersymmetry in the exact theory and we can compute a suitably weighted Witten index to count the number of ground states, which agrees with the large $N$ computation of the entropy.

In Chapter 5, we consider a continuous set of boson-fermion SYK models. The low energy scaling dimension can be tuned by the boson fermion number ratio $\alpha = M/N$. The four-point function is computed in this model, where in the OPE regime, one can read off the spectrum. We obtain a Schwarzian action for the $h = 2$ modes that give the maximal chaos and we also compute the other $h \neq 2$ modes, which accounts for the correction to the maximal Lyapunov exponent. We find that the system is less chaotic in the free fermion limit. And we discuss the relation between Lyapunov time and dephasing time.

In Chapter 6, we describe the phases of a solvable $t$-$J$ model of electrons with infinite-range, and random, hopping and exchange interactions, similar to those in the SYK models. The electron fractionalizes, as in an ‘orthogonal metal’, into a fermion $f$ which carries both the electron spin and charge, and a boson $\phi$. Both $f$ and $\phi$ carry emergent $\mathbb{Z}_2$ gauge charges. The model has a phase in which the $\phi$ bosons are gapped, and the $f$ fermions are gapless and critical, and so the electron spectral function is gapped. This phase can be considered as a toy model for the underdoped cuprates. The model also has an extended, critical, ‘quasi-Higgs’ phase where both $\phi$ and $f$ are gapless, and the electron operator $\sim f\phi$ has a Fermi liquid-like $1/\tau$ propagator in imaginary time, $\tau$. So while the electron spectral function has a
Fermi liquid form, other properties are controlled by $\mathbb{Z}_2$ fractionalization and the anomalous exponents of the $f$ and $\phi$ excitations. This ‘quasi-Higgs’ phase is proposed as a toy model of the overdoped cuprates. We also describe the critical state separating these two phases.
2

Numerical studies on the SYK models

2.1 Introduction

Fermion and boson models with infinite-range random interactions were studied in the 1990’s and later [6, 28, 73, 74, 151, 173] as models of quantum systems with novel non-Fermi liquid or spin glass ground states. More recently, it was proposed that such models are holographically connected to the dynamics of AdS$_2$ horizons of charged black holes [168, 169], and remarkable connections have since emerged to topics in quantum chaos and black hole physics [5, 98, 108, 116, 136, 162, 171, 196].

The model introduced by Sachdev and Ye [173] (SY model) was defined on $N$ sites, and each site had particles with $M$ flavors; then the double limit $N \to \infty$, followed by $M \to \infty$, was taken. In such a limit, the random interactions depend on 2 indices, each taking $N$ values. Taking the double limit is challenging in numerical studies, and so they have been restricted to $M = 2$ with increasing values of $N$ [6, 28]. It was found that the ground state for $N \to \infty$ at a fixed $M = 2$ was almost certainly a spin glass. So a direct numerical test
Chapter 2. Numerical studies on the SYK models

of the more exotic non-Fermi liquid states has not so far been possible.

Kitaev [116] has recently introduced an alternative large $N$ limit in which the random interaction depends upon 4 indices, each taking $N$ values; the saddle-point equations in the $N \to \infty$ limit are the same as those in Ref. [173]. No separate $M \to \infty$ is required, and this is a significant advantage for numerical study. The present chapter will study such Sachdev-Ye-Kitaev (SYK) models by exact diagonalization; some additional results will also be obtained in a high temperature expansion. Our numerical studies will be consistent the fermionic SYK model displaying a non-Fermi liquid state which has extensive entropy, and entanglement entropy, in the zero temperature limit. For the case of the bosonic SYK model, our numerical study indicates spin-glass order: this implies that the analytic study of the large $N$ limit will require replica symmetry breaking [73].

The outline of this chapter is as follows. In Section 2.2, we review the large $N$ solution of the SYK model, and present new results on its high temperature expansion. In Section 2.3 we present exact diagonalization results for the fermionic SYK model, while the hard-core boson case is considered later in Appendix A.1.

2.2 Model

2.2.1 Large $N$ limit for fermions

This section will introduce the SYK model for complex fermions, and review its large $N$ limit. We will obtain expressions for the fermion Green’s function and the free energy density. A high temperature expansion for these quantities will appear in Section 2.2.3.

The Hamiltonian of the SYK model is

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij,\ell} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_{i} c_i^\dagger c_i$$ (2.1)
where the $J_{ij;kl}$ are complex Gaussian random couplings with zero mean obeying

\[
J_{ji;kl} = -J_{ij;kl}, \quad J_{ij;lk} = -J_{ij;kl}, \quad J_{kl;ij} = J_{ij;kl}^*, \quad |J_{ij;kl}|^2 = J^2.
\] (2.2)

The above Hamiltonian can be viewed as a ‘matrix model’ on Fock space, with a dimension which is exponential in $N$. But notice that there are only order $N^4$ independent matrix elements, and so Fock space matrix elements are highly correlated. The conserved U(1) density, $Q$ is related to the average fermion number by

\[
Q = \frac{1}{N} \sum_i \langle c_i^\dagger c_i \rangle.
\] (2.3)

The value of $Q$ can be varied by the chemical potential $\mu$, and ranges between 0 an 1. The solution described below applies for the range of $\mu$ for which $0 < Q < 1$, and so realizes a compressible state.

Using the imaginary-time path-integral formalism, the partition function can be written as

\[
Z = \int Dc^\dagger Dc \exp (-S)
\] (2.4)

where

\[
S = \int_0^\beta d\tau (c^\dagger \partial_\tau c + H),
\] (2.5)

where $\beta = 1/T$ is the inverse temperature, and we have already changed the operator $c$ into a Grassman number.

In the replica trick, we take $n$ replicas of the system and then take the $n \to 0$ limit

\[
\ln Z = \lim_{n \to 0} \frac{1}{n} (Z^n - 1)
\] (2.6)

Introducing replicas $c_{ia}$, with $a = 1 \ldots n$, we can average over disorder and obtain the
replicated imaginary time ($\tau$) action

$$S_n = \sum_{ia} \int_0^\beta d\tau c_{ia}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) c_{ia} - \frac{J^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}(\tau)c_{ib}(\tau') \right|^4.$$

(here we neglect normal-ordering corrections which vanish as $N \to \infty$). Then the partition function can be written as

$$Z_n = \int \prod_a Dc_a Dc_a^\dagger \exp(-S_n)$$

Notice that the action has a global SU($N$) symmetry under $c_{ia} \to U_{ij}c_{ja}$. Also, if we ignore the time-derivative term in Eq. (2.7), notice that the action has a U(1) gauge invariance under $c_{ia} \to e^{i\delta_i(\tau)}c_{ja}$. And indeed, in the low energy limit leading to Eq. (2.19), the time-derivative term can be neglected. However, we cannot drop the time-derivative term at the present early stage, as it plays a role in selecting the manner in which the U(1) gauge invariance is ‘broken’ in the low energy limit. In passing, we note that this phenomenon appears to be analogous to that described in the holographic study of non-Fermi liquids by DeWolfe et al. [51]: there, the bulk fermion representing the low energy theory is also argued to acquire the color degeneracy of the boundary fermions due to an almost broken gauge invariance. As in Ref. [51], we expect the bulk degrees of freedom of gravitational duals to the SYK model to carry a density of order $N$ [171].

Following the earlier derivation [173], we decouple the interaction by two successive Hubbard-Stratonovich transformations. First, we introduce the real field $Q_{ab}(\tau, \tau')$ obeying

$$Q_{ab}(\tau, \tau') = Q_{ba}(\tau', \tau).$$

The equation above is required because the action is invariant under the reparameterization
$a \leftrightarrow b, \tau \leftrightarrow \tau'$. In terms of this field

$$S_n = \sum_{ia} \int_0^\beta d\tau c_{ia}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) c_{ia} + \sum_{ab} \int_0^\beta d\tau d\tau' \left\{ \frac{N}{4J^2} [Q_{ab}(\tau, \tau')]^2 - \frac{1}{2N} Q_{ab}(\tau, \tau') \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^2 \right\}. \quad (2.10)$$

A second decoupling with the complex field $P_{ab}(\tau, \tau')$ obeying

$$P_{ab}(\tau, \tau') = P_{ba}^*(\tau', \tau) \quad (2.11)$$

yields

$$S_n = \sum_{ia} \int_0^\beta d\tau c_{ia}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) c_{ia} + \sum_{ab} \int_0^\beta d\tau d\tau' \left\{ \frac{N}{4J^2} [Q_{ab}(\tau, \tau')]^2 + \frac{N}{2} Q_{ab}(\tau, \tau') |P_{ab}(\tau, \tau')|^2 \right. \nonumber$$

$$\left. - Q_{ab}(\tau, \tau') P_{ba}(\tau', \tau) \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right\} \quad (2.12)$$

Now we study the saddle point of this action in the large $N$ limit. After integrating out
fermion field and take $\frac{\delta S}{\delta P_{ba}} = 0$, we obtain

$$P_{ab}(\tau, \tau') = \frac{1}{N} \langle c_{ia}^\dagger(\tau) c_{ib}(\tau') \rangle \quad (2.13)$$

Note that we have combined $\frac{N}{2} Q_{ab} |P_{ab}|^2$ and $\frac{N}{2} Q_{ba} |P_{ba}|^2$ as one term. Similarly, taking
 derivative with respect to $Q_{ab}$, we have

$$Q_{ab}(\tau, \tau') = J^2 |P_{ab}(\tau, \tau')|^2. \quad (2.14)$$

If we only consider diagonal solution in the replica space (non spin-glass state), we can define
the self energy:

$$\Sigma(\tau, \tau') = -Q(\tau, \tau') P(\tau', \tau), \quad (2.15)$$
and the Green’s function
\[ G(\tau, \tau') = -\langle T_\tau c(\tau) c^\dagger(\tau') \rangle. \] (2.16)

Then we have
\[ P(\tau, \tau') = G(\tau', \tau), \] (2.17)

and the saddle point solution becomes
\[ G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)} \]
\[ \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau) \] (2.18)

The above equation shows a re-parameterization symmetry at low temperature if we ignore the \( i\omega_n \) term \([116, 171]\). At zero temperature, the low energy Green’s function is found to be\([171, 173]\)
\[ G(z) = C e^{-i(\pi/4+\theta)} \sqrt{z}, \quad \text{Im}(z) > 0, \ |z| \ll J, \ T = 0 \] (2.19)

where \( C \) is a positive number, and \(-\pi/4 < \theta < \pi/4\) characterizes the particle-hole asymmetry. A full numerical solution for Eq. (5.6) at zero temperature was also obtained in Ref. \([173]\), and is shown in Fig. 2.1. We can see the \( 1/\sqrt{z} \) behavior at low energy. However, it is not possible to work entirely within this low energy scaling limit to obtain other low temperature properties: the \( i\omega_n \) term is needed to properly regularize the ultraviolet, and select among the many possible solutions of the low-energy equations \([74, 171]\).

### 2.2.2 Free energy and thermal entropy

The free energy is defined to be
\[ \mathcal{F} = -\frac{1}{\beta} \ln Z_{\text{eff}} \] (2.20)
Figure 2.1: Figure, adapted from Ref [173], showing the imaginary part of Green’s function multiplied by $\sqrt{\omega}$ as a function of $\omega$ at particle-hole symmetric point $\theta = 0$. Our definition of the Green’s function, Eq. (2.16), differs by a sign from Ref. [173].

where $Z_{\text{eff}}$ has only one replica. So

$$Z_{\text{eff}} = \int Dc^\dagger Dc \exp(-S),$$

(2.21)

with

$$S = \sum_i \int_0^\beta d\tau c_i^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) c_i + \int_0^\beta d\tau d\tau' \left\{ \frac{N}{4J^2} [Q(\tau, \tau')]^2 + \frac{N}{2} Q(\tau, \tau') |P(\tau, \tau')|^2 
- Q(\tau, \tau') P(\tau', \tau) \sum_i c_i^\dagger(\tau)c_i(\tau') \right\},$$

(2.22)
Chapter 2. Numerical studies on the SYK models

For the free energy density $F/N$, we can just drop the site index $i$ to give the single site action, substituting the Green’s function and self energy

$$
S = \int_0^\beta d\tau d\tau' c^\dagger(\tau) \left( \frac{\partial}{\partial \tau} \delta(\tau - \tau') - \mu \delta(\tau - \tau') + \Sigma(\tau, \tau') \right) c(\tau')
+ \int_0^\beta d\tau d\tau' \left\{ \frac{1}{4J^2} [\Sigma(\tau, \tau')/G(\tau, \tau')]^2 - \frac{1}{2} \Sigma(\tau, \tau') G(\tau', \tau) \right\}
$$

(2.23)

After integrating out the fermion field

$$
S = -\text{Tr} \ln [(\partial_\tau - \mu) \delta(\tau - \tau') + \Sigma(\tau, \tau')] + \int_0^\beta d\tau d\tau' \left\{ \frac{1}{4J^2} [\Sigma(\tau, \tau')/G(\tau, \tau')]^2 - \frac{1}{2} \Sigma(\tau, \tau') G(\tau', \tau) \right\}
$$

(2.24)

To verify this result, we can vary with respect to $\Sigma(\tau, \tau')$ and $G(\tau, \tau')$, also using the fact that $\Sigma(\tau, \tau') = \Sigma^*(\tau', \tau)$, $G(\tau, \tau') = G^*(\tau', \tau)$, to obtain the equations of motions as before.

In the large $N$ limit, we can substitute in the classical solution, and then free energy density is

$$
\frac{\mathcal{F}}{N} = T \sum_n \ln (-\beta G(i\omega_n)) - \int_0^\beta d\tau \frac{3}{4} \Sigma(\tau) G(-\tau)
$$

(2.25)

The thermal entropy density can be obtained by

$$
\frac{S}{N} = -\frac{1}{N} \frac{\partial \mathcal{F}}{\partial T}
$$

(2.26)

2.2.3 High temperature expansion

Now we present a solution of Eqs. (5.6) by a high temperature expansion (HTE). Equivalently, this can be viewed as an expansion in powers of $J$.

We will limit ourselves to the simpler particle-hole symmetric case, $Q = 1/2$, for which both $G$ and $\Sigma$ are odd functions of $\omega_n$. We start with the high temperature limit

$$
G_0(i\omega_n) = \frac{1}{i\omega_n}
$$

(2.27)
and then expand both \( G \) and \( \Sigma \) in powers of \( J^2 \): \( G = G_0 + G_1 + \cdots \) and \( \Sigma = \Sigma_0 + \Sigma_1 + \cdots \). The successive terms can be easily obtained by iteratively expanding both equations in Eq. (5.6), and repeatedly performing Fourier transforms between frequency and time.

\[
\begin{align*}
\Sigma_1(i\omega_n) &= J^2 \frac{1}{4i\omega_n} \\
G_1(i\omega_n) &= J^2 \frac{1}{4(i\omega_n)^3} \\
\Sigma_2(i\omega_n) &= J^4 \frac{3}{16(i\omega_n)^3} \\
G_2(i\omega_n) &= J^4 \frac{1}{4(i\omega_n)^5} \\
\Sigma_3(i\omega_n) &= J^6 \left[ \frac{15}{32(i\omega_n)^5} + \frac{3}{128T^2(i\omega_n)^3} \right] \\
G_3(i\omega_n) &= J^6 \left[ \frac{37}{64(i\omega_n)^7} + \frac{3}{128T^2(i\omega_n)^5} \right] \\
\Sigma_4(i\omega_n) &= J^8 \left[ \frac{561}{256(i\omega_n)^7} + \frac{75}{512T^4(i\omega_n)^5} - \frac{1}{256T^4(i\omega_n)^3} \right] \\
G_4(i\omega_n) &= J^8 \left[ \frac{5}{2(i\omega_n)^9} + \frac{81}{512T^2(i\omega_n)^7} - \frac{1}{256T^4(i\omega_n)^5} \right]
\end{align*}
\]

(2.28)

The free energy density can be written in terms of \( G(i\omega_n) \) and \( \Sigma(i\omega_n) \)

\[
\frac{F}{N} = T \sum_n \ln (-\beta G(i\omega_n)) - \frac{3T}{4} \sum_n \Sigma(i\omega_n) G(i\omega_n)
\]

(2.29)

We also need to regularize the above free energy by subtracting and adding back the free particle part

\[
\frac{F}{N} = -T \ln 2 + T \sum_n \left[ \ln (-\beta G(i\omega_n)) - \ln (-\beta i\omega_n) \right] - \frac{3T}{4} \sum_n \Sigma(i\omega_n) G(i\omega_n)
\]

(2.30)

The series expansion of the entropy density is

\[
\frac{S}{N} = \ln 2 - \frac{1}{64} \frac{J^2}{T^2} + \frac{1}{512} \frac{J^4}{T^4} - \frac{11}{36864} \frac{J^6}{T^6} + \frac{599}{11796480} \frac{J^8}{T^8} + \cdots
\]

(2.31)
Next, we describe our numerical solution of Eq. (5.6) at non-zero temperature. We used a Fourier transform (FT) to iterate between the two equations, until we obtained a convergent solution. For faster convergence, we started at high temperature, and used the above high temperature expansion as the initial form. Then we decreased temperature to get the full temperature dependence. We compare the large $N$ exact numerical result with the high temperature expansion in Fig. 2.2. At high temperatures, all methods converge to $\ln 2$ as expected. The HTE results fit the exact numerics quite well for $T/J > 0.6$, but are no longer accurate at lower $T$. The exact numerics shows a finite entropy density in the limit of vanishing temperature, with a value consistent with earlier analytic results [74][171].

Figure 2.2: Entropy computation from exact large $N$ EOM and HTE: at high temperature, all approaches the infinite temperature limit $S/N = \ln 2$. HTE result fit the exact result quite well for $T/J > 0.6$. 


2.3 Results

2.3.1 Exact diagonalization for fermions

We now test the validity of the large $N$ results by comparing with an exact diagonalization (ED) computation at finite $N$. For the numerical setup, it was useful to employ the Jordan-Wigner transformation to map the Hamiltonian to a spin model

$$c_i = \sigma_i^- \prod_{j<i} \sigma_j^z, \quad c_i^\dagger = \sigma_i^+ \prod_{j<i} \sigma_j^z$$  \hspace{1cm} (2.32)

We built a matrix of the $N$ spins and diagonalized it numerically. After obtaining the full spectrum, we obtained both the imaginary part of Green’s function and thermal entropy, and our results are compared with the large $N$ results in Fig. 2.1 and Fig. 2.2.

In this note, we focus on the particle-hole symmetric point. But particle-hole symmetry does not correspond to the point $\mu = 0$ in the Hamiltonian Eq. (2.1), because there are quantum corrections to the chemical potential $\delta \mu \sim \mathcal{O}(N^{-1})$ coming from the terms in which $i, j, k, l$ are not all different from each other, because these terms are not particle-hole symmetric. So we use a Hamiltonian with extra correction terms that compensate $\delta \mu$:

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^N J_{ij;kl} (c_i^\dagger c_j^\dagger c_k c_\ell + \delta_{ik} n c_j^\dagger c_k - \delta_{il} n c_j^\dagger c_k - \delta_{jk} n c_i^\dagger c_\ell + \delta_{jl} n c_i^\dagger c_k),$$  \hspace{1cm} (2.33)

where we use $n = 1/2$ for the particle-hole symmetric case.

We define the on-site retarded Green’s function by

$$G_i^R(t, t') = -i\theta(t - t') \langle \{c_i(t), c_i^\dagger(t')\} \rangle.$$  \hspace{1cm} (2.34)
Using Lehmann representation

\[ G^R_i(\omega) = \frac{1}{Z} \sum_{nn'} \frac{\langle n|c_i|n'\rangle \langle n'|c_i^\dagger|n\rangle}{\omega + E_n - E_{n'} + i\eta} (e^{-\beta E_n} + e^{-\beta E_{n'}}), \] (2.35)

at zero temperature, we obtain

\[ G^R_i(0) = \sum_{n'} \frac{\langle 0|c_i|n'\rangle \langle n'|c_i^\dagger|0\rangle}{\omega + E_0 - E_{n'} + i\eta} + \frac{\langle 0|c_i^\dagger|n'\rangle \langle n'|c_i|0\rangle}{\omega - E_0 + E_{n'} + i\eta}. \] (2.36)

Using \((\omega + i\eta)^{-1} = P \frac{1}{\omega} - i\pi\delta(\omega)\)

\[ \text{Im} G^R_i(\omega) = -\pi \sum_{n'} \left[ \langle 0|c_i|n'\rangle \langle n'|c_i^\dagger|0\rangle \delta(\omega + E_0 - E_{n'}) + \langle 0|c_i^\dagger|n'\rangle \langle n'|c_i|0\rangle \delta(\omega - E_0 + E_{n'}) \right]. \] (2.37)

Numerically, we replace the delta function with a Lorentzian by taking a small \(\eta:\)

\[ \delta(E_0 - E_{n'} + \omega) = \lim_{\eta \to 0^+} \frac{1}{\pi} \frac{\eta}{(E_0 - E_{n'} + \omega)^2 + \eta^2}. \] (2.38)

A subtlety in the above numerics, when \(\mu = 0\), is the presence of an anti-unitary particle-hole symmetry. The ground state turns out to be doubly degenerate for some system sizes. If so, we will have two ground states \(|0\rangle\) and \(|0'\rangle\) in the expression of \(G^R_i(\omega)\), and we need to sum them up to get the correct Green’s function.

To better understand this degeneracy, we can define the particle-hole transformation operator

\[ P = \prod_i (c_i^\dagger + c_i) K \] (2.39)

where \(K\) is the anti-unitary operator. One can show that it is a symmetry of our Hamiltonian Eq. (2.33), \([H, P] = 0\). When the total site number \(N\) is odd, we know any eigenstate \(|\Psi\rangle\) and its particle-hole partner \(P|\Psi\rangle\) must be different and degenerate. For even site number, these two states may be the same state. However for \(N = 2 \mod 4\), one can show that \(P^2 = -1\), and then the degeneracy is analogous to the time-reversal Kramers doublet for
$T^2 = -1$ particles. We expect that all the eigenvalues must be doubly degenerate. For $N = 0 \mod 4$, $P^2 = 1$, there is no protected degeneracy in the half filling sector. These facts were all checked by numerics, and carefully considered in the calculation of the Green’s function.

A better understanding of the above facts can be reached from the perspective of symmetry-protected topological (SPT) phases. As shown recently in Ref. [196], the complex SYK model can be thought of as the boundary of a 1D SPT system in the symmetry class AIII. The periodicity of 4 in $N$ arises from the fact that we need to put 4 chains to gap out the boundary degeneracy without breaking the particle-hole symmetry. In the Majorana SYK case, the symmetric Hamiltonian can be constructed as a symmetric matrix in the Clifford algebra $Cl_{0,N-1}$, and the Bott periodicity in the real representation of the Clifford algebra gives rise to a $\mathbb{Z}_8$ classification[196]. Here, for the complex SYK case, we can similarly construct the Clifford algebra by dividing one complex fermion into two Majorana fermions, and then we will have a periodicity of 4.

2.3.2 Green’s function

From the above definition of retarded Green’s function, we can relate them to the imaginary time Green’s function as defined in Eq. (2.16), $G^R(\omega) = G(i\omega_n \rightarrow \omega + i\eta)$. In Fig. 2.3, we show a comparison between the imaginary part of the Green’s function from large $N$, and from the exact diagonalization computation. The spectral function from ED is particle-hole symmetric for all $N$, this is guaranteed by the particle-hole symmetry and can be easily shown from the definition of the spectral function Eq. (2.37). The two results agree well at high frequencies. At low frequencies, the deviations between the exact diagonalization and large $N$ results get smaller at larger $N$.

For a quantitative estimate of the deviations between the large $N$ and exact diagonalization results, we compute the areas under each curve in Fig. 2.3, and compare their difference:
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Figure 2.3: Imaginary part of the Green’s function in real frequency space from large $N$ and exact diagonalization. The inset figure is zoomed in near $\omega = 0$.

\[
\Delta \rho = \int d\omega |\text{Im} G_{ED}(\omega) - \text{Im} G_{N=\infty}(\omega)| \tag{2.40}
\]

As shown in Fig. 2.4, the convergence to the $N = \infty$ limit is slow, possibly with a power smaller than $1/N$.

2.3.3 Entropy

We can also compute the finite temperature entropy from ED. The partition function can be obtained from the full spectrum

\[
Z = \sum_n e^{-\beta E_n}, \tag{2.41}
\]
Figure 2.4: The difference of integrated spectral function between ED at different $N$ and large $N$ result. The difference appears to be tending to 0 as $N$ approaches infinity.

where $E_n$ is the many-body energy, and then free energy density is

$$\frac{F}{N} = -\frac{\beta}{N} \log Z. \quad (2.42)$$

We can obtain the entropy density from

$$\frac{S}{N} = \frac{1}{N} \frac{\langle E \rangle - F}{T} \quad (2.43)$$

where $\langle E \rangle = \sum_n \frac{E_n e^{-\beta E_n}}{Z}$ is the average energy.

We use this approach to compute the thermal entropy from the full spectrum, and compare it with the thermal entropy calculated from large $N$ equations of motion Eq. (5.6). As shown in Fig. 2.5, the finite size ED computation gives rise to the correct limit $s = \ln 2$ in the high temperature regime, and it agrees with the large $N$ result quite well for $T/J > 0.5$. Although there is a clear trend that a larger system size gives rise to larger thermal entropy at low
Figure 2.5: Thermal entropy computation from ED, and large \( N \). At high temperature, all results the infinite temperature limit \( S/N = \ln 2 \). At low temperature, all ED results go to zero, but do approach the \( N = \infty \) results with increasing \( N \). Note that the limits \( N \to \infty \) and \( T \to 0 \) do not commute, and the non-zero entropy as \( T \to 0 \) is obtained only when the \( N \to \infty \) is taken first.

As in Fig. 2.4, we estimate the deviation from the large \( N \) theory by defining

\[
\Delta S = \int dT |S_{ED}(T)/N - S_{N=\infty}(T)/N_{\infty}|,
\]

and plot the result in Fig. 2.6. The finite size correction goes to 0 as \( 1/N \) goes to zero.

### 2.3.4 Entanglement entropy

Finally, we compute the entanglement entropy in the ground state, obtained by choosing a subsystem \( A \) of \( N \) sites, and tracing over the remaining sites; the results are in Fig. 2.7. For \( N_A < N/2 \), we find that \( S_{EE} \) is proportional to \( N_A \), thus obeying the volume law, and so
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Figure 2.6: The difference of integrated thermal entropy between ED at different N and large N result. The difference goes to 0 as $1/N$ approaches 0.

Figure 2.7: Entanglement entropy for the ground state. We divide the system into two subsystems, A and B, we trace out part B and calculate the entropy for the reduced density matrix $\rho_A$. The x-axis is the size of subsystem A.
even the ground state obeys eigenstate thermalization [196].

2.3.5 Out-of-time-order correlations and scrambling

One of the interesting properties of the SYK model is that it exhibits quantum chaos [116, 162]. The quantum chaos can be quantified in terms of an out-of-time-ordered correlator \( \langle A(t)B(0)A(t)B(0) \rangle \) (OTOC) obtained from the cross terms in \( \langle [A(t), B(0)]^2 \rangle \) [136]. The exponential decay in the OTOC results in an exponential growth of \( \langle [A(t), B(0)]^2 \rangle \) at short times, and the latter was connected to analogous behavior in classical chaos. In particular, Ref. [136], established a rigorous bound, \( 2\pi/\beta \), for the decay rate, \( \lambda_L \), of the OTOC, and the Majorana SYK model is expected [116] to saturate this bound in the strong-coupling limit \( \beta J \gg 1 \). In the opposite perturbative limit, \( \beta J \ll 1 \), one expects \( \lambda_L \sim J \). Ref. [98] performed a ED calculation of the OTOC on the Majorana SYK model in the infinite temperature limit, \( \beta = 0 \). Here we will perform a similar calculation on the complex SYK model, and also obtain results at large \( \beta J \). We define our renormalized OTOC by

\[
\text{OTOC} = -\frac{\langle A(t)B(0)A(t)B(0) \rangle + \langle B(0)A(t)B(0)A(t) \rangle}{2\langle AA \rangle\langle BB \rangle}.
\]  

(2.45)

We choose the Hermitian Majorana operators \( A = c_1 + c_1^\dagger, \ B = c_2 + c_2^\dagger \). The negative sign gives a positive initial value for OTOC. At infinite temperature, the result is shown in Fig. 2.8. We observe the fast scrambling effect from the quick decay of OTOC, and the early time decay rate \( \lambda_L \) is proportional to \( J \) as expected. Similar behavior is found in the Majorana SYK model [98]. At finite temperature, although we can perform the computation in the strong coupling limit \( \beta J \gg 1 \), because of finite size effects, we do not get the predicted decay rate \( \lambda_L = 2\pi/\beta \). And the OTOC only has a weak dependence on \( \beta \) even in the strong coupling limit as shown in Fig. 2.9. Theoretically [116][136], in the large \( N \) and strong coupling conformal limit \( N \gg \beta J \gg 1 \), during the scrambling process \( \beta \ll t < \beta \log N \), we have \( 1 - \text{OTOC} \sim (\beta J/N)e^{(2\pi/\beta)t} \). Because of numerical computation
power constrain, we cannot reach the large $N$ conformal limit, and therefore we don’t have a well defined scrambling period. As a consequence, Fig. 2.9 does not display a large change in the exponent, and the pre-factor difference is also small. It is clearly that of our small system sizes, $J$ is the most relevant energy scale that controlling the chaos.

We can choose other operators for $A$ and $B$. For instance, we can choose them to be the same $A = B = c_1 + c_1^\dagger$. Then the initial value of OTOC will be -1 and after the scrambling process it will eventually saturate to 0 (this is also defined in Kitaev’s talk[116]). However if we choose $A = c_1^\dagger c_1$, $B = c_2^\dagger c_2$, OTOC will not change too much from its initial value. And if we choose other composite operators for $A$ and $B$, OTOC may decay but with different decay rate. In the integral models like the Ising model as discussed in Ref. [136], the OTOC depends on the monodromy behavior of conformal blocks when doing the analytical continuation. One can show that in RCFT models, the monodromy comes from the braiding data[29, 80]. One may or may not have this scrambling effect depends on the operators.
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Figure 2.9: OTOC as a function of time at different temperature with interaction strength $J = 1$. Here the total system size $N = 7$.

one choose. In the SYK model, it seems that the scrambling also depends on our choice of operators. The OTOC of composite operators may be thought as the sum of some single operator OTOC after the OPE and may exhibit different behavior.

2.4 Discussion

We have presented exact diagonalization results on the fermionic SYK model. The trends in the computed Green’s functions, high temperature expansion, entropy density, and entanglement entropy all support the conclusion that the large $N$ limit approaches the compressible non-Fermi liquid state obtained in the earlier $N = \infty$ analysis. Note that the entropy density approaches a non-zero value in the limit $T \to 0$ taken after the $N \to \infty$, and so the ground state itself exhibits eigenstate thermalization. This conclusion is also supported by the volume-law behavior of the entanglement entropy. The original model of Ref. [173] was argued [74] to have an instability to spin glass order at temperatures exponentially small
in $\sqrt{M}$; the consonance between large $N$ theory and our finite $N$ numerics indicates that the model in (2.1) (with a random interaction with 4 indices [116]) does not have such an instability.

For the SYK model for hard-core bosons, our results for the single-particle Green’s function were very different, and indicate the presence of spin glass order. Similar quantum spin glass states were examined in random models of bosons in Refs. [73, 74].

After the completion of this work, there are other papers with further discussions on related topics. Ref. [9] has discussed finite $N$ effect in two-point function using Liouville quantum mechanics. Ref. [99] has computed the entanglement entropy analytically which agrees with larger system size numerics in [129]. Ref. [121] has performed a larger size numerics in computing the OTOC and has a systematic improvement at low temperature towards the analytic result. There are more discussions about ETH in SYK in [88, 149, 180].
3.1 Introduction

Strange metal states are ubiquitous in modern quantum materials [24]. Field theories of strange metals [170] have largely focused on disorder-free models of Fermi surfaces coupled to various gapless bosonic excitations which lead to breakdown of the quasiparticle excitations near the Fermi surface, but leave the Fermi surface intact. On the experimental side [24, 40, 132, 197], there are numerous indications that disorder effects are important, even though many of the measurements have been performed in nominally clean materials. Theories of strange metals with disorder have only examined the consequences of dilute impurities perturbatively [93, 140, 153].

Disordered metallic states have been extensively studied [56, 126] under conditions in which the quasiparticle excitations survive. The quasiparticles are no longer plane wave states as they undergo frequent elastic scattering from impurities, and spatially random and extended quasiparticle states have been shown to be stable under electron-electron interactions [1]. In
contrast, the literature on quantum electronic transport is largely silent on the possibility of disordered conducting metallic states at low temperatures without quasiparticle excitations, when the electron-electron scattering length is of order or shorter than the electron-impurity scattering length. (Previous studies include a disordered doped antiferromagnet in which quasiparticles eventually reappear at low temperature [151], and a partial treatment of weak disorder in a model of a Fermi surface coupled to a gauge field [86].)

On the other hand, holographic methods do yield many examples of conducting quantum states in the presence of disorder and with no quasiparticle excitations [48, 54, 79, 89–92, 94, 95, 131, 165]. If we assume that the main role of disorder is to dissipate momentum, and we average over disorder to obtain a spatially homogeneous theory, then we may consider homogeneous holographic models which do not conserve momentum. Many such models have been studied [4, 11, 19, 22, 45–47, 52, 53, 77, 188], and their transport properties have been worked out in detail. However, there is not a clear quantum matter interpretation of these disordered, non-quasiparticle, metallic states.

The Sachdev-Ye-Kitaev (SYK) models [73, 74, 116, 151, 173] are theories of fermions with a label $i = 1 \ldots N$ and random all-to-all interactions. They have many interesting features, including the absence of quasiparticles in a non-trivial, soluble limit in the presence of disorder at low temperature ($T$). In the limit where $N$ is first taken to infinity and the temperature is subsequently taken to zero, the entropy $/N$ remains non-zero. Note, however, that such an entropy does not imply an exponentially large ground state degeneracy: it can be achieved by a many-body level spacing that is of the same order near the ground state as at a typical excited state energy. The SYK models were connected holographically to black holes with AdS$_2$ horizons, and the $T \to 0$ limit of the entropy was identified as the Bekenstein-Hawking entropy in Refs. [168, 169, 171]. Many recent studies have taken a number of perspectives, including the connections to two-dimensional quantum gravity [2, 3, 5, 9, 15, 39, 41, 43, 57, 64, 66, 78, 83, 98, 106–108, 116, 135, 137, 162, 171, 196]. The gravity duals of the SYK models are not in the category of the familiar AdS/CFT correspondence [182], and
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their low-energy physics is controlled by a symmetry-breaking pattern [135] which also arises in a generic two-derivative theory of dilaton gravity on a nearly AdS$_2$ spacetime [57, 106, 137].

With disordered metallic states in mind, in this chapter, we will study a class of SYK models [73, 74, 151, 173] which have a conserved fermion number$^1$. The SYK models have recently been extended to lattice models in one or more spatial dimensions [83] (see also [15]), which has opened an exploration into their transport properties. In this work, we shall further extend these higher-dimensional models to include a conserved fermion number. This will allow us to describe the thermoelectric response functions of a solvable metallic state without quasiparticles and in the presence of disorder. The transport properties can also be studied in Einstein-Maxwell-axion holographic theories, but we will not discuss this topic here. Readers can find details in the original paper.

3.2 Model

3.2.1 Complex SYK model

The zero-dimensional SYK model of interest to us has canonical complex fermions $f_i$ labeled by $i = 1 \ldots N$. We refer to them as the complex SYK models. The Hamiltonian is

$$H_0 = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_q \leq N, \\ 1 \leq i_{q+1} < i_{q+2} < \ldots < i_q \leq N}} J_{i_1,i_2\ldots,i_q} f_{i_1}^\dagger f_{i_2}^\dagger \cdots f_{i_{q/2}}^\dagger f_{i_{q/2+1}} \cdots f_{i_{q-1}} f_{i_q}. \quad (3.1)$$

$^1$A word about global symmetries is in order. The Majorana SYK model [116] with $2N$ Majorana fermions has a SO($2N$) symmetry only after averaging over disorder. However, this symmetry is not generated by a conserved charge. The model in Eq. (3.1) has a global U(1) symmetry for each realization of the disorder, and so this symmetry corresponds to a conserved charge. It is this symmetry which is of interest in this work. For completeness, we note that this model acquires an additional SU($N$) symmetry after averaging over disorder.
Here \( q \) is an even integer, and the couplings \( J_{i_1,i_2,...,i_q} \) are random complex numbers with zero mean obeying

\[
J_{i_1,i_2,...,i_{q/2},i_{q/2+1},...,i_q-1,i_q} = J_{i_q/2+1,i_q-1,i_q,i_1,i_2,...,i_{q/2}}^*,
\]

\[
\frac{|J_{i_1,i_2,...,i_q}|^2}{N^{q-1}} = \frac{J^2(q/2)!^2}{N^{q-1}}.
\]

Note that the case \( q = 2 \) is special and does have quasiparticles: this describes free fermions and the eigenstates of the random matrix \( J_{i_1,i_2} \) obey Wigner-Dyson statistics [12]. Our attention will be focused on \( q \geq 4 \), and \( N \gg q \), when the model flows to a phase without quasiparticles with an emergent conformal symmetry at low energies. We define the fermion number \(-1/2 < Q < 1/2\) by

\[
Q = \frac{1}{N} \sum_i \langle f_i^\dagger f_i \rangle - 1/2.
\]

We also define

\[
\Delta = \frac{1}{q},
\]

which will be the low-energy scaling dimension of the fermion \( f \).

### 3.2.2 LARGE \( N \) SADDLE POINT

Now we employ Green’s functions in the grand canonical ensemble at a chemical potential \( \mu \). Starting from a perturbative expansion of \( H_0 \) in Eq. (3.1), and averaging term-by-term, we obtain the following equations for the Green’s function and self energy in the large \( N \) limit:

\[
\Sigma(\tau) = -(-1)^{q/2} q J^2 [G(\tau)]^{q/2} [G(-\tau)]^{q/2-1}
\]

\[
G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)};
\]

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where $\omega_n$ is a Matsubara frequency $\omega_n = 2\pi(n + \frac{1}{2})$ and $\tau$ is imaginary time. As in Refs. [74, 173], we make the following IR ansatz at a complex frequency $z$

$$G(z) = C \frac{e^{-i(\pi\Delta+\theta)}}{z^{1-2\Delta}}, \quad \Im(z) > 0,$$

(3.7)

which is expressed in terms of three real parameters, $C$, $\Delta$ and $\theta$. Here the complex frequency is small compared to the disorder, $|z| \ll J$. Unitarity implies that the spectral weight is positive, which in turn implies that

$$-\pi\Delta < \theta < \pi\Delta.$$  

(3.8)

The particle-hole symmetric value is $\theta = 0$. Inserting Eq. (3.7) into (3.6), a straightforward analysis described in Appendix B.1 shows that $\Delta$ is given by (3.4), while

$$C = \left[ \frac{\Gamma(2(1 - 1/q))}{\pi q J^2} \right]^{1/q} \left[ \frac{\pi}{\Gamma(2/q)} \right]^{1-1/q} \left[ \sin(\pi/q + \theta) \sin(\pi/q - \theta) \right]^{1/q-1/2}.$$  

(3.9)

The value of $\theta$ remains undetermined in this IR analysis. Below, in Eq. (3.31), we find an exact relationship between $\theta$ and the density $Q$, as was first found in Ref. [74] for the $\Delta = 1/4$ theory.

Fourier transforming the fermion Green’s function Eq. (3.7) gives

$$G(\tau) \sim \begin{cases} 
-|\tau|^{-2\Delta}, & \tau > 0, \\
 e^{-2\pi \mathcal{E} |\tau|^{-2\Delta}}, & \tau < 0,
\end{cases}$$  

(3.10)

where the “spectral asymmetry” $\mathcal{E}$ is related to $\theta$ as

$$e^{2\pi \mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}.$$  

(3.11)

The asymmetry in (3.10) was argued in [152] (and reviewed in [171]) to fix the $T$ derivative
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of \( \mu \)

\[
\lim_{T \to 0} \left( \frac{\partial \mu}{\partial T} \right)_Q = -2\pi \mathcal{E} . \tag{3.12}
\]

So this \( \mathcal{E} \) is the same as that introduced in Eq. (3.16).

The \( T > 0 \) generalization of Eq. (3.10) is a saddle point of the action in Eq. (B.58)

\[
G_s(\tau) = -C \left( \frac{\Gamma(2\Delta) \sin(\pi \Delta + \theta)}{\pi} \right) e^{-2\pi \mathcal{E} \tau} \left( \frac{\pi T}{\sin(\pi T \tau)} \right)^{2\Delta} , \quad 0 < \tau < \frac{1}{T} , \tag{3.13}
\]

with \( \mathcal{E} \) remaining independent of \( T \) provided \( Q \) is held fixed. This result was found both in [152, 171], and in the AdS\(_2\) computation in [58]. After using the KMS condition and (3.11), the \( T \to 0 \) limit of (3.13) agrees with (B.7). In Appendix ??, we further discuss the kernel of the model and extract out the spectrum of the theory.

3.3 Results

3.3.1 Thermodynamics

We will show in this section that the canonical free energy, \( NF \), of the Hamiltonian \( H_0 \) in Eq. (3.1) has a low temperature (\( T \)) expansion

\[
F(Q,T) = E_0(Q) - TS(Q) + \ldots . \tag{3.14}
\]

where the ground state energy, \( E_0(Q) \), is not universal, but the zero-temperature entropy \( S(Q) \) is \textit{universal}, meaning that it depends only on the scaling dimension \( \Delta \) and is independent of high energy ("UV") details, such as higher order fermion interactions that could be added to Eq. (3.1). The value of \( E_0(Q) \) is not known analytically, but can only be computed numerically or in a large \( q \) expansion as we perform in Appendix B.3. However, remarkably, we can obtain exact results for the universal function \( S(Q) \) for all \( 0 < \Delta < 1/2 \), and for all \(-1/2 < Q < 1/2\) (see Fig. 3.3). These results agree with those obtained earlier for the
special cases $\Delta = 1/4$ and all $Q$ in Ref. [74], and for $Q = 0$ and all $\Delta$ in Ref. [116]. The higher-dimensional complex SYK models also have a free energy of the form Eq. (3.14), where $NF$ is now understood to be the free energy per site of the higher-dimensional lattice.

Because of the non-universality of $E_0(Q)$, the universal properties of the thermodynamics are more subtle in the grand canonical ensemble. The chemical potential, $\mu = (\partial F/\partial Q)_T$, has both universal and non-universal contributions. Consequently it requires a delicate computation to extract the universal portions of the grand potential $\Omega/N$,

$$\Omega(\mu, T) = F - \mu Q.$$ (3.15)

It is interesting to note that this universal dependence on $Q$, and not $\mu$, is similar to that in the Luttinger theorem for a Fermi liquid: there the Fermi volume is a universal function of $Q$, but the connection with $\mu$ depends upon many UV details. And indeed, the computation of $S(Q)$ in Ref. [74] for $\Delta = 1/4$ employs an analysis which parallels that used to prove the Luttinger theorem in Fermi liquids.

A quantity that will play a central role in our analyses of the SYK and holographic models is

$$2\pi \mathcal{E} = - \lim_{T \to 0} \frac{\partial^2 F}{\partial Q \partial T} = \frac{dS}{dQ} = - \lim_{T \to 0} \left( \frac{\partial \mu}{\partial T} \right)_Q.$$ (3.16)

Note that $\mathcal{E}$ is also universal. The factor of $2\pi$ has been inserted because then, for theories dual to gravity with an AdS$_2$ near-horizon geometry, $\mathcal{E}$ is the electric field in the AdS$_2$ region [171, 175, 176].

Now let us start the analysis. Ref. [135] has given an expression for the free energy of the Majorana SYK models as a functional of the Green’s function and the self energy; related expressions were given earlier for the complex SYK models [74, 171]. It is straightforward to obtain a similar result for the grand potential of the complex SYK models, which we give in Eq. (B.58). Here, we will only compute the $\Delta$ derivative of the grand potential in the $T \to 0$ limit, and then integrate with respect to $\Delta$ to obtain the low-temperature grand potential.
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The only term in the grand potential which explicitly depends on $q$ is (see Eq. (B.58))

$$\Omega = \ldots - J^2 \int_0^{1/T} d\tau [G(\tau)]^{q/2} [G(1/T - \tau)]^{q/2}. \quad (3.17)$$

Substituting $\Delta = 1/q$ and using (3.9), (3.11) and (3.13) the leading low-temperature derivative of $\Omega$ with respect to $\Delta$ comes from this term,

$$\frac{d\Omega}{d\Delta} = -2(2\Delta - 1) \frac{\sin(\pi \Delta + \theta) \sin(\pi \Delta - \theta)}{\sin(2\pi \Delta)} \int_0^{1/T} d\tau \left( \frac{\pi T}{\sin(\pi T \tau)} \right)^2 \ln \left( \frac{\Lambda}{\sin(\pi T \tau)} \right)$$

$$= -2\pi T(2\Delta - 1) \frac{\sin(\pi \Delta + \theta) \sin(\pi \Delta - \theta)}{\sin(2\pi \Delta)} + \text{a term of order } J, \quad (3.18)$$

where $\Lambda$ is some $\tau$-independent constant. We subtract the term in $\Omega$ of order $J$, which we call $\Omega_0$. Using Eq. (3.11) to express the result in terms of $E$ we obtain

$$\frac{d(\Omega - \Omega_0)}{d\Delta} = -\pi T(2\Delta - 1) \frac{\sin(2\pi \Delta)}{\cos(2\pi \Delta) + \cosh(2E \pi)}$$

$$= -\frac{\pi T(2\Delta - 1)}{2} \left[ \tan(\pi(\Delta - iE)) + \tan(\pi(\Delta + iE)) \right]. \quad (3.19)$$

The relation (3.16) between $E$, $\mu$, and $T$, implies that $E$ is fixed when $\mu$ and $T$ are also fixed. Thus in writing $d\Omega/d\Delta$ we treat $E$ and $\Delta$ as independent variables, in particular, we keep $E$ fixed when integrating over $\Delta$. Note that the relationship between $E$ and $Q$ in (3.30) depends upon $q$, and so varying $q$ at fixed $E$ implies that $Q$ will vary. We can now integrate (3.19) to obtain

$$\Omega - \Omega_0 = \frac{(2\Delta - 1)T}{2} \ln [2(\cos(2\pi \Delta) + \cosh(2\pi E))]$$

$$+ \frac{iT}{4\pi} \left[ \text{Li}_2 \left( -e^{2\pi(\varepsilon - i\Delta)} \right) + \text{Li}_2 \left( -e^{2\pi(\varepsilon + i\Delta)} \right) - \text{Li}_2 \left( -e^{2\pi(\varepsilon - i\Delta)} \right) - \text{Li}_2 \left( -e^{2\pi(\varepsilon + i\Delta)} \right) \right]$$

$$\equiv -T \mathcal{G}(E). \quad (3.20)$$

The integration constant is fixed by the boundary conditions that the singular part of the grand potential, $\Omega - \Omega_0$ vanishes at the free fermion point $\Delta = 1/2$. The last line in Eq. (3.20)
defines the function $G(E)$.

**Separating the universal and non-universal parts**

The grand potential $\Omega$ computed in (3.20) depends upon $E$ and $T$. But, in the grand canonical ensemble at fixed $\mu$ and $T$, $E$ has an unknown dependence upon $\mu$ and $T$. It is therefore better to convert to the canonical ensemble at fixed $Q$ and $T$, where we know the $Q$ and $T$ dependence of $\mu$ from (3.12),

$$\mu(Q,T) = \mu_0 - 2\pi E(Q)T + \ldots$$

as $T \to 0$. Here $E$ depends only on $Q$, and $\mu_0$ is the contribution from the ground state energy, $E_0$, with

$$\mu_0 = \frac{dE_0}{dQ}.$$  \hfill (3.22)

The complete grand potential, including the contribution of the ground state energy, is

$$\Omega = E_0 - \mu_0 Q - T G(E) + \ldots,$$

where the functional form of the singular term $G(E)$ was given in (3.20). The free energy in the canonical ensemble, $F$, is

$$F(Q,T) = \Omega + \mu Q.$$  \hfill (3.24)

Now we use the thermodynamic identity

$$\mu = \left( \frac{\partial F}{\partial Q} \right)_T,$$

(3.25) becomes

$$\mu_0 - 2\pi E T = \frac{dE_0}{dQ} - T \frac{dG}{dE} \frac{dE}{dQ} - 2\pi E T - 2\pi T \frac{dE}{dQ} Q,\hfill (3.26)$$
which gives us

\[ Q = -\frac{1}{2\pi} \frac{dG}{d\delta}. \]  

(3.27)

Similarly, the entropy is

\[ S = -\left( \frac{\partial F}{\partial T} \right)_Q = G + 2\pi \delta Q \] 

(3.28)

Eqs. (3.27) and (3.28) show that \( S(Q) \) and \( G(\delta) \) are a Legendre pair, and so

\[ \frac{dS}{dQ} = 2\pi \delta. \] 

(3.29)

This equality is equivalent to Eq. (3.12) by the Maxwell relation in Eq. (3.16), and this supports the validity of our analysis.

Appendix B.3 presents a computation of the thermodynamics at large \( q \). In this limit, we explicitly verify the above decompositions into universal and non-universal components.

**Charge**

We compute the density from Eqs. (3.20) and (3.27) to obtain

\[ Q = \frac{(2\Delta - 1) \sinh(2\pi \delta)}{2(\cos(2\pi \delta) + \cosh(2\pi \delta))} - \frac{i}{4\pi} \ln \left[ \frac{(1 + e^{2\pi(\delta - i\Delta)})}{(1 + e^{2\pi(\delta + i\Delta)})} \right]. \] 

(3.30)

This simplifies considerably when expressed in terms of \( \theta \) via (3.11)

\[ Q = -\frac{\theta}{\pi} + \left( \Delta - \frac{1}{2} \right) \frac{\sin(2\theta)}{\sin(2\pi \Delta)}. \] 

(3.31)

This agrees with Appendix A of Ref. [74] at \( q = 4 \). Note that \( Q = \pm 1/2 \) at the limiting values \( \theta = \mp \pi \Delta \). A plot of the density appears in Fig. 3.1.
Entropy

We can compute the entropy $S$ from (3.20), (3.28) and (3.30). It can be verified that $S \to 0$ as $\mathcal{E} \to \pm \infty$. A plot of the entropy as a function of $\mathcal{E}$ appears in Fig. 3.2. We can combine Figs. 3.1 and 3.2 to obtain the entropy as a function of density, and this is shown in Fig. 3.3. In Appendix B.4, we present the results of the numerical solution of the saddle point equations in Eqs. (6.38) and (3.6) for $q = 4$, and find good agreement with the analytic results above.

At the particle-hole symmetric point, $\mathcal{E} = Q = 0$, this yields from (3.20)

$$S(0) = \frac{(1 - 2\Delta)}{2} \ln [4 \cos^2(\pi \Delta)] - \frac{i}{2\pi} \left[ \text{Li}_2 (-e^{-2\pi i \Delta}) - \text{Li}_2 (-e^{2\pi i \Delta}) \right], \quad (3.32)$$

which agrees with Kitaev’s result [116].
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Figure 3.2: The entropy $S$ as a function of $\mathcal{E}$ and $\Delta$

Figure 3.3: The entropy $S$ as a function of $\mathcal{Q}$ and $\Delta$
3.3.2 Effective action

We will also examine aspects of $1/N$ fluctuations about the saddle point which led to the thermodynamic results in this section. Here we will follow Ref. [135], who argued that the dominant fluctuations of the Majorana SYK model at low $T$ are controlled by a Schwarzian effective action with $\text{PSL}(2, \mathbb{R})$ time reparameterization symmetry. This effective action can be used to compute energy fluctuations, and hence the specific heat, in the canonical ensemble. In our analysis of the complex SYK model, we find that an additional $U(1)$ phase field, $\phi$, is needed; $\phi$ is conjugate to $Q$ fluctuations in the grand canonical ensemble. A similar phase field also appeared in the original paper of the Chapter 4, which studies SYK models with $\mathcal{N} = 2$ supersymmetry with the mean $Q$ close to zero. We propose a combined action for energy and $Q$ fluctuations at a generic mean $Q$, with both $\text{PSL}(2, \mathbb{R})$ and $U(1)$ symmetry; for the zero-dimensional complex SYK models, the action is by

$$S_{\phi\epsilon} \frac{K}{N} = \frac{K}{2} \int_{0}^{1/T} d\tau \left[ \partial_{\tau} \phi + i(2\pi\epsilon T)\partial_{\tau} \epsilon \right]^{2} - \frac{\gamma}{4\pi^{2}} \int_{0}^{1/T} d\tau \left\{ \tan(\pi T(\tau + \epsilon(\tau)), \tau) \right\}. \quad (3.33)$$

Here $\tau$ is imaginary time, $\tau \rightarrow \tau + \epsilon(\tau)$ is the time reparameterization, $\{f, \tau\}$ is the Schwarzian derivative (given explicitly in Eq. (3.42)), and $K$ and $\gamma$ are non-universal thermodynamic parameters determining the compressibility and the specific heat respectively. The off-diagonal coupling between energy and $Q$ fluctuations is controlled by the value of $\mathcal{E}$. Our effective action will play a central role in the structure of thermoelectric transport, as described in the next subsection.

Now let us start the analysis. While solving the equations for the Green’s function and the self energy, Eqs. (6.38) and (3.6), we found that, at $\omega, T \ll J$, the $i\omega + \mu$ term in the inverse Green’s function could be ignored in determining the IR solution Eq. (3.7). After dropping the $i\omega + \mu$ term, it is not difficult to show that Eqs. (6.38) and (3.6) have remarkable, emergent, time reparameterization and $U(1)$ invariances. This is clearest if we write the Green’s function in a two-time notation, i.e. $G(\tau_1, \tau_2)$ ; then Eqs. (6.38) and (3.6)
are invariant under \[116, 171\]

\[
G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^\Delta \frac{g(\tau_2)}{g(\tau_1)} G(f(\tau_1), f(\tau_2))
\]
\[
\Sigma(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1-\Delta} \frac{g(\tau_1)}{g(\tau_2)} \Sigma(f(\tau_1), f(\tau_2))
\]

(3.34)

where \(f(\tau)\) and \(g(\tau)\) are arbitrary functions representing the reparameterizations of time and U(1) transformations respectively.

Next, we observe that these approximate symmetries are broken by the saddle-point solution, \(G_s\) in Eq. (3.13). So, following Ref. \[135\], we deduce an effective action for the associated Nambu-Goldstone modes by examining the action of the symmetries on the saddle-point solution,

\[
G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^\Delta G_s(f(\tau_1), f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}.
\]

(3.35)

Here, we find it convenient to parameterize \(g(\tau) = e^{-i\phi(\tau)}\) in terms of a phase field \(\phi\). We will soon see that its derivative is conjugate to density fluctuations. Our remaining task \[135\] is to (i) find the set of \(f(\tau)\) and \(\phi(\tau)\) which leave Eq. (3.35) invariant i.e. Eq. (3.35) holds after we replace the l.h.s. by \(G_s(\tau_1, \tau_2)\); and (ii) propose an effective action which has the property of remaining invariant under the set of \(f(\tau)\) and \(\phi(\tau)\) which leave Eq. (3.35) invariant.

For the first task, we find \[116, 135\] that only reparameterizations, \(f(\tau)\) belonging to PSL(2, \(\mathbb{R}\)) leave Eq. (3.35) invariant. At \(T > 0\), we need PSL(2, \(\mathbb{R}\)) transformations which map the thermal circle \(0 < \tau < 1/T\) to itself. These are given by

\[
\frac{1}{\pi T} \tan(\pi Tf(\tau)) = \frac{a \tan(\pi T\tau) + b\pi T}{c \tan(\pi T\tau) + d\pi T}, \quad ad - bc = 1,
\]

(3.36)

where \(a, b, c, d\) are real numbers. This transformation is more conveniently written in terms
of unimodular complex numbers

\[ z = e^{2\pi iT} \quad , \quad z_f = e^{2\pi iT f(\tau)} \]  
(3.37)

as

\[ z_f = \frac{w_1 z + w_2}{w_2^* z + w_1^*} \quad , \quad |w_1|^2 - |w_2|^2 = 1, \]  
(3.38)

where \( w_{1,2} \) are complex numbers. Applying Eq. (3.38) to Eqs. (3.13) and (3.35), we find that Eq. (3.35) remains invariant only for the particle-hole symmetric case \( \mathcal{E} = 0 \), which was considered previously [135]. However, Eq. (3.35) is invariant under PSL(2, \( \mathbb{R} \)) transformations when the phase field \( \phi(\tau) \) is related to the PSL(2, \( \mathbb{R} \)) transformation \( f(\tau) \) as

\[ -i\phi(\tau) = 2\pi\mathcal{E}T(\tau - f(\tau)) \]  
(3.39)

The effective action is required to vanish when \( \phi(\tau) \) satisfies Eq. (3.39): this is the key result of this subsection, and is the origin of the constraints on thermoelectric properties described in this paper.

Now we can turn to the second task of obtaining an effective for \( f(\tau) \) and \( \phi(\tau) \) which is invariant Eqs. (3.38) and (3.39). It is more convenient to use the parameterization

\[ f(\tau) \equiv \tau + \epsilon(\tau), \]  
(3.40)

and express the action in terms \( \phi(\tau) \) and \( \epsilon(\tau) \). Generalizing the reasoning in Ref. [135], we propose the action

\[ \frac{S_{\phi,\epsilon}}{N} = \frac{K}{2} \int_0^{1/T} d\tau \left[ \partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau \epsilon \right]^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T(\tau + \epsilon(\tau)), \tau) \}, \]  
(3.41)

which appeared earlier in Eq. (3.33). Higher powers of the first term in square brackets can also be present, but we do not consider them here. The curly brackets in the second term
represent a Schwarzian derivative

\[ \{f, \tau\} = f''' - \frac{3}{2} \left( \frac{f''}{f'} \right)^2, \quad (3.42) \]

which has the important property of vanishing under PSL(2, \mathbb{R}) transformations.

Our reasoning above falls short of a complete derivation of the structure of the effective action in Eq. (3.41). The missing ingredient is our assumption that it is permissible to expand the action in gradients of \( \phi \) and \( \epsilon \), when the saddle-point action contains long-range power-law interactions in time. If this assumption was not valid, then the phenomenological couplings \( K \) and \( \gamma \) would diverge in the \( T \to 0 \) limit. We compute the values of \( K \) and \( \gamma \) in Appendix B.3 using a large \( q \) expansion and find that they are finite as \( T \to 0 \). This a posteriori justifies our gradient expansion. We also present a normal-mode analysis of fluctuations of the underlying path integral for the SYK model in Appendix B.5; this follows the analysis of Ref. [135], and uses their reasoning to provide an alternative motivation of Eq. (3.41).

We now relate the phenomenological couplings, \( \gamma \) and \( K \), to thermodynamic quantities by computing the fluctuations of energy and number density implied by \( S_{\phi,\epsilon} \) in the large \( N \) limit. The energy and density operators are defined by

\[ \delta E(\tau) - \mu \delta Q(\tau) = \frac{1}{N} \frac{\delta S_{\phi,\epsilon}}{\delta \epsilon'(\tau)}, \quad \delta Q(\tau) = i \frac{\delta S_{\phi,\epsilon}}{\delta \phi'(\tau)}. \quad (3.43) \]

Introducing,

\[ \tilde{\phi}(\tau) = \phi(\tau) + i2\pi T \epsilon(\tau) \quad (3.44) \]

and expanding (3.41) to quadratic order in \( \phi \) and \( \epsilon \), we obtain the Gaussian action

\[ \frac{S_{\phi,\epsilon}}{N} = \frac{K T}{2} \sum_{\omega_n \neq 0} \omega_n^2 \left| \tilde{\phi}(\omega_n) \right|^2 + \frac{T \gamma}{8\pi^2} \sum |\omega_n| \neq 0, \pm 2\pi T \omega_n^2 (\omega_n^2 - 4\pi^2 T^2) |\epsilon(\omega_n)|^2 + \ldots \quad (3.45) \]

where \( \omega_n \) is a Matsubara frequency. Note the restrictions on \( n = 0, \pm 1 \) frequencies in
(3.45), which are needed to eliminate the zero modes associated with \( \text{PSL}(2, \mathbb{R}) \) and \( \text{U}(1) \) invariances. In terms of \( \widetilde{\phi}(\tau) \) and \( \epsilon(\tau) \), Eq. (3.43) is

\[
\delta Q(\tau) = iK \tilde{\phi}'(\tau)
\]

\[
\delta E(\tau) - \mu_0 \delta Q(\tau) = -\frac{\gamma}{4\pi^2} \left[ \epsilon''(\tau) + 4\pi^2 T^2 \epsilon'(\tau) \right] + i2\pi K \epsilon T \tilde{\phi}'(\tau).
\]

(3.46)

Now we compute the correlators of these observables in the Gaussian action in Eq. (3.45), following the methods of Ref. [135]. We have for the two-point correlator of \( \tilde{\phi}(\tau) \)

\[
\langle \tilde{\phi}(\tau)\tilde{\phi}(0) \rangle = \frac{T}{NK} \sum_{\omega_n \neq 0} \frac{e^{i\omega_n \tau}}{\omega_n^2}
\]

\[
= \frac{1}{NKT} \left[ \frac{1}{2} \left( T\tau - \frac{1}{2} \right)^2 - \frac{1}{24} \right] \quad \text{for } 0 < T\tau < 1,
\]

(3.47)

and extended periodically for all \( \tau \) with period \( 1/T \). Similar for \( \epsilon(\tau) \)

\[
\langle \epsilon(\tau)\epsilon(0) \rangle = \frac{4\pi^2 T}{N\gamma} \sum_{|\omega_n| \neq 0, 2\pi T} \frac{e^{i\omega_n \tau}}{\omega_n^2 (\omega_n^2 - 4\pi^2 T^2)}
\]

\[
= \frac{1}{N\gamma T^3} \left[ \frac{1}{24} + \frac{1}{4\pi^2} - \frac{1}{2} \left( T\tau - \frac{1}{2} \right)^2 + \frac{5}{8\pi^2} \cos(2\pi T\tau) + \frac{1}{2\pi} \left( T\tau - \frac{1}{2} \right) \sin(2\pi T\tau) \right]
\]

(3.48)

for \( 0 < T\tau < 1 \).

Inserting Eqs. (3.47) and (6.67) into Eq. (3.46), we confirm that the correlators of the conserved densities are \( \tau \)-independent; their second moment correlators, which define the
matrix of static susceptibility correlators by (3.51), are given by

\[
\chi_s = \frac{1}{N} \begin{pmatrix}
-(\partial^2\Omega/\partial\mu^2)_T & -(\partial^2\Omega/\partial\mu\partial T)_{\mu} \\
-T(\partial^2\Omega/\partial\mu\partial T)_{\mu} & -T(\partial^2\Omega/\partial T^2)_{\mu}
\end{pmatrix}
\]

\[
= \frac{1}{T} \begin{pmatrix}
\langle(\delta Q)^2\rangle & \langle(\delta E - \mu\delta Q)\delta Q\rangle / T \\
\langle(\delta E - \mu\delta Q)\delta Q\rangle & \langle(\delta E - \mu\delta Q)^2\rangle / T
\end{pmatrix}
\]

(3.49)

\[
= \frac{1}{N} \begin{pmatrix}
K & 2\pi K E \\
2\pi K E T & (\gamma + 4\pi^2 E^2 K) T
\end{pmatrix}
\]

From Eq. (3.49) we obtain the relationship between the couplings \(K\) and \(\gamma\) in the effective action in Eq. (3.41). After application of some thermodynamic identities, we can write these as

\[
K = \left( \frac{\partial Q}{\partial \mu} \right)_T, \quad \gamma = -\left( \frac{\partial^2 F}{\partial T^2} \right)_Q
\]

(3.50)

and also confirm the thermodynamic definitions of \(E\) in Eqs. (3.16), (3.12), and (3.29).

Appendix B.6 presents another argument for the results in Eq. (3.50) without computation of fluctuations of the effective action.

### 3.3.3 Transport

We will characterize transport by two-point correlators of the conserved number density, which we continue to refer to as \(Q\), and the conserved energy density \(E = H_0/N\). For the zero-dimensional SYK model in (3.1), both of these quantities are constants of the motion, and so have no interesting dynamics. So we consider here the higher dimensional SYK models, for which \(Q\) and \(E\) are defined per site of the higher-dimensional lattice. Then their correlators do have an interesting dependence of wavevector, \(k\), and frequency, \(\omega\). We define
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the dynamic susceptibility matrix, $\chi(k, \omega)$, where

$$
\chi(k, \omega) = \begin{pmatrix}
    \langle Q; Q \rangle_{k,\omega} & \langle E - \mu Q; Q \rangle_{k,\omega} / T \\
    \langle E - \mu Q; Q \rangle_{k,\omega} & \langle E - \mu Q; E - \mu Q \rangle_{k,\omega} / T
\end{pmatrix},
$$

(3.51)

and we use the notation

$$
\langle A; B \rangle_{k,\omega} \equiv -i \int_0^\infty dt \int d^d x \langle [A(x, t), B(0, 0)] \rangle e^{-i k x + i \omega t}.
$$

(3.52)

As in the standard analysis of Kadanoff and Martin [112], we expect the low energy and long distance form of these correlators to be fully dictated by the hydrodynamic equations of motion for a diffusive metal [89]. From such an analysis, we obtain, at low frequency and wavenumber,

$$
\chi(k, \omega) = \left[ i \omega (-i \omega + D k^2)^{-1} + 1 \right] \chi_s,
$$

(3.53)

where $D$ and $\chi_s$ are $2 \times 2$ matrices. The diffusivities are specified by $D$, and the static susceptibilities are, as usual, $\chi_s = \lim_{k \to 0} \lim_{\omega \to 0} \chi(k, \omega)$. The values of $\chi_s$ are related by standard thermodynamic identities to second derivatives of the grand potential $\Omega$, as shown in Eq. (3.49).

One of our main results is that the low $T$ limit of the diffusivity matrix, $D$, takes a specific form

$$
D = \begin{pmatrix}
    D_1 & 0 \\
    2\pi \mathcal{E} T (D_1 - D_2) & D_2
\end{pmatrix},
$$

(3.54)

where $D_1$ and $D_2$ are temperature-independent constants. We will show that the result in Eq. (3.54) is obeyed in the higher-dimensional SYK models. And in the original paper, the same form also appears in the holographic theories. It is a consequence of the interplay between the global $U(1)$ fermion number symmetry and the emergent $\text{PSL}(2, \mathbb{R})$ symmetry of the scaling limit of the SYK model. In holography, $\text{PSL}(2, \mathbb{R})$ is the isometry group of $\text{AdS}_2$; while transport properties of the holographic theories have been computed earlier, the
specific form the diffusivity matrix in Eq. (3.54) was not noticed. This form will be crucial for the mapping between the holographic and SYK models.

We can use the Einstein relation to define a matrix of conductivities

\[
\begin{pmatrix}
\sigma & \alpha \\
\alpha T & \kappa
\end{pmatrix} = D \chi_s,
\]

where \( \sigma \) is the electrical conductivity, \( \alpha \) is the thermoelectric conductivity, and \( \kappa = \pi - T \alpha^2/\sigma \) is the thermal conductivity. The matrix in Eq. (3.55) is constrained by Onsager reciprocity. Details of the derivation of Eq. (3.55) are discussed in Appendix B.8. From Eqs. (3.54) and (3.55) we find the following result for the low \( T \) limit of the thermopower; the Seebeck coefficient \( S \) is given in both the SYK and holographic models by

\[
\lim_{T \to 0} S \equiv \frac{\alpha}{\sigma} = \frac{dS}{dQ}.
\]

Since \( dS/dQ = 2\pi \mathcal{E} \), we see that the Seebeck coefficient is entirely determined by the particle-hole asymmetry of the fermion spectral function.

Eq. (3.56) has been proposed earlier as the ‘Kelvin formula’ by Peterson and Shastry [158] using very different physical arguments. Earlier holographic computations of transport did not notice the result in Eq. (3.56). The remarkable aspect of this expression is that it relates a transport quantity to a thermodynamic one, the derivative of the entropy with respect to particle number. Such a relation is in general only approximate, see Ref. [158] and the discussion and applications in Ref [144]. Remarkably, this relation holds exactly here: it is an exact consequence of the PSL(2, \( \mathbb{R} \)) symmetry of both SYK and holographic models. We note that the form in Eq. (3.54) is implied by Eq. (3.56) and Onsager reciprocity.

We also obtain an interesting result for the Wiedemann-Franz ratio, \( L \), of the SYK model.
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For the particular higher-dimensional generalization in Eq. (3.60), we find the exact result

\[ L \equiv \lim_{T \to 0} \frac{\kappa}{T} = \frac{4\pi^2}{3q^2}. \]  

(3.57)

We comments on aspects of this result:

- For the free fermion case, \( q = 2 \), this reduces to the universal Fermi liquid Lorenz number \( L_0 = \pi^2 k_B^2 / (3e^2) \) (re-inserting fundamental constants). Although expected, this agreement with \( L_0 \) at \( q = 2 \) is non-trivial and remarkable: rather than the usual arguments based upon integrals over the Fermi function, Eq. (3.57) arises from the structure of bosonic normal modes of the \( 1/N \) fluctuations, as discussed in Appendix B.5.

- The decrease of \( L \) for large \( q \) can be understood as follows. As we will see in the large \( q \) solution in Appendix B.3, the energy bandwidth for fermion states vanishes as \( q \to \infty \). Consequently, fermion hopping transfers little energy, and the thermal conductivity \( \kappa \) is suppressed. In contrast, fermion hopping continues to transfer unit charge, and hence the conductivity does not have a corresponding suppression.

- Although the result in Eq. (3.57) appears universal, it is not so: there are other higher-dimensional generalizations of the inter-site coupling term in Eq. (3.60) which will lead to corrections to the value of \( L \). Needless to say, these corrections will not modify the universal value \( L_0 \) at \( q = 2 \). The non-universality of \( L \) for higher \( q \) is connected to the non-renormalization of inter-site disorder in the present large \( N \) limit.

Let’s start the analysis. Gu et al. have defined a set of higher-dimensional SYK models [83] for Majorana fermions and computed their energy transport properties. Here we extend their results to the case of complex fermions at a general \( \mu \), and discuss their thermoelectric transport.

We will limit our presentation to one spatial dimension (although the results easily gen-
eralize to all spatial dimensions). We consider the model

\[ H = \sum_x (H_x + \delta H_x) \]  

(3.58)

The on-site term \( H_x \) is equivalent to a copy of Eq. (3.1) on each site \( x \)

\[ H_x = \sum_{1 \leq i_1 < i_2 \ldots < i_{q/2} \leq N, \atop 1 \leq i_{q/2+1} < i_{q/2+2} \ldots < i_q \leq N} J_{x,i_1,i_2\ldots,i_q} f_{x,i_1}^\dagger \ldots f_{x,i_{q/2}}^\dagger f_{x,i_{q/2+1}} \ldots f_{x,i_q} \]  

(3.59)

The nearest neighbor coupling term \( \delta H_x \) denotes nearest-neighbor interactions as shown in Fig 3.4,

\[ \delta H_x = \sum_{1 \leq i_1 < i_2 \ldots < i_{q/2} \leq N, \atop 1 \leq i_{q/2+1} < i_{q/2+2} \ldots < i_q \leq N} J'_{x,i_1,i_2\ldots,i_q} f_{x,i_1}^\dagger \ldots f_{x,i_{q/2}}^\dagger f_{x+1,i_{q/2+1}} \ldots f_{x+1,i_q} + \text{H.c.} \]  

(3.60)

The couplings \( \{J_{x,i_1,i_2\ldots,i_q}\} \) and \( \{J'_{x,i_1,i_2\ldots,i_q}\} \) are all independent random variables\(^2\) with zero mean, and variances given by

\[ |J_{x,i_1,i_2\ldots,i_q}|^2 = \frac{J_0^2(q/2)!^2}{N^{q-1}}, \quad |J'_{x,i_1,i_2\ldots,i_q}|^2 = \frac{J_1^2(q/2)!^2}{N^{q-1}}. \]  

(3.61)

We remark that the particular interaction we choose here is just one possible example, and this particular choice produces the Wiedemann-Franz ratio discussed in Eq. (3.57). In general, we can choose \( p \) fermions from one site (with \( p_1 \) many \( f^\dagger \) and \( p_2 \) many \( f \), \( p_1 \) and \( p_2 \) can be chosen arbitrarily) to couple \((q-p)\) fermions in the nearest neighbor site (with \((q/2-p_1)\) many \( f^\dagger \) and \((q/2-p_2)\) many \( f \)). For example, for \( q = 4 \) case, we are allowed to

\(^2\)except that \( J_{x,i_1,i_2\ldots,i_q} \) need to satisfy the hermitian condition as shown in Eq. (3.2).
Figure 3.4: A chain of coupled SYK sites with complex fermions (in this figure we draw \( q = 4 \) case): each site contains \( N \gg 1 \) fermions with on-site interactions as in (3.1). The coupling between nearest neighbor sites are four-fermion interaction with two from each site. In general, one can consider other types of \( q \)-body interactions (\( q = 4 \) in this caption), e.g. \( f_{x,i_1}^\dagger f_{x,i_2} f_{x+1,i_3} f_{x+1,i_4} \). Such terms will only change the ratio between \( D_1 \) and \( D_2 \), i.e. Eq. (3.63) by a non-universal coefficient depends on the details of the model. In particular, if we only have \( f_{x,i_1}^\dagger f_{x,i_2} f_{x+1,i_3} f_{x+1,i_4} \)-type terms to couple the nearest neighbour sites, charge diffusion \( D_1 \) will vanish due to the local charge conservation.

If we include these couplings with different coefficient, the Wiedemann-Franz ratio will be non-universal. However, for \( q = 2 \) model, the only term we can add is \( f_{i_1,x}^\dagger f_{i_2,x+1} \) and therefore the Wiedemann-Franz ratio goes back to \( \pi^2/3 \) as discussed previously.

Following the analysis in Ref. [83], the effective action for the higher dimensional model can be deduced from that of the zero dimensional model. Using the results in Appendices B.5 and B.7, we find that the Gaussian action for energy and density fluctuations in higher dimensions generalizes from Eq. (3.45) to

\[
\frac{S_{\phi,\epsilon}}{N} = \frac{KT}{2} \sum_{k,\omega, n \neq 0} \left| \omega_n \right| (D_1 k^2 + |\omega_n|) \left| \tilde{\phi}(k, \omega_n) \right|^2,
\]

\[
+ \frac{T \gamma}{8\pi^2} \sum_{k, |\omega_n| \neq 0, 2\pi T} \left| \omega_n \right| (D_2 k^2 + |\omega_n|) (\omega_n^2 - 4\pi^2 T^2) |\epsilon(k, \omega_n)|^2,
\]

\[
(3.62)
\]
where $D_1$ and $D_2$ are the diffusion constants of the conserved charges. In Appendix B.7 we find that their ratio obeys

$$\frac{D_2}{D_1} = \frac{4\pi^2 \Delta^2 K}{3 \gamma}. \quad (3.63)$$

Following Ref. [83], from Eq. (3.62), and including a contact term as described in Ref. [163], we can obtain the long-wavelength and low frequency dynamic susceptibilities

$$\langle Q; Q \rangle_{k,\omega} = K \frac{D_1 k^2}{-i\omega + D_1 k^2},$$

$$\langle E - \mu Q; Q \rangle_{k,\omega} = 2\pi K \mathcal{E} T \frac{D_1 k^2}{-i\omega + D_1 k^2},$$

$$\langle E - \mu Q; E - \mu Q \rangle_{k,\omega} / T = \gamma T \frac{D_2 k^2}{-i\omega + D_2 k^2} + 4\pi^2 \mathcal{E}^2 K T \frac{D_1 k^2}{-i\omega + D_1 k^2}. \quad (3.64)$$

We note that the form of the thermoelectric correlators in Eq. (3.64) is not generic to incoherent metals [89], and the simple structure here relies on specific features of our effective action in Eq. (3.62); in holographic models as in the original paper, we will again obtain Eq. (3.64) by holographic methods, where its structure is linked to the presence of AdS$_2$ factor in the near-horizon metric. Now comparing (3.64) with (3.53), and using the susceptibility matrix (3.49), we can work out the diffusion matrix, $D$, leading to the result presented earlier in Eq. (3.54). Using (3.55), we find the conductivity matrix

$$\begin{pmatrix} \sigma & \alpha \\ \alpha T & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} D_1 K & 2\pi K \mathcal{E} D_1 \\ 2\pi K \mathcal{E} D_1 T & (\gamma D_2 + 4\pi^2 \mathcal{E}^2 K D_1) T \end{pmatrix}. \quad (3.65)$$

From this result we obtain the Seebeck coefficient presented in Eq. (3.56). Also, we have the thermal conductivity

$$\kappa = \bar{\kappa} - \frac{T \alpha^2}{\sigma} = \gamma D_2 T. \quad (3.66)$$
All of these hydrodynamic results are in accord with the linearized equations of motion

\[
\begin{pmatrix}
\partial Q / \partial t \\
\partial E / \partial t - \mu \partial Q / \partial t
\end{pmatrix} = D \begin{pmatrix}
\nabla^2 Q \\
\nabla^2 E - \mu \nabla^2 Q
\end{pmatrix},
\]

with the diffusion matrix as in Eq. (3.54). The dynamic susceptibilities (3.64) can be diagonalized by using the operators \( Q \) and \( E - Q(\mu + 2\pi \mathcal{E} T) \). The first of these is carried only by the mode with diffusion constant \( D_1 \), and the second is carried only by the mode with diffusion constant \( D_2 \). \( D_1 \) is the charge diffusion constant. We call \( D_2 \) the thermal diffusion constant as it’s very simply related to the thermal conductivity via Eqs. (3.66) and (3.50).

We can also compute the Wiedemann-Franz ratio, \( L \), from the above results and the computations in Appendix B.7. Eq. (3.63) leads directly to Eq. (3.57).

The Lyapunov exponent \( \lambda_L \) and butterfly velocity \( v_B \), which characterize early-time chaotic growth through the growth of connected out-of-time-ordered thermal four-point functions

\[
\langle V^\dagger(t, \vec{x}) W^\dagger(0) V(t, \vec{x}), W(0) \rangle_\beta \sim e^{\lambda_L(t - |\vec{x}|/v_B)},
\]

can be computed in this model just as in Ref. [83]. The key observation here is that these properties are associated with the fluctuations of the \( \epsilon \) mode, and the \( \phi \) mode is mostly a spectator. Taking \( V \) and \( W \) to be the SYK fermions, the present model has a Lyapunov exponent given by

\[
\lambda_L = 2\pi T,
\]

saturating the chaos bound of [136]. As in the works [20, 21, 83, 154], we find that the butterfly velocity is simply related to the thermal diffusivity as

\[
D_2 = \frac{v_B^2}{2\pi T}.
\]

From Eq. (3.63), we observe that the relationship between \( v_B \) and the charge diffusion...
constant $D_1$ is not universal \cite{130}: it depends upon the specific parameters of the SYK model.

### 3.4 Discussion

This chapter has presented the thermodynamic and transport properties of the higher-dimensional SYK models of fermions with random $q/2$-body interactions. The model conserves total energy and a U(1) charge, $Q$, but do not conserve total momentum, which describes a diffusive metallic state without quasiparticle excitations. We found that the model also shared a number of common properties with holographic models of black branes with an AdS$_2$ near-horizon geometry:

- The low $T$ thermodynamics is described by the free energy in Eq. (3.14), with the entropy $S(Q)$ universal, and the ground state energy $E_0(Q)$ non-universal. For the SYK models, universality implies dependence only on the IR scaling dimension of the fermion, and independence from possible higher-order interactions in the Hamiltonian. In the holography, universality implies independence from the geometry far from the AdS$_2$ near-horizon geometry.

- The thermoelectric transport is constrained by a simple expression (Eq. (3.56)) equating the Seebeck coefficient to the $Q$-derivative of the entropy $S$. This is the ‘Kelvin formula’ proposed in Ref. \cite{158} by different approximate physical arguments. In our analysis, the Kelvin formula was the consequence of an emergent PSL(2, $\mathbb{R}$) symmetry shared by both classes of models.

- As has also been discussed earlier \cite{171}, the correlators of non-conserved local operators have a form (see Eq. (3.13)) constrained by conformal invariance, and characterized by a spectral asymmetry parameter, $\mathcal{E}$, which is defined by Eq. (3.16). In the holographic context, $\mathcal{E}$ also has the interpretation as the strength of an electric field in AdS$_2$. 

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Both classes of models [116, 135, 177] saturate the bound [136] on the Lyapunov rate which characterizes the growth of quantum chaos, \( \lambda_L = 2\pi T \).

For the SYK models, the butterfly velocity, \( v_B \), was found to be universally related to the thermal diffusivity, \( D_2 \) by Eq. (3.70), as in Ref. [83]. On the other hand, the SYK models do not display a universal relation between \( v_B \) and the charge diffusivity, \( D_1 \).

So the universal connection between chaos and transport is restricted to energy transport, as was also found in the study of a critical Fermi surface [154]. Chaos is naturally connected to energy fluctuations, because the local energy determines the rate of change of the phase of the quantum state, and phase decoherence is responsible for chaos. This physical argument finds a direct realization in the computation on the SYK model. In the holographic axion model with \( \mu = 0 \), the relationship between \( D_{1,2} \) and \( v_B \) was investigated in Ref. [20], and \( D_2 \) was found to obey Eq. (3.70).
4.1 Introduction

In this chapter, we introduce supersymmetric generalizations of the SYK models. Like previous models, the supersymmetric models have random all-to-all interactions between fermions on $N$ sites. There are no canonical bosons in the underlying Hamiltonian, and in this respect, our models are similar to the supersymmetric lattice models in Refs. [59–61, 100–103]. As we describe below, certain structures in the correlations of the random couplings of our models lead to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. Supersymmetric models with random couplings that include both bosons and fermions were considered in [5].

Let us discuss now the model with $\mathcal{N} = 1$ supersymmetry, and defer presentation of the $\mathcal{N} = 2$ case to Section 4.3.2. For the $\mathcal{N} = 1$ case, we introduce the supercharge

$$Q = i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^i \psi^j \psi^k,$$  \hspace{1cm} (4.1)
where $\psi_i$ are Majorana fermions on sites $i = 1 \ldots N$,

$$\{\psi^i, \psi^j\} = \delta^{ij}, \quad (4.2)$$

and $C_{ijk}$ is a fixed real $N \times N \times N$ antisymmetric tensor so that $Q$ is Hermitian. We will take the $C_{ijk}$ to be independent gaussian random variables, with zero mean and variance specified by the constant $J$:

$$C_{ijk} = 0, \quad \overline{C^2_{ijk}} = \frac{2J}{N^2} \quad (4.3)$$

where $J$ is positive and has units of energy. As is the case in supersymmetric theories, the Hamiltonian is the square of the supercharge

$$\mathcal{H} = Q^2 = E_0 + \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \psi^i \psi^j \psi^k \psi^l \quad (4.4)$$

where

$$E_0 = \sum_{1 \leq i < j < k \leq N} C^2_{ijk}, \quad J_{ijkl} = -\frac{1}{8} \sum_a C_{ijkl} C_{kl} C_{kl} \quad (4.5)$$

with $[\ ]$ representing all possible anti-symmetric permutations. Note that the $J_{ijkl}$ are not independent gaussian random variables, and this is formally the only difference from the Hamiltonian of the non-supersymmetric SYK models $[9, 43, 98, 107, 108, 116, 135, 162, 171, 196]$. These particular correlations change the structure of the large $N$ equations and lead to a solution where the fermion has dimension $\Delta_f = 1/6$. In addition, there is a supersymmetric partner of this operator which is bosonic and has dimension $\Delta_b = 2/3 = 1/2 + \Delta_f$. This large $N$ solution has unbroken supersymmetry, and we have checked this numerically by comparing with exact diagonalization of the Hamiltonain. We have also computed the large $N$ ground state entropy from a complete numerical solution of the saddle-point equations. In the exact diagonalization we find that the lowest energy state has non-zero energy, and therefore,
broken supersymmetry. However, this energy is estimated to be of order $e^{-\alpha N}$ where $\alpha$ is a numerical constant. We have also generalized the model to include a supercharge of the schematic form $Q \sim \psi^{\hat{q}}$, and we also solved this model in the large $\hat{q}$ limit. We also formulated the model in superspace, and show that the large $N$ equations have a super-reparameterization invariance, which is both spontaneously as well as explicitly broken by the appearance of a superschwarzian action, which we describe in detail.

We have also analyzed the eigenvalues of the ladder kernel which appears in the computation of the four point function. There are both bosonic and fermionic operators that can propagate on this ladder. There is a particular eigenvalue of the kernel which is a zero mode and corresponds to the degrees of freedom described by the Schwarzian. They are a bosonic mode with dimension $h = 2$ and a fermionic one with $h = 3/2$. The other eigenvalues of the kernel should describe operators appearing in the OPE. These also come in boson-fermion pairs and have a structure similar to the usual SYK case. One interesting feature is the appearance of a boson fermion pair with dimensions $h = 1$ and $h = 3/2$, which is associated to an additional symmetry of the low energy equations. These do not give rise to extra zero modes but simply correspond to other operators in the theory.

We have also analyzed the $\mathcal{N} = 2$ version of the theory. In this case we can also compute a kind of Witten index. More precisely, the model has a discrete $Z_{\hat{q}}$ global symmetry that commutes with supersymmetry, so that we can include the corresponding discrete chemical potential in the Witten index, which turns out to be non-zero. These are generically expected to be lower bounds on the large $N$ ground state entropy; it turns out that the largest Witten index is, in fact, equal to the large $N$ ground state entropy. The model also has a $U(1)_R$ symmetry. The exact diagonalization analysis also suggests a conjecture for number of ground states for each value of the $U(1)_R$ charge. For the $\hat{q} = 3$ case, they are concentrated at very small values of the $U(1)$ R-charge, within $|Q| \leq 1/3$.

This chapter is organized as follows. In section 4.2 we define the $\mathcal{N} = 1$ supersymmetric model, write the large $N$ effective action and the corresponding classical equations. We
determine the dimensions of the operators in the IR and we derive a constraints imposed by unbroken supersymmetry on the correlators. We also present a generalization of the model where the supercharge is a product of \( \hat{q} \) fermions and solve the whole flow in the \( \hat{q} \to \infty \) limit. In section 4.3 we present some results on exact numerical diagonalization of the Hamiltonian. This includes results on the ground state energy and two point correlation functions. We then generalize the model with \( \mathcal{N} = 2 \) supersymmetry. We compute the Witten index and use it to argue that the model has a large exact degeneracy at zero energy. In the end, we discuss the ladder diagrams that contribute to the four point function. We use them to determine the eigenvalues of the ladder kernel and use it to determine the spectrum of dimensions of composite operators. We have not included derivations of the low-energy effective theory using superspace formalism, readers can find details in the original paper. In section 4.4, we discuss our results.

4.2 Model

4.2.1 Definition of the Model and the Large \( N \) Effective Action

To set up a path integral formulation of \( H \), we first note that the supercharge acts on the fermion as

\[
\{Q, \psi^i\} = i \sum_{1 \leq j < k \leq N} C_{ijk} \psi^j \psi^k. \tag{4.6}
\]

We introduce a non-dynamical auxiliary boson \( b^i \) to linearize the supersymmetry transformation and realize the supersymmetry algebra off-shell. The Lagrangian describing \( H \) is

\[
\mathcal{L} = \sum_i \left[ \frac{1}{2} \psi^i \partial_\tau \psi^i - \frac{1}{2} b^i b^i + i \sum_{1 \leq j < k \leq N} C_{ijk} b^i \psi^j \psi^k \right]. \tag{4.7}
\]

Under the transformation \( Q \psi^i = b^i, Q b^i = \partial_\tau \psi^i \) it changes as

\[
Q \mathcal{L} = \partial_\tau \left( -\frac{1}{2} \sum_i \psi^i b^i + \frac{i}{3} \sum_{1 \leq j < k \leq N} C_{ijk} \psi^i \psi^j \psi^k \right) + i \sum_{1 \leq j < k \leq N} C_{ijk} b^i b^j \psi^k. \tag{4.8}
\]
This implies that the action is invariant as long as the structure constants $C_{ijk}$ in (4.7) are totally anti-symmetric.

Now we proceed to obtain the effective action. This can be done by averaging over the Gaussian random variables $C_{ijk}$ in the replica formalism. In this model, as in SYK, the interaction between replicas is suppressed by $1/N^2$, so that we can simply average over disorder by treating it as an additional field with time independent two point functions as in (4.3). Averaging over disorder, we obtain

$$S_{\text{eff}} = \int_0^\beta d\tau \left( \frac{1}{2} \psi^i \partial_\tau \psi^i - \frac{1}{2} b^i b^i \right) - \frac{J}{2N^2} \int_0^\beta d\tau_1 d\tau_2 \left( b^i(\tau_1)b^i(\tau_2) \right) \left( \psi^j(\tau_1)\psi^j(\tau_2) \right)^2 +$$
$$- \frac{J}{N^2} \int_0^\beta d\tau_1 d\tau_2 \left( b^i(\tau_1)\psi^j(\tau_2) \right) \left( \psi^j(\tau_1)\psi^j(\tau_2) \right) \left( \psi^k(\tau_1)\psi^k(\tau_2) \right). \quad (4.9)$$

Note that this action contains terms in which the bosons and fermions carry the same index, and which should be omitted e.g. $b^i(\tau_1)b^i(\tau_2)\psi^j(\tau_1)\psi^j(\tau_2)$; however they are subdominant in the large $N$ limit, and so we ignore this issue.

Notice further that the relative coefficient between the last two terms is determined by the supersymmetry requirement that the structure constants $C_{ijk}$ are totally anti-symmetric, so that

$$\langle C_{ijk}C_{i'j'k'} \rangle \sim \delta_{ii'}\delta_{jj'}\delta_{kk'} + \delta_{ij'}\delta_{jk'}\delta_{kk'} + \delta_{ik'}\delta_{jj'}\delta_{kk'} + (j \leftrightarrow k). \quad (4.10)$$

The purpose of this section is to discuss the large $N$ saddle-point equations for the diagonal Green’s functions

$$G_{\psi \psi}(\tau_1, \tau_2) = \frac{1}{N} \psi^i(\tau_1)\psi^i(\tau_2),$$
$$G_{bb}(\tau_1, \tau_2) = \frac{1}{N} b^i(\tau_1)b^i(\tau_2), \quad (4.11)$$

where we have a sum over $i$. We will thus drop the last term in (4.9), which only affects the
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saddle-point equations for the off-diagonal Green’s functions

\[ G_{b\psi}(\tau_1, \tau_2) = \frac{1}{N} b^j(\tau_1) \psi^i(\tau_2), \]
\[ G_{\psi b}(\tau_1, \tau_2) = \frac{1}{N} \psi^i(\tau_1) b^j(\tau_2), \]
\hspace{1cm} (4.12)

We will restore it in a later section ???, where we write the saddle-point equations in a manifestly super-symmetric fashion.

We introduce the Lagrange multipliers \( \Sigma_{\psi \psi} \)

\[ 1 = \int \mathcal{D}G_{\psi \psi} \mathcal{D}\Sigma_{\psi \psi} \exp \left( - \frac{N}{2} \Sigma_{\psi \psi}(\tau_1, \tau_2) \left( G_{\psi \psi}(\tau_1, \tau_2) - \frac{1}{N} \psi^i(\tau_1) \psi^j(\tau_2) \right) \right), \]
\hspace{1cm} (4.13)

and \( \Sigma_{bb} \)

\[ 1 = \int \mathcal{D}G_{bb} \mathcal{D}\Sigma_{bb} \exp \left( - \frac{N}{2} \Sigma_{bb}(\tau_1, \tau_2) \left( G_{bb}(\tau_1, \tau_2) - \frac{1}{N} \psi^i(\tau_1) b^j(\tau_2) \right) \right). \]
\hspace{1cm} (4.14)

As the notation suggests, these Lagrange multipliers will eventually become the self energies. Inserting these factors of 1 in the fermion path integral with the action (4.9), using the delta functions implied by the integration over \( \Sigma_{\psi, b} \) to express the interaction terms in (4.9), and integrating out the fermions we obtain

\[ Z = \int \mathcal{D}G_{\psi \psi} \mathcal{D}\Sigma_{\psi \psi} e^{-S_{\text{eff}}(G_{\psi \psi}, G_{bb}, \Sigma_{\psi \psi}, \Sigma_{bb})} \]
\[ S_{\text{eff}}(G_{\psi}, G_{bb}, \Sigma_{\psi \psi}, \Sigma_{bb})/N = -\log \text{Pf}[\partial_\tau - \Sigma_{\psi \psi}(\tau)] + \frac{1}{2} \log \det[-1 - \Sigma_{bb}(\tau)] + \]
\[ + \frac{1}{2} \int d\tau_1 d\tau_2 \left[ \Sigma_{\psi \psi}(\tau_1, \tau_2) G_{\psi \psi}(\tau_1, \tau_2) + \Sigma_{bb}(\tau_1, \tau_2) G_{bb}(\tau_1, \tau_2) - J G_{bb}(\tau_1, \tau_2) G_{\psi \psi}(\tau_1, \tau_2)^2 \right], \]
\hspace{1cm} (4.15)

which becomes a classical action when \( N \) is large. Let us look at the classical equations for
the action in (4.15). Taking derivatives with respect to $G_\psi$ and $G_{bb}$, we obtain

$$
\Sigma_{\psi\psi}(\tau_1, \tau_2) = 2J G_{bb}(\tau_1, \tau_2) G_{\psi\psi}(\tau_1, \tau_2)
$$
$$
\Sigma_{bb}(\tau_1, \tau_2) = J G_{\psi\psi}(\tau_1, \tau_2)^2,
$$
(4.16)

Taking derivatives with respect to $\Sigma_{\psi\psi}$ and $\Sigma_{bb}$, assuming time translation symmetry and going to Fourier space, we obtain

$$
G_\psi(i\omega)^{-1} = -i\omega - \Sigma_{\psi\psi}(i\omega)
$$
$$
G_{bb}(i\omega)^{-1} = -1 - \Sigma_{bb}(i\omega),
$$
(4.17)

which confirms that $\Sigma_{\psi, b}$ are the self energies.

In temporal space, the saddle point equations take the form

$$
\partial_\tau G_{\psi\psi}(\tau_1, \tau_3) - \int d\tau_2 \left( 2J G_{bb}(\tau_1, \tau_2) G_{\psi\psi}(\tau_1, \tau_2) \right) G_{\psi\psi}(\tau_2, \tau_3) = \delta(\tau_1 - \tau_3)
$$
$$
-G_{bb}(\tau_1, \tau_3) - \int d\tau_2 \left( J G_{\psi\psi}(\tau_1, \tau_2)^2 \right) G_{bb}(\tau_2, \tau_3) = \delta(\tau_1 - \tau_3),
$$
(4.18)

These equations can be solved numerically, and we can see some plots in figure (4.5).
Once we find a solution to these equations, we can compute the on-shell action, which can be written as

$$
\log \frac{Z}{N} = -\frac{S_{\text{eff}}}{N} = \frac{1}{2} \log 2 - \sum_{n \in \text{half integer}} \frac{1}{2} \log \left[-i\omega_n G_{\psi\psi}(i\omega_n)\right] + \sum_{n \in \text{integer}} \frac{1}{2} \log G_{bb}(i\omega_n)
$$
$$
- \frac{J\beta}{2} \int_0^\beta G_{bb}(\tau) G_{\psi\psi}(\tau)^2
$$
(4.19)

where $\omega_n$ are the Matsubara frequencies for the fermion and boson cases. From this we can compute the entropy through the usual thermodynamic formula. A plot of the entropy as a function of the temperature can be found in figure (4.1).
Figure 4.1: Thermal entropy obtained by numerically solving the large $N$ equations of motion (4.16)(4.17). At high temperatures we have just the log of the dimension of the Hilbert space, $S_N = \frac{1}{2} \log 2$. The zero temperature entropy is approximately $\frac{S}{N} \approx 0.2745 + 0.0005$, where the error is estimated by the convergence of the FFT (Fast Fourier Transform) algorithm. The analytical result $\frac{S}{N} = \frac{1}{2} \log \left[ 2 \cos \frac{\pi}{6} \right]$ (4.38) also lies in this range.

We can now determine the low energy structure of the solutions of (4.16) and (4.17), as in [173], by making a power law ansatz at late times ($1 \ll J \tau \ll N$)

$$G_{\psi\psi} \propto \frac{1}{\tau^{2\Delta_{\psi}}}, \quad G_{bb} \propto \frac{1}{\tau^{2\Delta_{b}}},$$

(4.20)

where $\Delta_{\psi}$ and $\Delta_{b}$ are the scaling dimensions of the fermion and the boson. We then insert (4.20) into (4.16), (4.17) in order to fix the values of $\Delta_{\psi}$ and $\Delta_{b}$. Matching the power-laws in the saddle point equations yields only the single constraint

$$2\Delta_{\psi} + \Delta_{b} = 1.$$

(4.21)

As we will see later the dimension can be determined by looking at the constant coefficients. Before showing this, let us discuss a simpler way to obtain another condition.
4.2.2 SUPERSYMMETRY CONSTRAINTS

Further analytic progress can be made if we assume that the solutions of the saddle point
equations (4.16), (4.17) preserve supersymmetry. With such an assumption, we now show
that the scaling dimensions $\Delta_\psi$ and $\Delta_b$ can be easily determined.

If supersymmetry is not spontaneously broken, then

$$G_{bb}(\tau_1 - \tau_2) = \langle Q \psi(\tau_1) b(\tau_2) \rangle = \langle \psi(\tau_1) Q b(\tau_2) \rangle$$

$$= \partial_{\tau_2} \langle \psi(\tau_1) \psi(\tau_2) \rangle = -\partial_{\tau_1} G_{\psi \psi}(\tau_1 - \tau_2) \quad (4.22)$$

This relationship together with

$$\Sigma_{\psi \psi}(\tau_1 - \tau_2) = -\partial_{\tau_1} G_{bb}(\tau_1 - \tau_2). \quad (4.23)$$

is compatible with the saddle-point equations in Section 4.2.1.

(4.22) together with the ansatz (4.20) leads to

$$\Delta_b = \Delta_\psi + \frac{1}{2}. \quad (4.24)$$

Together with Eq. (4.21), we can now determine the scaling dimensions

$$\Delta_\psi = \frac{1}{6}, \quad \Delta_b = \frac{2}{3}. \quad (4.25)$$

4.2.3 SIMPLE GENERALIZATION

We now show how to derive the $\Delta_b = \Delta_\psi + \frac{1}{2}$ constraint directly from the saddle point
equations without assuming that the solution preserves supersymmetry.

It is useful to consider a simple generalization of Eq. (4.1) to case where the supercharge
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$Q$ is the sum over products of $\tilde{q}$ fermions\(^{1}\). The Hamiltonian $\mathcal{H} = Q^2$ involves sums of terms with up to $2\tilde{q} - 2$ fermions. $\tilde{q} = 3$ corresponds to the case discussed above (4.1).

The large $N$ equations are (4.17) and

$$
\Sigma_{\psi\psi}(\tau_1, \tau_2) = (\tilde{q} - 1)JG_{\psi\psi}(\tau_1, \tau_2)G_\psi(\tau_1, \tau_2)\tilde{q}^{-2}, \quad \Sigma_{bb}(\tau_1, \tau_2) = JG_\psi(\tau_1, \tau_2)\tilde{q}^{-1} \quad (4.26)
$$

We can explore them at low energy by making the ansatz

$$
G_\psi(\tau_1, \tau_2) = \frac{b_\psi \text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_\psi}}, \quad G_{bb}(\tau_1, \tau_2) = \frac{b_b}{|\tau_{12}|^{2\Delta_b}}, \quad \tau_{12} \equiv \tau_1 - \tau_2 \quad (4.27)
$$

where $b_\psi, b_b$ are some constants.

Again, if we assume supersymmetry we immediately derive $\Delta_\psi = 1/(2\tilde{q})$ and $\Delta_b = \Delta_\psi + \frac{1}{2}$.

Doing so without that assumption requires us to look at the equations for $b_\psi$ and $b_b$.

Using the Fourier transforms for symmetric and antisymmetric functions

$$
\int dt e^{i\omega t} \frac{\text{sgn}(t)}{|t|^{2\Delta}} = c_f(\Delta) \text{sgn}(\omega)|\omega|^{2\Delta - 1}, \quad \int dt e^{i\omega t} \frac{1}{|t|^{2\Delta}} = c_b(\Delta)|\omega|^{2\Delta - 1}, \quad (4.28)
$$

$$
c_f(\Delta) \equiv 2i \cos(\pi\Delta)\Gamma(1 - 2\Delta), \quad c_b(\Delta) \equiv 2\sin(\pi\Delta)\Gamma(1 - 2\Delta) \quad (4.29)
$$

The following relations are useful

$$
c_f(\Delta)c_f(1 - \Delta) \equiv -\frac{2\pi \cos \pi\Delta}{(1 - 2\Delta) \sin \pi\Delta}, \quad c_b(\Delta)c_b(1 - \Delta) \equiv -\frac{2\pi \sin \pi\Delta}{(1 - 2\Delta) \cos \pi\Delta} \quad (4.30)
$$

Then (4.26) , together with the low energy approximation of (4.17), which is $G_\psi(i\omega)\Sigma_{\psi\psi}(i\omega) = -1$ and , $G_{bb}(i\omega)\Sigma_{bb}(i\omega) = -1$, gives the conditions

$$
Jb_b^{\tilde{q} - 1}b_b(\tilde{q} - 1)c_f(\Delta_\psi)c_f((\tilde{q} - 2)\Delta_\psi + \Delta_b)|\omega|^{2(\tilde{q} - 1)\Delta_\psi + 2\Delta_b - 2} = -1 \, ,
$$

\(^{1}\)In detail $Q = i^{\tilde{q}/2} \sum_{j_1, j_2 < \cdots < \tilde{q}} \mathcal{C}_{j_1, j_2, \ldots, j_\tilde{q}} \psi^{j_1} \psi^{j_2} \cdots \psi^{j_\tilde{q}}$, with $(\mathcal{C}_{j_1, j_2, \ldots, j_\tilde{q}}^2) = \frac{(\tilde{q} - 1)!J}{N^{\tilde{q} - 1}}$.  

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\[ J b^\psi_{\tilde{\psi}}^{-1} b_{\psi} c_{\psi}(\Delta_b) c_{\psi}((\tilde{q} - 1)\Delta_\psi) |\omega|^{2(\tilde{q} - 1)\Delta_\psi + 2\Delta_b - 2} = -1 \]  

(4.31)

Matching the frequency dependent part we get the condition \( \Delta_b = 1 - (\tilde{q} - 1)\Delta_\psi \). The equations for the coefficients reduce to

\begin{align*}
2\pi J b^\psi_{\tilde{\psi}}^{-1} b_{\psi} (\tilde{q} - 1) &= (1 - 2\Delta_\psi) \frac{\sin \pi \Delta_\psi}{\cos \pi \Delta_\psi}, \\
2\pi J b^\psi_{\tilde{\psi}}^{-1} b_{\psi} &= (2(\tilde{q} - 1)\Delta_\psi - 1) \frac{\sin \pi (\tilde{q} - 1)\Delta_\psi}{\cos \pi (\tilde{q} - 1)\Delta_\psi} \tag{4.32}
\end{align*}

The ratio between the two equations gives another condition for \( \Delta_\psi \), with one rational solution obeying \( \Delta_b = \Delta_\psi + \frac{1}{2} \), which is also independently implied by supersymmetry, see (4.23). In the range where \( \Delta_\psi \) and \( \Delta_b \) are both positive there is a second, irrational solution to the equations which has higher \( \Delta_\psi \). This second solution breaks supersymmetry, since it does not obey (4.24). It would be nice to understand it further, but we leave that to the future.

We also see that the low energy equations have a symmetry

\[ G_{\psi\psi} \to \lambda^2 G_{\psi\psi}, \quad G_{bb} \to \lambda^{2 - 2\tilde{q}} G_{bb} \]  

(4.33)

Indeed (4.31) involves only the combination \( J b^\psi_{\tilde{\psi}}^{-1} b_{\psi} \). This symmetry of the IR equations is broken by the UV boundary conditions that arise from considering the full equations in (4.17).

In fact, the supersymmetry relation (4.23) also fixes this freedom of rescaling, by setting \( b_b = 2\Delta_\psi b_\psi \).

In the end this fixes the coefficients to

\[ J b^\psi_{\tilde{\psi}}^{-1} b_b = \frac{\tan \frac{\pi}{2\tilde{q}}}{2\tilde{q}\pi}, \quad b_b = \frac{1}{\tilde{q}} b_\psi, \quad \Rightarrow \quad b_\psi = \left[ \tan \frac{\pi}{2\tilde{q}} \right]^{\frac{1}{\tilde{q}}} \]  

(4.34)

This coefficient (for \( \tilde{q} = 3 \)) is used in the plot of figure 4.4. Of course the finite temperature
version is

\[ G_{\psi\psi}(\tau) = b_\psi \left[ \frac{\pi}{\beta \sin \frac{\pi \tau}{\beta}} \right]^{2\Delta_\psi} \]  

(4.35)

This generalization makes it easy to compute the ground state entropy. In principle this can be done by inserting these solutions into the effective action

\[
\log Z_N = \frac{1}{2} \log \det(\partial_\tau - \Sigma \psi) - \frac{1}{2} \log \det(\delta - \Sigma b) + \frac{1}{2} \int d\tau d\tau' \left[ -\Sigma b b - \Sigma \psi \psi + JG b G b - G \psi \psi + J G b G \psi \psi \right]
\]

(4.36)

It is slightly simpler to take the derivative with respect to \( b \), ignoring any term that involves derivatives of \( G_{\psi\psi} \) since those terms vanish by the equations of motion. This gives

\[
\partial_b \log Z_N = \frac{J}{2} \beta \int d\tau G_{\psi}(\tau) G_{\psi}^{-1} \log G_{\psi\psi} = \beta(\text{constant}) + \frac{\pi^2}{2\hat{q} b} \beta(\text{constant}) + \frac{\pi \tan \frac{\pi \beta}{2\hat{q}^2}}{4\hat{q}^2}
\]

(4.37)

Where we inserted (4.35) and used (4.34). The constant term includes UV divergencies which are \( \beta \) independent. This term contributes to the ground state energy\(^2\), but not to the entropy. Integrating (4.37) we obtain the ground state entropy

\[
\frac{S}{N} = \frac{1}{2} \log[2 \cos \pi \Delta_f] = \frac{1}{2} \log[2 \cos \frac{\pi}{2\hat{q}^2}]
\]

(4.38)

where in integrating we used the boundary condition that the entropy should be the entropy of free fermion system at \( \hat{q} = \infty \), a fact we will check below. For \( \hat{q} = 3 \) this matches the numerical answer, see figure 4.1.

\(^2\)If we computed it using the exact solution (as opposed to the conformal solution) of the equations we expect the ground state energy to vanish due to supersymmetry.

We have also discussed the large \( \hat{q} \) limit of the model, see Appendix. C.1.
Chapter 4. Supersymmetric SYK models

4.3 Results

4.3.1 Numerics from exact diagonalization

This section presents results from the exact numerical diagonalization of the Hamiltonian in Eq. (4.4). We examined samples with up to \( N = 28 \) sites, and averaged over 100 or more realizations of disorder. This exact diagonalization allows us to check the validity of the answer we obtained using large \( N \) methods.

Supersymmetry

An important purpose of the numerical study was to examine whether supersymmetry was unbroken in the \( N \to \infty \) limit. In Fig. 4.2 we test the basic relationship in Eq. (4.22) between the fermion and boson Green’s functions. The agreement between the boson Green’s function and the time derivative of the fermion Green’s function is evidently excellent.

We also computed the value of the ground state energy \( E_0 = \langle 0 | QQ | 0 \rangle \). Supersymmetry is unbroken if and only if \( E_0 = 0 \). We have found that \( E_0 \) is non-zero in the exact theory,
but it becomes very small for large $N$. Indeed fig. 4.3 shows that $E_0$ does become very small, and the approach to zero is compatible with an exponential decrease of $E_0$ with $N$. This is then compatible with a supersymmetric large $N$ solution, supersymmetry is then broken non-perturbatively in the $1/N$ expansion. The combination of Figs. 4.2 and 4.3 is strong numerical evidence for the preservation of supersymmetry in the $N \to \infty$ limit (with supersymmetry breaking at finite $N$). The ground state energy can be fitted well by $E_0 \propto e^{-\alpha S_0}$ with $\alpha = 1.9 \pm .2$, which is compatible with $\alpha = 2$. Here $S_0$ is the ground state entropy, (4.38). This is smaller than the naive estimate for the interparticle level spacing which is $e^{-S}$.

Note that the breaking of supersymmetry is also compatible with the Witten index of this model which is $Tr[(-1)^F] = 0$. This can be easily computed in the free theory. For $N$ odd we defined the Hilbert space by adding an extra Majorana mode that is decoupled from the ones appearing in the Hamiltonian.

Figure 4.3: Ground state energy as a function of $N$ in a log-linear plot, where we have averaged over 100 samples. The plot is compatible with an exponential decrease of $E_0$ with $N$. Notice also the structure in $E_0$ dependent on $N \pmod{8}$.

As in Ref. [196], we found a ground state degeneracy pattern that depended upon $N \pmod{8}$. 

As in Ref. [196], we found a ground state degeneracy pattern that depended upon $N \pmod{8}$. 

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8). The pattern in our case is (for \( N \geq 3 \))

\[
\begin{array}{cccccccc}
N \mod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\text{Degeneracy} & 2 & 4 & 4 & 4 & 4 & 2 & 2 & .
\end{array}
\]  

\hspace{1cm} (4.39)

For odd \( N \) this degeneracy includes all the states in the Hilbert space defined by adding an extra decoupled fermion. We also found that the value of \( E_0 \) has structure dependent upon \( N \mod 8 \), as is clear from Fig. 4.3. The symmetry classification in the \( \mathcal{N} = 1 \) theory for \( \hat{q} = 3 \) is further discussed in Ref. [127] using random matrix theory. Cases with general \( \hat{q} \) are discussed in Ref. [113].

**Scaling**

We also compared our numerical results for the Green’s functions with the conformal scaling structure expected at long times and low temperatures. From Eq (4.34), with \( \hat{q} = 3 \), we expect that at \( T = 0 \) and large \( \tau \)

\[
G_\psi^c(\tau) = \frac{\text{sgn}(\tau)}{(2\pi \sqrt{3})^{1/3}} |J\tau|^{-1/3}, \quad G_\psi(\tau) = \frac{1}{3(2\pi \sqrt{3})^{1/3}} J |J\tau|^{-4/3}. \quad (4.40)
\]

Fig 4.4 shows that Eq. (4.40) is obeyed well for large \( J\tau \).

We also extended this comparison to \( T > 0 \), where we expect the generalization of Eq. (4.40) to

\[
G_\psi^c(\tau) = \frac{\text{sgn}(\tau)}{(2\pi \sqrt{3})^{1/3}} \left|\frac{\pi T}{J \sin(\pi \tau T)}\right|^{1/3}, \quad G_\psi(\tau) = \frac{1}{3(2\pi \sqrt{3})^{1/3}} J \left|\frac{\pi T}{J \sin(\pi \tau T)}\right|^{4/3} \quad (4.41)
\]

The comparison of these results with the numerical data appears in Fig. 4.5.
4.3.2 $\mathcal{N} = 2$ supersymmetry

This section turns to the generalization to $\mathcal{N} = 2$ supersymmetry. The real fermion $\psi^i$ is replaced by complex fermions $\psi^i$ and $\bar{\psi}_i$, and the supercharge $Q$ in Eq. (4.1) is replaced by a pair of charges $Q$ and $\bar{Q}$. The defining relations are

$$\{\psi^i, \bar{\psi}_j\} = \delta^i_j, \quad \{\psi^i, \psi^j\} = 0, \quad \{\bar{\psi}_i, \bar{\psi}_j\} = 0$$

$$Q = i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^i \psi^j \bar{\psi}^k$$

$$\bar{Q} = i \sum_{1 \leq i < j < k \leq N} \bar{C}^{ijk} \bar{\psi}_i \bar{\psi}_j \psi_k$$

which imply $Q^2 = \bar{Q}^2 = 0$. The theory has a $U(1)_R$ R-symmetry, under which the fermions $\psi^i$ and $\bar{\psi}_i$ carry charges $1/3$ and $-1/3$. As is customary, we normalize the $U(1)_R$ charge so that the supercharges carry charge $\pm 1$. The supersymmetry acts on the fermionic variables as

$$[Q, \psi^i] = 0 \quad [Q, \bar{\psi}_i] = \bar{b}^i \equiv i \sum_{1 \leq j < k \leq N} C_{ijk} \bar{\psi}_j \psi^k$$

Figure 4.4: Imaginary time Green’s function at $T = 0$ for $N = 24$ Majorana fermions averaged over 100 samples. Left panel: blue solid line is $G_{\psi\psi}(\tau)$, red dotted-dashed line is the conformal solution $G^c_{\psi\psi}(\tau)$ in Eq. (4.40); right panel: blue solid line is $G_{bb}(\tau)$, red dotted-dashed line is the conformal solution $G^c_{bb}(\tau)$ in Eq. (4.40).
Figure 4.5: Imaginary time Green’s function at finite temperature for $N = 20$ Majorana fermions averaged over 100 samples. Left panel is $G_{\psi\psi}(\tau)$ while right panel is $G_{bb}(\tau)$. Solid lines are the exact diagonalization result; dashed lines are conformal results as in Eq. (4.41); dotted line are large $N$ result by numerically solving Eq. (4.16) and Eq. (4.17). Different colors correspond to different interaction strength: the blue one is $\beta J = 5$; the red one is $\beta J = 20$ and the black one is $\beta J = 200$.

The Hamiltonian replacing Eq. (4.4) is now

$$\mathcal{H} = \{Q, \overline{Q}\} = |C|^2 + \sum_{i,j,k,l} J_{ij}^{kl} \psi_i \psi^j \overline{\psi}_k \overline{\psi}_l$$  \hspace{1cm} (4.44)

We note this Hamiltonian has the same form as the complex SYK model introduced in Ref. [171], but now the complex couplings $J_{ij}^{kl}$ are not independent random variables. Instead we take the $C_{ijk}$ to be independent random complex numbers, with the non-zero second moment

$$\overline{C_{ijk} C_{ijkl}} = \frac{2J}{N^2}$$  \hspace{1cm} (4.45)

replacing Eq. (4.3).

The subsequent analysis of Eq. (4.44) closely parallels the $\mathcal{N} = 1$ case. The main difference is that we now introduce complex non-dynamical auxiliary bosonic fields $b^i$ and $\overline{b}_i$ to linearize the supersymmetry transformations. The model can also be generalized so that $Q$ is built from products of $\tilde{\psi}$ fermions so that the Hamiltonian involves up to $2\tilde{q} - 2$ fermions.

The equations of motion are a complexified version of the $\mathcal{N} = 1$ equations. We will
describe them momentarily. The fermion also has scaling dimension \( \Delta_\psi = 1/(2\tilde{q}) \) and R-charge \( 1/\tilde{q} \). Notice that the R-charge of \( \psi \) is twice its scaling dimension, which is as expected for a super-conformal chiral primary field. As is conventional the \( U(1)_R \) charge is normalized so that the supercharge has charge one. The \( U(1)_R \) charge does not commute with the supercharges. There is however a \( Z_{\tilde{q}} \) group of this \( U(1) \) symmetry that acts on the fermions as \( \psi^j \rightarrow e^{2\pi ir/\tilde{q}} \psi^j \) which does leave the supercharge invariant and is a global symmetry commuting with supersymmetry. Note that the quantization condition on the \( U(1)_R \) charge, \( Q_R \), is that \( \tilde{q}Q_R \) should be an integer.

This fact enables us to compute an simple generalization of the Witten index defined as

\[
W_r = Tr[(-1)^F e^{2i\pi rQ_R}] = Tr[(-1)^F g^r] = \left[1 - e^{2\pi i r/\tilde{q}}\right]^N = e^{iN\pi(\frac{r}{\tilde{q}}-\frac{1}{2})} \left[2\sin\frac{\pi r}{\tilde{q}}\right]^N
\] (4.46)

where \( g \) is the generator of the \( Z_{\tilde{q}} \) symmetry, and \( Q_R \) is the \( U(1)_R \) charge. In the third equality we have used that the index is invariant under changes of the coupling and computed it in the free theory, with \( J = 0 \). The Witten index is maximal for \( r = (\tilde{q} \pm 1)/2 \) where its absolute value is greatest and equal to

\[
\log |W_r| = N \log [2 \cos \frac{\pi}{2\tilde{q}}].
\] (4.47)

The right hand side happens to be the same as the value of the ground state entropy computed using the large \( N \) solution, which is the same as \( (4.38) \), up to an extra overall factor of two because now the fermions are complex. In general, these Witten indices should be a lower bound on the number of ground states, and also a lower bound on the large \( N \) ground state entropy (recall that in the \( \mathcal{N} = 1 \) case we had that the Witten index was zero). The fact that the bound is saturated tells us that most of the states contributing to the large \( N \) ground state entropy are actually true ground states of the model. Thus, in this case supersymmetry is not broken by \( e^{-N} \) effects.

We have also looked at exact diagonalization of the theory, and computed the number of
states for different values of the $U(1)_R$ charge. Let us define the R-charge so that it goes between $-N/\tilde{q} \leq Q_R \leq N/\tilde{q}$, in increments of $1/\tilde{q}$. We have looked at the case $\tilde{q} = 3$ and we found the following degeneracies, $D(N, Q_R)$, as a function of $N$ and the charge

$$D(N, 0) = 2 \cdot 3^{N/2-1}, \quad D(N, \pm \frac{1}{3}) = 3^{N/2-1}, \quad \text{for } N \text{ even}$$

$$D(N, \pm \frac{1}{6}) = 3^{(N-1)/2}, \quad \text{for } N \equiv 3 \mod 4$$

$$D(N, \pm \frac{1}{6}) = 3^{(N-1)/2}, \quad D(N, \pm \frac{3}{6}) = 1 \text{ or } 3, \quad \text{for } N \equiv 1 \mod 4$$

And we have $D(N, Q_R) = 0$ outside the cases mentioned above. Therefore, we see that the degeneracies are concentrated on states with very small values of the $R$ charge. Of course, these values are consistent with the Witten index in (4.46) for $\tilde{q} = 3$. The agreement/disagreement of the ground state degeneracy and Witten index is further discussed in Ref. [113, 174].

### 4.3.3 Four point function and the spectrum of operators

![Diagrams](image)

Figure 4.6: (a) Diagram contributing to a correction to the fermion propagator. Full lines are fermions and dotted lines are bosons. (b) Correction to the boson propagator. (c) A simple ladder diagram contributing to the four-point function in the fermionic channel, where the intermediate state obtained when we cut the ladder is a fermion. (d), (e), (f) Diagrams contributing in the bosonic channel, with either a pair of bosons or a pair of fermions. The full ladders are obtained by iterating these diagrams. These are the diagrams for $\tilde{q} = 3$ and they look similar in the general case.

3Recall that the ground states of a quantum mechanics with $\mathcal{N} = 2$ supersymmetry are in one-to-one correspondence with the cohomology of the $Q$ supercharge. This is easier to compute than the eigenvalues and eigenstates of the Hamiltonian.

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The four point function can be computed by techniques similar to those discussed in [116, 135, 162]. We should sum a series of ladder diagrams, see figure 4.6. There are various types of four point functions we could consider. The simplest kind has the form

$$\langle \psi^i(\tau_1)\phi^i(\tau_2)\psi^j(\tau_3)\phi^j(\tau_4) \rangle$$  \hspace{1cm} (4.49)

In this case the object propagating along the ladder is fermionic, produced by a boson and fermion operator. We will not present the full form of the four point function in detail, but we will note the dimensions of the operators appearing in the singlet channel OPE (the $\tau_1 \to \tau_2$ limit). As in [116, 135, 162] these dimensions are computed by using conformal symmetry to diagonalize the ladder kernel in terms of a basis of functions of two variables with definite conformal casimir specified by a conformal dimension $h$. Then setting the kernel equal to one gives us the spectrum of dimensions that can appear in the OPE. The problem can be separated into contributions where the intermediate functions are essentially symmetric or antisymmetric under the exchange of variables. This gives us two sets of fermionic operators specified by the conditions

$$1 = k_s(h) \equiv -2^{-1 + \frac{b}{2}} \frac{\Gamma(2 - \frac{1}{q}) \Gamma(\frac{1}{4} + \frac{1}{2q} - \frac{h}{2}) \Gamma(\frac{1}{4} + \frac{1}{2q} + \frac{h}{2})}{\Gamma(1 + \frac{b}{2}) \Gamma(\frac{3}{4} - \frac{1}{2q} - \frac{h}{2}) \Gamma(\frac{3}{4} - \frac{1}{2q} + \frac{h}{2})}$$

$$1 = k_a(h) \equiv -2^{-1 + \frac{b}{2}} \frac{\Gamma(2 - \frac{1}{q}) \Gamma(\frac{3}{4} + \frac{1}{2q} - \frac{h}{2}) \Gamma(-\frac{1}{4} + \frac{1}{2q} + \frac{h}{2})}{\Gamma(1 + \frac{b}{2}) \Gamma(\frac{3}{4} - \frac{1}{2q} - \frac{h}{2}) \Gamma(\frac{3}{4} - \frac{1}{2q} + \frac{h}{2})}$$  \hspace{1cm} (4.50)

From the first and second we get eigenvalues of the form

$$h_{s,m} = \frac{3}{2}, \frac{3}{2} - \frac{5}{2}, \frac{5}{2} - \frac{5}{2}, \cdots \quad h_{s,m} = \Delta_\psi + \Delta_\phi + 2m + \gamma_m$$

$$h_{a,m} = \frac{3}{2}, \frac{3}{2} - \frac{5}{2}, \frac{5}{2} - \frac{5}{2}, \cdots \quad h_{a,m} = \Delta_\psi + \Delta_\phi + 2m + 1 - \gamma_m$$  \hspace{1cm} (4.51)

Except for $h = 3/2$ the numbers do not appear to be rational. They approach the values we indicated above for large $m$, with small positive $\gamma_m$ or $\gamma_m$ for large $m$. These operators can be viewed as having the rough form $\psi^i \partial^n \phi^j$ with $n = 2m$, $2m + 1$ respectively.
It is also possible to look at the ladder diagrams corresponding to four point functions of the form \(\langle \psi^i \psi^j \psi^i \psi^j \rangle\). When we compute the ladders these are mixed with ones with structures like \(\langle \psi^i \psi^j b^i b^j \rangle\) or \(\langle b^i b^j b^i b^j \rangle\), see figure 4.6. So the kernel even for a given intermediate \(h\) is a two by two matrix. Diagonalizing this matrix we find that the operators split into two towers which are the partners of the above one. These bosonic partners have conformal dimensions given by \(h_{s,m} + \frac{1}{2}\) and \(h_{a,m} - \frac{1}{2}\) for each of the two fermionic towers. Of course, it should be possible to directly use super-graphs so that we can preserve manifest supersymmetry.

Now, we expect that the case where \(h_s = 3/2\) and its bosonic partner with \(h = 2\) lead to a divergence in the computation of the naive expression for the four point function and that the proper summation would reproduce what we obtain from the super-Schwarzian action.

The pair of modes with \(h_a = 3/2\) and its bosonic partner at \(h = 1\) are more surprising. The origin of the \(h = 1\) mode is due to the rescaling symmetry of the IR equations mentioned in (4.33). In fact, one can extend that symmetry to a local symmetry of the form

\[
G_{\psi\psi}(t, t') \to \lambda(t)\lambda(t')G_{\psi\psi}(t, t'), \quad G_{bb}(t, t') \to [\lambda(t)\lambda(t')]^{1-\bar{q}}G_{bb}(t, t')
\]

which would naively suggest the presence of an extra set of zero modes. However, we noted that this symmetry is broken by the UV boundary conditions. Of course this was also true of the reparameterization symmetry. However, (4.52) changes the short distance form of the correlators, which leads us to expect terms in the effective action of the form \(J \int d\tau (\lambda(t) - 1)^2\), which strongly suppress the deviations from the value of \(\lambda\) given by the short distance solution. Thus, in the low energy theory we do not expect a zero mode from these. Indeed, when we look at the ladders with the boson exchanges, we see that the basis of functions we are summing over when we express the four point function should be the same as the one for the usual SYK model, (see [135]). Namely, the expansion for the four point function can be expressed as an integral over \(h = 1/2 + is\) and a sum over even values of \(h\). Since \(h = 1\) is not even, it does not lead to a divergence. Then we conclude that it
corresponds to an operator of the theory. It looks like a marginal deformation, since it has \(h = 1\). In the UV, it looks like the operator corresponds to a relative rescaling of the boson and fermion field. We think that the transformation simply corresponds to a rescaling of \(J\), which breaks the original supersymmetry but preserves a new rescaled supersymmetry. We have not studied in detail the meaning of its supersymmetric partner which is a dimension 3/2 operator.

The case with \(\mathcal{N} = 2\) supersymmetry leads to similar operators in the singlet channel with zero \(U(1)_R\) charge. The fermions have the same dimensions as in (4.50), (4.50), but each with a factor of two degeneracy arising from the fact that now we change \(\psi^i b^j \to \psi^i b^j\) and \(\bar{\psi}^i b^j\). The bosonic operators fill a whole \(\mathcal{N} = 2\) multiplet with dimensions \((h_{s,m} - \frac{1}{2}, h_{s,m}, h_{s,m} + \frac{1}{2})\) and \((h_{a,m} - \frac{1}{2}, h_{a,m}, h_{a,m} + \frac{1}{2})\). In this model the functions we need to sum over in order to get the four point function are more general than the ones in the SYK model, since now that the basic two point function \(\langle \psi^i(t_1) \bar{\psi}^j(t_2) \rangle\) does not have a definite symmetry. So now the expression for the four point function should include a sum over all values of \(h\), including both even and odd values, depending on whether we consider symmetric or antisymmetric parts. Though we have not filled out all the details we expect that by supersymmetry we will have that the multiplet coming from the symmetric tower with dimensions \((1, 3/2, 2)\) should lead to the superschwarzian while the second one, coming from \(h_{a,m}\), also with dimensions \((1, 3/2, 2)\) should correspond to operators in the IR theory. As before these arise from symmetries of the low energy equations. Let us discuss in detail the ones corresponding to the dimension two operators. The low energy equations have the form

\[
G_{\psi \psi} \star G_{\bar{\psi} \psi}^{-1} = -\delta , \quad G_{\psi \bar{\psi}} \star [(\bar{q} - 1)G_{\bar{\psi} \bar{\psi}} G_{\bar{\psi} \psi}^{-2}] = -\delta \tag{4.53}
\]

where \(\star\) is a convolution and we think of each side as a function of two variables. The right hand side is a delta function that sets these two variables equal. We also have complex conjugate equations obtained by replacing \(G_{\psi \psi} \leftrightarrow G_{\bar{\psi} \bar{\psi}}, G_{b \bar{b}} \leftrightarrow G_{\bar{b} b}\). We can then check that
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The following is a symmetry

\[
G_{\psi\psi} \rightarrow G_{\psi\psi}'(\tau_1, \tau_2) = [f'(\tau_1)h'(\tau_2)]^{\Delta} G_{\psi\psi}(f(\tau_1), h(\tau_2))
\]

\[
G_{\bar{\psi}\psi} \rightarrow G_{\bar{\psi}\psi}'(\tau_1, \tau_2) = [h'(\tau_1)f'(\tau_2)]^{\Delta} G_{\bar{\psi}\psi}(f(\tau_1), h(\tau_2))
\]

\[
G_{\psi\bar{\psi}} \rightarrow G_{\psi\bar{\psi}}'(\tau_1, \tau_2) = [f'(\tau_1)h'(\tau_2)]^{\Delta} G_{\psi\bar{\psi}}(f(\tau_1), h(\tau_2))
\] (4.54)

and similarly for \(G_{\bar{\psi}\bar{\psi}}\). If \(G\) is a solution of (4.53), then \(G'\) is also a solution. The reparameterizations are nearly zero modes of the full problem are those that obey \(h = f\) . The ones where they are different, are far from being zero modes of the full problem. The reality condition sets that \(h(\tau) = f(\tau)^*\). These look similar to two independent coordinate transformations that preserve conformal gauge in a two dimensional space, with a boundary condition that restricts them to be equal. More details about the four point functions and spectrum of the supersymmetric SYK models are discussed in Ref. [157, 194].

4.4 Discussion

We have studied supersymmetric generalizations of the SYK model. We studied models with \(\mathcal{N} = 1, 2\) supersymmetry. Both models are very similar to the SYK system, with a large ground state entropy and a large \(N\) solution that is scale invariant in the IR. In these super versions, the scale invariance becomes a superconformal symmetry and the leading order classical solutions preserve supersymmetry. These large \(N\) solutions were also checked against numerical exact diagonalization results. As in SYK, there is also an emergent superconformal symmetry that is both spontaneously and explicitly broken. This action gives the leading corrections to the low energy thermodynamics and should produce the largest contributions to the four point function. Besides the ordinary reparameterizations, we have fermionic degrees of freedom and, in the \(\mathcal{N} = 2\) case, a bosonic degree of freedom associated to a local \(U(1)\) symmetry, which is related to the \(U(1)_R\) symmetry. A similar bosonic degree of freedom arises in other situations with a \(U(1)\) symmetry, such as the model studied in [171].
We also analyzed the operators in the “singlet” channel. These operators have anomalous dimensions of order one. Therefore, in these models, supersymmetry is not enough to make those dimensions very high.

In the $\mathcal{N} = 1$ case, the exact diagonalization results allowed us to show that the ground state energy is non-zero and of order $E_0 \propto e^{-2S_0}$. This means that supersymmetry is non-perturbatively broken. On the other hand, in the $\mathcal{N} = 2$ case, supersymmetry is not broken and there is a large number of zero energy states which matches the ground state entropy computed using the large $N$ solution. Furthermore, these zero energy states can have non-zero R charge, but with an R charge parametrically smaller than $N$, and even smaller than one.

These results offer some lessons for the study of supersymmetric black holes. In supergravity theories there is a large variety of extremal black holes that are supersymmetric in the gravity approximation. The fact that supersymmetry can be non-perturbatively broken offers a cautionary tale for attempts to reproduce the entropy using exactly zero energy states (see eg. [30, 115]). Of course, in situations where there is an index reproducing the entropy, as in [185], this is not an issue. The authors of [14] have argued that the ground states of supersymmetric black holes carry zero R charge, where the $R$ charge is the IR one that appears in the right hand side of the superconformal algebra. In our case there is only one continuous $U(1)_R$ symmetry and we find that the ground states do not have exactly zero charge. A possible loophole is that the R-symmetry appearing in the superconformal algebra leaves invariant the thermofield double, not each copy individually. Perhaps a modified version of the argument might be true since in our case the R charges are relatively small. Also the discrete chemical potential we introduced in (4.46), looks like a discrete version of the maximization procedure discussed in [14]. It seems that this is a point that could be understood further.

Another surprise in the model is the emergence of additional local symmetries of the equations, beyond the ones associated to super-reparameterizations. Similar symmetries arise in
some of the non-supersymmetric models discussed in [78]. A common feature of these IR symmetries is that they change the short distance structure of the bilocals. Namely, they change the functions $G(t, t')$ even when $t \to t'$. Since this is a region where the conformal approximation to the effective action develops divergencies, we see that now these divergencies will depend on the symmetry generator. For this reason these symmetries do not give rise to zero modes, but are related to operators of the IR theory. Amusingly, in the $\mathcal{N} = 2$ case we also have an additional reparameterization symmetry of this kind. This symmetry together with the usual reparameterization symmetry look very similar to the conformal symmetries we would have in two dimensional $AdS_2$ space in conformal gauge.

We can wonder whether we can get models with $\mathcal{N} > 2$ supersymmetry. It would be interesting to see if one can find models of this sort with only fermions. A model with $\mathcal{N} = 4$ supersymmetry that also involves dynamical bosons was studied in [5].

After the completion of this work, there are other papers with further discussions on related topics. Ref. [113, 174] discussed the symmetry classification of the model using random matrix theory. Ref. [157, 194] discussed correlators, spectrum and effective action of the model. Ref. [147] generalized the supersymmetric model with global symmetry. Ref. [145] generalized the model to one higher dimension. Ref. [149] discussed the ETH in supersymmetric SYK models.
A continuous set of boson-fermion SYK models

5.1 Introduction

As discussed in the previous two chapters, SYK models describe a non-fermi liquid state, which does not have a quasiparticle description. The absence of quasiparticles are argued to give a relative short dephasing time [170]:

\[ \tau_\phi \sim \frac{\hbar}{k_B T} \]  

(5.1)

and this is the conjectured "Planckian" bound on relaxation time discovered in many materials [24]. While on the other hand, SYK models also exhibit chaotic behavior. One can define a Lyapunov time as the inverse of Lyapunov exponent:

\[ \tau_L = \frac{1}{\lambda_L} = \frac{1}{2\pi} \frac{\hbar}{k_B T} \]  

(5.2)
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A natural question to ask is whether these two quantities are related. Physically, we expect there should be some relation: a state with quasiparticle should not be too chaotic. However do we have the conclusion that the closer a system is to a free system, the smaller the Lyapunov exponent will be (or in another way, the longer the Lyapunov time). In this chapter, we shall engineer a boson-fermion SYK model with tunable scaling dimension and compute the Lyapunov exponent. We indeed find that when the anomalous scaling dimension is smaller, the correction to the maximal Lyapunov exponent gets larger.

5.2 Model

5.2.1 Definition of the model and the large $N/M$ limit

The model we want to consider first appears in [18]

$$H = \frac{u}{2} \sum_{a=1}^{M} C_{ij}^{a} C_{kl}^{a} \chi_i \chi_j \chi_k \chi_l$$

(5.3)

with the following renormalization

$$N^{3/2} M^{1/2} \langle C_{ij}^{a} C_{kl}^{a} \rangle = J \delta_{ab} \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}$$

(5.4)

Notice that $J$ is positive and has units of energy. As a remark, if we choose $u$ to be positive, there is an instability towards a pairing state [18]. On the other hand, if we choose negative $u$ with $M = N$, we get back to the supersymmetric SYK models as in Chapter 4. In the following, we will keep $u = -1$, since rescaling $u$ can be restored by rescaling $J$. The interaction term can be linearized by $M$ bosons, the effective action can be written in terms of bilocal fields $G$ and $\Sigma$:

$$S_{\text{eff}} = - \log \text{Pf}(-\sigma_f - \Sigma_f) + \frac{\alpha}{2} \log \det(-\sigma_b - \Sigma_b) +$$

$$\frac{1}{2} \int d\tau^2 \left( G_f(\tau_1, \tau_2) \Sigma_f(\tau_1, \tau_2) + \alpha G_b(\tau_1, \tau_2) \Sigma_b(\tau_1, \tau_2) - 2 \sqrt{\alpha} J G_b(\tau_1, \tau_2) G_f(\tau_1, \tau_2)^2 \right)$$

(5.5)

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In the large $N$ and large $M$ limit, the theory is described by its saddled point and we can obtain the Schwinger-Dyson equations

$$G_f(i\omega_n)^{-1} = -i\omega_n - \Sigma_f(i\omega_n) , \quad \Sigma_f(\tau) = 4\sqrt{\alpha} J G_b(\tau) G_f(\tau)$$  
$$G_b(i\omega_n)^{-1} = -1 - \Sigma_b(i\omega_n) , \quad \Sigma_b(\tau) = \frac{2}{\sqrt{\alpha}} J G_f(\tau)^2$$

(5.6)

with $\alpha = \frac{M}{N}$. This model can also be generalized to a $2(q-1)$ body interaction Hamiltonian, and the SD equations will reduce to

$$G_f(i\omega_n)^{-1} = -i\omega_n - \Sigma_f(i\omega_n) , \quad \Sigma_f(\tau) = 2\sqrt{\alpha}(q-1) J G_b(\tau) G_f(\tau)^{q-2}$$  
$$G_b(i\omega_n)^{-1} = -1 - \Sigma_b(i\omega_n) , \quad \Sigma_b(\tau) = \frac{2}{\sqrt{\alpha}} J G_f(\tau)^{q-1}$$

(5.8)

We will work with the general model (5.8). In the IR limit, we can ignore the $-i\omega_n$ and $-1$ terms. We then assume the low-energy scaling ansatz

$$G^c_f(\tau) = \frac{F \text{sgn}(\tau)}{|\tau|^{2\Delta_f}} , \quad G^c_b(\tau) = \frac{B}{|\tau|^{2\Delta_b}}$$

(5.9)

From the SD equations, we can obtain

$$2\sqrt{\alpha} J (q-1) F^{q-1} B c_f(\Delta_f) c_f((q-2)\Delta_f + \Delta_b)|\omega|^{2(q-1)\Delta_f + 2\Delta_b - 2} = -1$$

(5.10)

$$\frac{2}{\sqrt{\alpha}} J F^{q-1} B c_b(\Delta_b) c_b((q-1)\Delta_f)|\omega|^{2(q-1)\Delta_f + 2\Delta_b - 2} = -1$$

(5.11)

where

$$c_f(\Delta) = 2i \cos \pi \Delta \Gamma(1 - 2\Delta) , \quad c_b(\Delta) = 2 \sin (\pi \Delta) \Gamma(1 - 2\Delta)$$

(5.12)

\(^1\)Now the renormalization for the random interaction is defined as

$$N^{g-3/2} M^{1/2} \langle C^a_i C^a_k \rangle = J^2 \delta_{ab}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

(5.7)
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Figure 5.1: $\alpha$ as a function of $\Delta_f$ from Eq. (5.15) for a fixed $q = 5$. $\Delta = \frac{1}{2q}$ gives $\alpha = 1$. When $\alpha \to 0$, we also have a solution $\Delta_f \to 0$. Notice that only the line on the left side of the divergence makes sense $\Delta_f < \frac{1}{2(q-1)}$, which makes $\Sigma_b(i\omega)$ more relevant at IR.

It appears in the Fourier transform of a symmetric/asymmetric scaling function:

$$\int d\tau e^{i\omega\tau} \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta}} = c_f(\Delta) \text{sgn}(\omega)|\omega|^{2\Delta-1}, \quad \int d\tau e^{i\omega\tau} \frac{1}{|\tau|^{2\Delta}} = c_b(\Delta)|\omega|^{2\Delta-1} \quad (5.13)$$

The power of $\omega$ gives us one relation

$$(q - 1)\Delta_f + \Delta_b = 1 \quad (5.14)$$

The other relation is obtained from the consistent value of $F^{q-1}B$ in Eq. (5.11):

$$\alpha = -\frac{1}{q-1} \frac{1 - 2\Delta_f}{2(q-1)\Delta_f - 1} \frac{\sin \pi \Delta_f \sin \pi (q-1)\Delta_f}{\cos \pi \Delta_f \cos \pi (q-1)\Delta_f} \quad (5.15)$$

We see that in the case $\alpha = 1$, the supersymmetric solution $\Delta_f = \frac{1}{2q}$ satisfies the above equation. When $\alpha \neq 1$, we don’t have a supersymmetric solution, which is as expected, since we have different numbers of bosons and fermions. So with different boson/fermion number ratio, we have a continuous set of conformal theories. To see how $\Delta_f$ depends on $\alpha$, we plot the solution in Figure 5.1, as we can see as $\alpha$ goes down below $\alpha = 1$, $\Delta_f$ also goes down. In the small $\alpha$ limit when $\alpha \ll \frac{1}{q}$, we have the solution $\Delta_f = \frac{\sqrt{\alpha}}{n}$. Compared to the free theory $\Delta_f = 0$, the anomalous scaling dimension gets smaller as the ratio $\alpha$ gets smaller. So we will approach a free fermion limit in the small $\alpha$ limit.
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Figure 5.2: Figure, adapted from Ref. [18], showing the scaling dimensions as a function of $M/N$. The dots are obtained by numerically solving Eq. (5.8). Solid lines are obtained from Eqs. (5.14)(5.15).

The coefficient combination $F^{q-1}B$ is fixed by the low-energy ansatz:

$$2\sqrt{\alpha} J F^{q-1} B = \frac{1 - 2\Delta_f}{2\pi(q - 1)} \frac{\sin \pi \Delta_f}{\cos \pi \Delta_f}$$

(5.16)

although we still can not individually fix $F$ and $B$. This also happens in the supersymmetric SYK models, it corresponds to a symmetry in the IR SD equations

$$G_f(\tau) \to \lambda G_f(\tau), \quad G_b(\tau) \to \lambda^{1-q} G_b(\tau)$$

(5.17)

This corresponds to an $h = 0$ mode and will be broken by the UV terms as discussed in Chapter 4. We have also discussed the large $q$ limit in Appendix. D.1.

To verify this continuous set of dimensions are correct, one needs to verify the solution numerically. And we have shown the consistency in Figure 5.2.
5.3 Results

5.3.1 Imaginary time kernel

In this model, we can define an imaginary time kernel which will be used when computing the four-point function. There are different ways to derive the kernel. The usual way is to consider the ladder diagram and find the kernel to that ladder summation. Here we use another approach by considering the change of the conformal solution from a UV perturbation:

\[ G_f * \Sigma_f + G_f * s_f = -\delta(\tau - \tau''), \quad G_b * \Sigma_b + G_b * s_b = -\delta(\tau - \tau'') \] (5.18)

where the UV perturbation source is \( s_f = -\delta'(\tau - \tau') \) and \( s_b = \delta(\tau - \tau') \), we consider the change caused by this source

\[ G_f = G_f^c + \delta G_f, \quad G_b = G_b^c + \delta G_b \] (5.19)
\[ \Sigma_f = \Sigma_f^c + \delta \Sigma_f, \quad \Sigma_b = \Sigma_b^c + \delta \Sigma_f \] (5.20)

where

\[
\begin{align*}
\delta \Sigma_f &= 2\sqrt{\alpha}(q - 1) J \left[ \delta G_b G_f^c(\tau)^{q-2} + (q - 2) G_b^c G_f^c(\tau)^{q-3} \delta G_f \right] \\
\delta \Sigma_b &= \frac{2}{\sqrt{\alpha}} J(q - 1) G_f^c(\tau)^{q-2} \delta G_f
\end{align*}
\] (5.21)

Plugging this into Eq. (5.18), we can obtain

\[(K_c - I)\delta G = -G_c * s * G_c \] (5.23)

where \( \delta G \) is a vector

\[ v = (\delta G_f, \delta G_b)^T \] (5.24)
Similarly we find that $K_c$ element as an example:

$$
K_c = \begin{pmatrix}
2\sqrt{\alpha}J(q-1)(q-2)G_f^*(\tau_{13})G_f^*(\tau_{42})G_b^*(\tau_{34})G_f^*(\tau_{34})^{q-3} & 2\sqrt{\alpha}J(q-1)G_f^*(\tau_{13})G_f^*(\tau_{42})G_f^*(\tau_{34})^{q-2} \\
\frac{2}{\sqrt{\alpha}}J(q-1)G_b^*(\tau_{13})G_b^*(\tau_{42})G_f^*(\tau_{34})^{q-2} & 0
\end{pmatrix}
$$

We can see that this form can also be obtained from the diagrams as in Ref. [157]. The kernel can be generally diagonalized by

$$
v_h = \left( \frac{\text{sgn}(\tau_{34})}{|\tau_{34}|^{2\Delta_f-\bar{h}}}, \frac{1}{|\tau_{34}|^{2\Delta_b-\bar{h}}} \right)^T
$$

(5.26)

If we use the UV source in the SD equations, it suggests that $\delta G_f \sim \frac{1}{|r|^{2\Delta_f}}$ and $\delta G_b \sim \frac{1}{|r|^{2\Delta_b}}$, which can be induced by an operator with $h = -2\Delta_f$ and $h = 1 - 2\Delta_b$. Later we will see that this gives the eigenvalue of the kernel to be $\infty$ and indicating that this mode won’t propagate to IR.

Now we will use the general vector (5.26) to diagonalize the kernel, we take the first element as an example:

$$
K_c^{11} \delta G_f^h
= 2\sqrt{\alpha}J(q-1)(q-2)BF^{q-1} \int d\tau_3 d\tau_4 \frac{\text{sgn}(\tau_{13}) \text{sgn}(\tau_{42}) \text{sgn}(\tau_{34})}{|\tau_{13}|^{2\Delta_f} |\tau_{24}|^{2|\Delta_f|} |\tau_{34}|^{(q-2)|\Delta_f|+2\Delta_b-\bar{h}}} \\
= (q-2) \frac{1 - 2\Delta_f}{2\pi} \frac{\sin \pi \Delta_f}{\cos \pi \Delta_f} c_f(\Delta_f)^2 \int \frac{d\omega d\omega'}{(2\pi)^2} \int d\tau_3 d\tau_4 \text{sgn}(\omega) \text{sgn}(\omega') |\omega \omega'|^{2\Delta_f-1} e^{-i\omega \tau_{13} - i\omega' \tau_{12}} \text{sgn}(\tau) \\
= (q-2) \frac{1 - 2\Delta_f}{2\pi} \frac{\sin \pi \Delta_f}{\cos \pi \Delta_f} c_f(\Delta_f)^2 c_f(1 - \Delta_f - \frac{h}{2}) \int \frac{d\omega}{2\pi} |\omega|^{2\Delta_f-1} e^{-i\omega \tau_{12}} \\
= (q-2) \frac{1 - 2\Delta_f}{2\pi} \frac{\sin \pi \Delta_f}{\cos \pi \Delta_f} c_f(\Delta_f)^2 c_f(1 - \Delta_f - \frac{h}{2}) \frac{\text{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta_f-\bar{h}}} \\
= -(q-2) c_f(\Delta_f) c_f(1 - \Delta_f - \frac{h}{2}) \delta G_f^h
$$

(5.27)

Similarly we find that

$$
k_c^{12} = -\frac{F c_f(\Delta_f) c_f(1 - \Delta_f - \frac{h}{2})}{B c_f(\Delta_f - \frac{h}{2}) c_f(1 - \Delta_f)}, \quad k_c^{21} = -(q-1) \frac{B c_b(\Delta_b) c_b(1 - \Delta_b - \frac{h}{2})}{F c_b(\Delta_b - \frac{h}{2}) c_b(1 - \Delta_b)}
$$

(5.28)
With $h = 2$, the kernel matrix reduces to

$$
\begin{pmatrix}
\frac{(g-2)\Delta_f}{1-\Delta_f} & \frac{\Delta_f}{1-\Delta_f} & F \\
\frac{(g-1)\Delta_b}{1-\Delta_b} & B & F \\
\frac{1}{F} & 0
\end{pmatrix}
$$

There are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -\frac{\Delta_b}{1-\Delta_f}$. When we take $h = -2\Delta_f$, $k_c^{11} = \infty$ and $k_c^{12} = \infty$, there will be no solution; when we take $h = 1 - 2\Delta_b$, $k_c^{21} = \infty$, also there is no solution. Interestingly, only when $\alpha = 1$, the supersymmetric solution, $h = -2\Delta_f = 1 - 2\Delta_b$ will give all the three elements $\infty$.

To see the $h \rightarrow 1 - h$ symmetry in the kernel more explicitly, we can write $c_f(\Delta)$ and $c_b(\Delta)$ as

$$
c_f(\Delta) = 2i2^{-2\Delta}\sqrt{\pi}\frac{\Gamma(1-\Delta)}{\Gamma(\frac{1}{2} + \Delta)}, \quad c_b(\Delta) = 2\tan(\pi\Delta)2^{-2\Delta}\sqrt{\pi}\frac{\Gamma(1-\Delta)}{\Gamma(\frac{1}{2} + \Delta)}
$$

We can plug this in the kernel and find that the matrix element is invariant under $h \rightarrow 1 - h$. This also indicates that the $h = -1$ mode gives $\lambda = 1$ and this corresponds to the UV perturbation that can propagate to IR.

There is another mode $h = 0, 1$ that gives $\lambda = 1$, this corresponds to the rescaling symmetry as in (5.17).

We can compute the kernel eigenvalue of the $h \neq 2$ modes in the small $\Delta_f$ limit. Consider $h = 2n$, and we plug into the kernel matrix with $\Delta_f \rightarrow 0$

$$
\begin{pmatrix}
-\frac{(g-2)\Delta_f}{n(1-2n)} \mathcal{O}(\Delta_f^2) + O(\Delta_f^3) & -\frac{(g-2)\Delta_f}{n(1-2n)} \left[ 1 + a(n)\Delta_f + \mathcal{O}(\Delta_f^2) \right] \frac{F}{F} \\
-\frac{a(n-2)}{\Delta_f} \left[ 1 - (g-1)a(n)\Delta_f + \mathcal{O}(\Delta_f^2) \right] \frac{B}{F} & 0
\end{pmatrix}
$$

with $a(n) = b(n) + b(n-1) + b(n - \frac{1}{2}) + b(n - \frac{3}{2}) - 2$ and $b(x)$ is the generalized Hamonic series

$$
b(x) = \sum_{i=0}^{x-i>0} \frac{1}{x-i}
$$
From this we can obtain there is an eigenvalue quite close to 1

$$\lambda_{2n} = 1 - \frac{q - 2}{2} \left[ a(n) + \frac{1}{n(1-2n)} \right] \Delta_f$$  \hspace{1cm} (5.33)

This is an indication that the $h \neq 2$ piece will give more contribution to the four-point function as $\Delta_f$ becomes small. As we will see in later sections, the coefficient is proportional to $\frac{1}{\Delta_f}$, and the $h = 2n$ piece will be proportional to $\Delta_f^{-1}$. Since in this model, we still have the $\text{Diff}(S^1)$ breaks to $\text{PSL}(2, \mathbb{R})$ picture, we will later show that the conformal piece will still give maximal chaos. However we expect that as $\Delta_f$ becomes smaller, the correction to the maximal Lyapunov exponent becomes larger due to the $h \neq 2$ modes.

This can be compared to the large $q$ normal SYK case, there we find that

$$k_c(2n) \approx \frac{1}{2n(n-1/2)}$$  \hspace{1cm} (5.34)

is not close to 1 and does not contribute much although the scaling dimension $\Delta_f = \frac{1}{q}$ is small.

### 5.3.2 Retarded kernel and chaos

The simplest way to obtain Lyapunov exponent is by diagonalizing the retarded kernel. We will need the retarded and Wightman Green’s function

$$G^R_f(t) = 2\cos(\pi \Delta_f)F\left(\frac{\pi}{\beta \sinh(\frac{\pi t}{\beta})}\right)^{2 \Delta_f} \Theta(t) \hspace{1cm} G^R_b = -2i \sin(\pi \Delta_b)B\left(\frac{\pi}{\beta \sinh(\frac{\pi t}{\beta})}\right)^{2 \Delta_b} \Theta(t)$$

$$G^{lr}_f(t) = F\left(\frac{\pi}{\beta \cosh(\frac{\pi t}{\beta})}\right)^{2 \Delta_f} \hspace{1cm} G^{lr}_b(t) = B\left(\frac{\pi}{\beta \cosh(\frac{\pi t}{\beta})}\right)^{2 \Delta_b}$$  \hspace{1cm} (5.35)
The retarded kernel is defined as

\[
K_R = \begin{pmatrix}
2\sqrt{\alpha}J(q-1)(q-2)G^R_f(t_{13})G^R_f(t_{24})G^R_b(t_{34})G^R_f(t_{34})q^{-3} & 2\sqrt{\alpha}J(q-1)G^R_f(t_{13})G^R_b(t_{24})G^R_f(t_{34})q^{-2} \\
-\frac{2}{\sqrt{\alpha}}J(q-1)G^R_b(t_{13})G^R_b(t_{24})G^R_f(t_{34})q^{-2} & 0
\end{pmatrix}
\]  

We can diagonalize the retarded kernel with an exponential growing ansatz

\[
v_h = \begin{pmatrix}
E^{-h\frac{\pi}{\beta}(t_3 + t_4)} & E^{-h\frac{\pi}{\beta}(t_3 + t_4)} \\
(cosh \frac{\pi t_{34}}{\beta})^{2\Delta_f - h} & (cosh \frac{\pi t_{34}}{\beta})^{2\Delta_b - h}
\end{pmatrix}\]  

After some algebra, we find that

\[
k_{R1} = (q - 2)\Gamma(2 - 2\Delta_f)\Gamma(2\Delta_f - h) \quad k_{R2} = \frac{\Gamma(2 - 2\Delta_f)\Gamma(2\Delta_f - h) F}{\Gamma(2\Delta_f)\Gamma(2 - 2\Delta_f - h) B},
\]

\[
k_{R3} = (q - 1)\Gamma(2 - 2\Delta_b)\Gamma(2\Delta_b - h) B \quad k_{R4} = \frac{\Gamma(2 - 2\Delta_b)\Gamma(2\Delta_b - h) F}{\Gamma(2\Delta_b)\Gamma(2 - 2\Delta_b - h) B}
\]  

If we plug in \( h = -1 \), the retarded kernel reduces to

\[
\begin{pmatrix}
\frac{(q-2)\Delta_f}{1-\Delta_f} & \frac{\Delta_f F}{1-\Delta_f B} \\
\frac{(q-1)\Delta_b B}{F} & 0
\end{pmatrix}
\]  

This is exactly the same as \( K_{c}^{h=2} \) as in (5.29).

From the retarded kernel, we can actually estimate the contribution from the \( h \neq 2 \) pieces. By contour deformation, the \( h \neq 2 \) piece will sum together and we will see later that it results in a double pole residue of a function proportional to \( \frac{1}{\tan(\pi h/2)} \frac{1}{1-k_R(1-h)} \) at \( h = 2 \), if other prefactors do not have diverging behavior when \( \Delta_f \to 0 \) the coefficient will be proportional to \( \frac{1}{k_R(1-h)^2} \). In the \( q = 3 \) case, when \( \Delta_f \) is small, we find that

\[
k_R(\Delta_f, h = -1)' = \left(-\frac{1}{2} + \frac{\pi^2}{3}\right)\Delta_f + \mathcal{O}(\Delta_f^2)
\]  

This means that the \( h \neq 2 \) part is proportional to \( \Delta_f^{-1} \). To compare with the \( h = 2 \) part,
we also need to know the coefficient of the $h = 2$ piece. From Ref.[119][135], there is an $1/k_c(-1)'$ prefactor in the $h = -1$ four-point function together with the coefficient of the correction to the conformal two-point function and other factors. We find that:

$$k_c(\Delta_f, h = -1)' = \frac{1}{4\Delta_f} + O(1) \quad (5.41)$$

This indicates that as we lower $\Delta_f$, the correction from the $h \neq 2$ piece will become more important. And we will have a smaller Lyapunov exponent. We put more careful analysis in the next two sections.

### 5.3.3 RG flow and the Schwarzian action

We follow the analysis in Ref.[119] and generalize to our model. The goal is to find the relation between $\alpha_S$ (coefficient of the Schwarzian action) and $\alpha_G$ (coefficient of the correction piece of the two-point function from the conformal solution). We change the coordinate to $\theta = \frac{2\pi \tau}{\beta}$ and the effective action can be written as:

$$\frac{S_{\text{eff}}}{N} = -\log \text{Pf}(-\sigma_f - \Sigma_f) + \frac{\alpha}{2} \log \det(-\sigma_b - \Sigma_b)$$

$$+ \frac{1}{2} \int d\tau_1 d\tau_2 \left( G_f(\tau_1, \tau_2) \Sigma_f(\tau_1, \tau_2) + \alpha G_b(\tau_1, \tau_2) \Sigma_b(\tau_1, \tau_2) - 2\sqrt{\alpha} J G_b(\tau_1, \tau_2) G_f(\tau_1, \tau_2)^{q-1} \right)$$

$$- \log \text{Pf}(-\tilde{\Sigma}_f) + \frac{\alpha}{2} \log \det(-\tilde{\Sigma}_b) + \frac{1}{2} \int d\theta_1 d\theta_2 \left( \tilde{G}_f \tilde{\Sigma}_f + \alpha \tilde{G}_b \tilde{\Sigma}_b - 2\sqrt{\alpha} \tilde{G}_b \tilde{G}_f^{q-1} \right)$$

$$- \frac{1}{2} \int d\theta_1 d\theta_2 \left( \tilde{\sigma}_f \tilde{G}_f + \alpha \tilde{\sigma}_b \tilde{G}_b \right) \quad (5.42)$$

where

$$G_f(\tau_1, \tau_2) = \tilde{G}_f(\theta_1, \theta_2) e^{2\Delta_f}, \quad G_b(\tau_1, \tau_2) = \tilde{J} \tilde{G}_b(\theta_1, \theta_2) e^{2\Delta_b}$$

$$\Sigma_f(\tau_1, \tau_2) = J^2 \left( \tilde{\Sigma}_f(\theta_1, \theta_2) - \tilde{\sigma}_f(\theta_1, \theta_2) \right) e^{2-2\Delta_f}, \quad \Sigma_b(\tau_1, \tau_2) = J \left( \tilde{\Sigma}_b(\theta_1, \theta_2) - \tilde{\sigma}_b(\theta_1, \theta_2) \right) e^{2-2\Delta_b}$$

$$\sigma_f(\tau_1, \tau_2) = J^2 \tilde{\sigma}_f(\theta_1, \theta_2) e^{2-2\Delta_f}, \quad \sigma_b(\tau_1, \tau_2) = J \tilde{\sigma}_b(\theta_1, \theta_2) e^{2-2\Delta_b} \quad (5.43)$$
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with $\epsilon = \frac{2\pi}{\beta J}$. Notice that this is just a change of the definition rather than a conformal transformation and under this definition all the bilocal fields are dimensionless. Later when we use $G_{f/b}^c$, we mean the function obtained from $G_{f/b}^c$ using the above transformation.

We consider a weak perturbation with some window function at intermediate time scales ($J^{-2} \ll \tau \ll \beta$) that would mimic the IR effect produced by the true source. Then we want to see how this perturbation will generate the change in the two-point function.

Expand the action to the quadratic order around the conformal saddle point:

$$
\frac{\delta S_{\text{eff}}}{N} = \frac{1}{4} \left( \begin{array}{ccc}
\delta \Sigma_f & \delta \tilde{G}_f & \delta \tilde{G}_b \\
\delta \Sigma_f & I & -\tilde{C} \\
\delta \Sigma_b & \alpha \tilde{A}_b & \alpha I
\end{array} \right) \left( \begin{array}{c}
\delta \Sigma_f \\
\delta \tilde{G}_f \\
\delta \tilde{G}_b
\end{array} \right) - \frac{1}{2} \sigma_f \delta \tilde{G}_f - \frac{\alpha}{2} \sigma_b \delta \tilde{G}_b
$$

(5.44)

where

$$
\tilde{A}_f(\theta_1, \theta_2; \theta_3, \theta_4) = G_{f}^c(\theta_1 \theta_3) G_{f}^c(\theta_2 \theta_4) , \quad \tilde{A}_b(\theta_1, \theta_2; \theta_3, \theta_4) = G_{b}^c(\theta_1 \theta_3) G_{b}^c(\theta_2 \theta_4)
$$

(5.45)

and

$$
\tilde{B}_f(\theta_1, \theta_2; \theta_3, \theta_4) = 2\sqrt{\alpha(q - 1)(q - 2)} G_{b}^c(\theta_{12}) G_{f}^c(\theta_{12})^{q - 3} \delta(\theta_{13}) \delta(\theta_{24})
$$

(5.46)

$$
\tilde{C}(\theta_1, \theta_2; \theta_3, \theta_4) = 2\sqrt{\alpha(q - 1)} G_{b}(\theta_{12}) G_{f}(\theta_{12})^{q - 2} \delta(\theta_{13}) \delta(\theta_{24})
$$

(5.47)
After integrating out $\delta \Sigma_f$ and $\delta \Sigma_b$

$$
\frac{\delta S_{\text{eff}}}{N} = \frac{1}{4} \begin{pmatrix} \delta \tilde{G}_f & \delta \tilde{G}_b \end{pmatrix} \begin{pmatrix} \tilde{A}_f^{-1} - \tilde{B}_f & -\tilde{C} \\ -\tilde{C} & -\alpha \tilde{A}_b^{-1} \end{pmatrix} \begin{pmatrix} \delta \tilde{G}_f \\ \delta \tilde{G}_b \end{pmatrix} - \frac{1}{2} \tilde{\sigma}_f \delta \tilde{G}_f - \frac{\alpha}{2} \tilde{\sigma}_b \delta \tilde{G}_b
$$

$$
= \frac{1}{4} \begin{pmatrix} \delta \tilde{G}_f & \delta \tilde{G}_b \end{pmatrix} \begin{pmatrix} \tilde{A}_f^{-1} \\ \alpha \tilde{A}_b^{-1} \end{pmatrix} (I - \tilde{K}_c) \begin{pmatrix} \delta \tilde{G}_f \\ \delta \tilde{G}_b \end{pmatrix} - \frac{1}{2} \tilde{\sigma}_f \delta \tilde{G}_f - \frac{\alpha}{2} \tilde{\sigma}_b \delta \tilde{G}_b
$$

(5.48)

where $\tilde{K}_c$ with arguments $\theta$ is the re-definition of the imaginary time kernel as in (5.25).

Consider the scaling ansatz as in (5.26), we have

$$
(1 - k_c(h)) \begin{pmatrix} \tilde{A}_f^{-1} \\ \alpha \tilde{A}_b^{-1} \end{pmatrix} \begin{pmatrix} \delta \tilde{G}_f \\ \delta \tilde{G}_b \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_f \\ \alpha \tilde{\sigma}_b \end{pmatrix}
$$

(5.49)

We normalize the function $\delta G$, $\sigma$

$$
\delta \tilde{G}_f = |\tilde{G}_f^c|^{\alpha_f} g_f , \quad \delta \tilde{G}_b = |\tilde{G}_b^c|^{\alpha_b} g_b
$$

(5.50)

$$
\tilde{\sigma}_f = |\tilde{G}_f^c|^{-\alpha_f} s_f , \quad \tilde{\sigma}_b = |\tilde{G}_b^c|^{-\alpha_b} s_b
$$

(5.51)

$$
\alpha_{f/b} = 1 - \frac{1}{2 \Delta_{f/b}} , \text{then } g, s \in (\mathcal{F}_{1/2} \otimes \mathcal{F}_{1/2})^1. \text{Using } \tilde{A}_f^{-1} = \Sigma_{f/b}^{c} (\theta_{13}) \Sigma_{f/b}^{c} (\theta_{42}). \text{We assume the ansatz}
$$

$$
\tilde{G}_f^c (\theta) = \frac{\tilde{F} \text{ sgn } \theta}{(2 \sin \frac{\theta}{2})^{2 \Delta_f}} , \quad \tilde{G}_b^c (\theta) = \frac{\tilde{B}}{(2 \sin \frac{\theta}{2})^{2 \Delta_b}}
$$

(5.52)

which gives

$$
a_f(h)(1 - k(h)) \tilde{F}^{-\frac{1}{2\Delta_f}} g_f = s_f , \quad a_b(h)(1 - k(h)) \tilde{B}^{-\frac{1}{2\Delta_b}} g_b = s_b
$$

(5.53)
with

\[
a_f(h) = \frac{c_f(1 - \Delta_f)^2 c_f(\Delta_f - \frac{h}{2})}{c_f(1 - \Delta_f - \frac{h}{2})} \left( \frac{1 - 2\Delta_f \sin \pi \Delta_f}{2\pi(q - 1) \cos \pi \Delta_f} \right)^2 \tag{5.54}
\]

\[
a_b(h) = \frac{1}{\alpha^2(q - 1)^2} \frac{c_b(1 - \Delta_b)^2 c_b(\Delta_b - \frac{h}{2})}{c_b(1 - \Delta_b - \frac{h}{2})} \left( \frac{1 - 2\Delta_f \sin \pi \Delta_f}{2\pi(q - 1) \cos \pi \Delta_f} \right)^2 \tag{5.55}
\]

This also indicates that

\[
s_f(\theta) \sim \theta^{h-1} \text{sgn}(\theta) \quad , \quad s_b(\theta) \sim \theta^{h-1} \tag{5.56}
\]

But the power-law source is not that physical because it directly influences the Green’s function at intermediate times, whereas the physical effect is due to RG flow. So we consider the source is supported by an extended UV region, where \(|\delta\tau|\) is bounded by some large constant times \(J^{-1}\). In the case of resonance, i.e. \(k_c(h) = 1\), we will see that the response extends to longer times and eventually flow to IR. We assume the perturbation has the form

\[
s_f(\theta) = -b_f \epsilon^{-h_0} |\theta|^{h_0-1} \text{sgn}(\theta) u_f(\xi) \quad , \quad s_b(\theta) = -b_b \epsilon^{-h_0} |\theta|^{h_0-1} u_b(\xi) \tag{5.57}
\]

with \(\xi = \ln(\frac{|\theta|}{\epsilon})\) and \(h_0 = -1\) corresponds to the irrelevant perturbation. And \(u_{f/b}(\xi)\) is a normalized window function \(\int u(\xi)d\xi = 1\). Let us consider the Fourier transform of the window function

\[
\tilde{u}_{f/b}(\eta) = \int d\xi u_{f/b}(\xi)e^{in\xi} \tag{5.58}
\]

Then

\[
s_f(\theta) = -b_f \epsilon^{-h_0} \int \frac{d\eta}{2\pi} \epsilon^{-in} |\theta|^{h_0-1+in} \text{sgn}(\theta) \tilde{u}_f(\eta)
\]

\[
s_b(\theta) = -b_b \epsilon^{-h_0} \int \frac{d\eta}{2\pi} \epsilon^{-in} |\theta|^{h_0-1+in} \tilde{u}_b(\eta) \tag{5.59}
\]
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We can see that the window function serves to smear the power $h_0 - 1$ over a small imaginary width. For each given eta, the right-hand side is the eigenfunction of the kernel with eigenvalue $k_c(h = h_0 + i\eta)$. We find that

$$g_f(\theta) \approx -\frac{b_f}{a_f(h_0)}F^{\frac{1}{k_c}}\epsilon^{-h_0}\epsilon^{h_0-1}\text{sgn}(\theta)v_f(\xi), \quad g_b(\theta) \approx -\frac{b_b}{a_b(h_0)}\tilde{B}^{\frac{1}{k_c}}\epsilon^{h_0-1}\epsilon^{h_0}v_b(\xi)$$

(5.60)

with

$$v_f/b(\xi) = \frac{1}{-k'_c(h_0)}\int\frac{d\eta}{2\pi}w_{f/b}(\eta)\frac{1}{i\eta}e^{i\eta\xi}$$

(5.61)

We can thus derive an RG equation

$$\frac{dv_{f/b}(\xi)}{d\xi} = -\frac{1}{k'_c(h_0)}u_{f/b}(\xi)$$

(5.62)

From this we have $v_{f/b}(\xi) = (-k'_c(h_0))^{-1}\int_{-\infty}^{\xi}u(\zeta)d\zeta$, and in the intermediate region in which $u$ has been fully integrated,

$$g_f(\theta) \approx \frac{b_f}{a_f(h_0)}F^{\frac{1}{k'_c}}\epsilon^{-h_0}\epsilon^{h_0-1}\text{sgn}(\theta), \quad g_b(\theta) \approx \frac{b_b}{a_b(h_0)}\tilde{B}^{\frac{1}{k'_c}}\epsilon^{h_0-1}\epsilon^{h_0}$$

for $\epsilon \ll |\theta| \ll 1$

(5.63)

This means

$$\frac{\delta G_f(\tau)}{G_f^c(\tau)} = \frac{b_f}{a_f(h_0)k'_c(-1)}\frac{1}{J\tau} = \frac{\alpha'_G}{J\tau}, \quad \frac{\delta G_b(\tau)}{G_b^c(\tau)} = \frac{b_b\tilde{B}^{\frac{1}{k'_c}}}{a_b(h_0)k'_c(-1)}\frac{1}{J\tau} = \frac{\alpha'_B}{J\tau}$$

(5.64)

Now let us derive the Schwarzian action. We consider the Green’s function deformed by the
soft mode, i.e. reparametrization $\varphi(\theta)$. We denote the deformed Green’s function by $\tilde{G}^{\text{IR}}$

$$\tilde{G}^{\text{IR}}_{f/b}(\theta_1, \theta_2) = \tilde{G}^{\varphi}(\varphi(\theta_1), \varphi(\theta_2))\varphi'(\theta_1)\varphi'(\theta_2)\Delta_{f/b}$$

$$\approx \tilde{G}^{\beta=\infty}_{f/b}(\theta_1, \theta_2) \left(1 + \frac{\Delta_{f/b}}{6} \text{Sch}(e^{i\varphi(\theta_+)}(\theta_+)(\theta_1 - \theta_2)^2)\right) \quad \text{for } |\theta_1 - \theta_2| \ll 1$$

(5.65)

where

$$\tilde{G}^{\beta=\infty}_{f}(\theta_1, \theta_2) = \frac{\tilde{F}\text{sgn}(\theta_1 - \theta_2)}{|\theta_1 - \theta_2|2\Delta_f}, \quad \tilde{G}^{\beta=\infty}_{b}(\theta_1, \theta_2) = \frac{\tilde{B}}{|\theta_1 - \theta_2|2\Delta_b} \quad (5.66)$$

The coupling between the soft modes and the UV perturbation is given by

$$\delta S_{\text{eff.}} = -\frac{N}{2} \int d\theta_1 d\theta_2 \left[ \tilde{G}^{\text{IR}}_f - \tilde{G}^{\beta=\infty}_{f} \right] = -N\alpha^{f}_{S}\epsilon \int \text{Sch}(e^{i\varphi(\theta)}, \theta)d\theta - N\alpha^{b}_{S}\epsilon \int \text{Sch}(e^{i\varphi(\theta)}, \theta)d\theta$$

(5.67)

where

$$\alpha^{f}_{S} = -\tilde{F}\frac{\Delta_f}{12}b f \int_{-2\pi}^{2\pi} d\theta_1 d\theta_2 |\theta_1|^{-1}u_f(\ln|\theta_1/\epsilon|) = -b f \tilde{F}\frac{\Delta_f}{12} = -\frac{\alpha^{f}_G a_f(-1)k'_f(-1)\Delta_f}{6}$$

$$\alpha^{b}_{S} = -\alpha^{b}_{S}\epsilon \int \text{Sch}(e^{i\varphi(\theta)}, \theta)d\theta - N\alpha^{b}_{S}\epsilon \int \text{Sch}(e^{i\varphi(\theta)}, \theta)d\theta$$

(5.68)

We also know that $\delta G_f$ and $\delta G_b$ will be related by the coefficient of the $h = -1$ eigenvector.

In the $q = 3$ case, at small $\alpha$ limit, we find that $a_f(-1) = \frac{1}{8} + \mathcal{O}(\Delta f)$, $a_b(-1) = \frac{1}{8\pi^2} + \mathcal{O}(\Delta f)$. From (5.29), we have $\frac{\delta G_f}{G_f} = \frac{\alpha^f_{G}}{\alpha^b_{G}} \sim \Delta f$, we have $\alpha^f_{S} = \alpha^b_{S}$ that contribute the same.

5.3.4 Four-point functions

To diagonalize the four-point function, we can make use of the $SL(2)$ algebra, then represent the four-point function using the cross-ratio $\xi = \frac{\tau_{12} \tau_{34}}{\tau_{13} \tau_{24}}$. The same analysis has been done in
Ref. [157]. The difference here is that the scaling dimension depends on $\alpha$. We define our fermion four point function:

$$
\frac{1}{N^2} \sum_{i,j=1}^{N} \langle \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4) \rangle = G_f(\tau_{12})G_f(\tau_{34}) + \frac{1}{N} \mathcal{F}^{xxxx}(\tau_1, \tau_2; \tau_3, \tau_4) + \mathcal{O}(\frac{1}{N^2}) \\
= G_f(\tau_{12})G_f(\tau_{34}) \left[ 1 + \frac{1}{N} \mathcal{F}^{xxxx}(\xi) + \mathcal{O}(\frac{1}{N^2}) \right] \quad (5.69)
$$

we can use eigenfunctions of the Casimir to diagonalize the four-point functions:

$$
F_1(\xi) = \frac{\xi^h \Gamma(h)^2}{\Gamma(2h)} \ {}_2F_1(h, h; 2h; \xi), \quad (5.70)
$$

$$
F_1(\xi) = \frac{\xi^{1-h} \Gamma(1-h)^2}{\Gamma(2-2h)} \ {}_2F_1(1-h, 1-h; 2-2h; \xi) \quad (5.71)
$$

In the region $\xi > 1$, the solution with definite parity under $\xi \to \frac{\xi}{\xi-1}$ is

$$
\Phi_{s>1}^s(\xi) = \frac{\Gamma(\frac{1}{2} - \frac{h}{2})\Gamma(\frac{h}{2})}{\sqrt{\pi}} \ {}_2F_1\left(\frac{h}{2}, \frac{1-h}{2}; \frac{1}{2}; \frac{(\xi-2)^2}{\xi^2}\right) \quad (5.72)
$$

here $s$ stands for symmetric. By matching the behavior at $\xi = 1$, we can extend the solution to $0 < \xi < 1$

$$
\Phi_{0<\xi<1}^s(\xi) = AF_1(\xi) + BF_2(\xi) \quad (5.73)
$$

with

$$
A = \frac{1}{2} \tan(\pi h) \cot\left(\frac{\pi h}{2}\right), \quad B = -\frac{1}{2} \tan(\pi h) \tan\left(\frac{\pi h}{2}\right) \quad (5.74)
$$

The eigenfunction in the region $\xi < 0$ is defined by

$$
\Phi_{\xi<0}^s = \Psi_{0<\xi<1}^s\left(\frac{\xi}{\xi-1}\right) \quad (5.75)
$$
The allowed $h$ is determined by requiring that the Casimir be hermitian with respect to the inner product, one finds that

$$h = \frac{1}{2} + is, \quad s > 0, \quad \text{or} \quad h = 2n, n \in \mathbb{Z}^+ \quad (5.76)$$

The norm of the eigenfunctions can be directly computed

$$\langle \Phi_h^s, \Phi_h^{s'} \rangle = \begin{cases} \frac{\pi \tan(\pi h)}{4h-2}, & h = \frac{1}{2} + is \\ \frac{\pi^2 \delta_{hh'}}{4h-2}, & h = 2n \end{cases} \quad (5.77)$$

Another way to compute these symmetric eigenfunctions other than the boundary condition matching is by the direct integral in the shadow representation\[25].

Now we can decompose any function $f(\xi)$ with the symmetric eigenfunctions above

$$f(\xi) = \sum_h \Phi_h^s(\xi) \frac{\langle \Phi_h^s, f \rangle}{\langle \Phi_h^s, \Phi_h^s \rangle} = \int_0^{\infty} ds \frac{4h - 2}{2\pi \tan(\pi h)} \langle \Phi_h^s, f \rangle \Phi_h^s(\xi) + \sum_{n=1}^{\infty} \left[ \frac{4h - 2}{\pi^2} \langle \Phi_h^s, f \rangle \Phi_h^s(\xi) \right]_{h=2n} \quad (5.78)$$

The four-point function can be obtained by summing over all the $n$-rung ladders. The 0-rung ladder is

$$F_{0}^{xxxx}(\tau_1, \tau_2; \tau_3, \tau_4) = -G_f(\tau_{13})G_f(\tau_{24}) + G_f(\tau_{14})G_f(\tau_{23}) \quad (5.79)$$

which gives

$$F_0^{xxxx}(\xi) = -\text{sgn}(\xi)|\xi|^{2\Delta_f} + \text{sgn}(\xi)\frac{\xi}{\xi - 1}|\xi - 1|^{2\Delta_f} \quad (5.80)$$

Using the symmetric eigenfunctions, we find

$$F_0^{xxxx}(\xi) = \int_0^{\infty} ds \frac{4h - 2}{2\pi \tan(\pi h)} f_0(h)\Phi_h^s(\xi) + \sum_{n=1}^{\infty} \left[ \frac{4h - 2}{\pi^2} f_0(h)\Phi_h^s(\xi) \right]_{h=2n} \quad (5.81)$$

where

$$f_0(h) = \langle \mathcal{F}_0^{xxxx}, \Phi_h^s \rangle = \frac{c_f(1 - \Delta_f - h/2)c_f(\Delta_f)^2}{c_f(\Delta_f - h/2)} \quad (5.82)$$
here we used the integral representation of $\Phi^s_h$:

$$\Phi^s_h(\xi) = \frac{1}{2} \int_{-\infty}^{\infty} dy \frac{|\xi|^h}{|y|^h|\xi - y|^h|1 - y|^{1-h}} \quad (5.83)$$

Together with (5.80), we have$^2$

$$f_0(h) = -\int dy d\xi \frac{\text{sgn}(\xi)}{|\xi|^{2-h-2\Delta_f}|\xi - y|^h|1 - y|^{1-h}|y|^h} \quad (5.84)$$

This integral can be transformed to the kernel integral, which leads to (5.82). Since the kernel also connects bosons with fermions, we also need to consider $\mathcal{F}_0^{bbXX}$, so we denote $b_0(h) = \langle \mathcal{F}_0^{bbXX}, \Phi^s_h \rangle = 0$. Starting from the 0-rung ladder, we generate the $n$-rung ladder by convolving $n$ times with the kernel matrix (5.25). The full four-point function is determined by

$$f(h) = \sum_{n=0}^{\infty} f_n(h), \quad b(h) = \sum_{n=0}^{\infty} b_n(h) \quad (5.85)$$

The action of the kernel matrix on an $n$-rung ladder is

$$\begin{pmatrix} f_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} k_{c11}^1 & k_{c12}^1 \\ k_{c21}^2 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ b_n \end{pmatrix} \quad (5.86)$$

The similar analysis has been carried out in Ref. [157], here we just copy the results:

$$f(h) = \frac{f_0(h)}{(1 - \overline{k}_c^-)(1 - \overline{k}_c^+)} \quad b(h) = \frac{k_{c21}^2 f_0(h)}{(1 - \overline{k}_c^-)(1 - \overline{k}_c^+)} \quad (5.87)$$

where $\overline{k}_c^\pm = \frac{1}{2} k_{c11}^\pm \pm \sqrt{\frac{1}{4}(k_{c11}^\pm)^2 + k_{c12}^1 k_{c21}^2}$.

$^2$In Ref. [135] there seem to be a typo, there should be no prefactor of $\frac{1}{2}$ in (3.81)
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$h \neq 2$ modes

One can see that the $h = 2$ mode will cause divergence in the denominator, and needed to be regularized beyond the conformal limit. We group all the other contribution from $h \neq 2$ modes, and by substituting

$$\frac{2}{\tan(\pi h)} = \frac{1}{\tan\frac{\pi h}{2}} - \frac{1}{\tan\frac{\pi(1-h)}{2}}$$

(5.88)

We have

$$\mathcal{F}^{\chi_\chi_\chi_\chi}_{h \neq 2}(\xi) = \int_{-\infty}^{\infty} ds \frac{2h - 1}{2\pi \tan(\pi h/2)} f(h)\Phi_h^s(\xi) + \sum_{n=2}^{\infty} \text{Res}_{h=2n} \left[ \frac{2h - 1}{\pi \tan(\pi h/2)} f(h)\Phi_h^s(\xi) \right]_{h=2n}$$

(5.89)

This can be understood as a single integral

$$\frac{1}{2\pi i} \int_C dh = \int_{-\infty}^{\infty} ds + \sum_{n=1}^{\infty} \text{Res}_{h=2n}$$

(5.90)

Note that $\Phi_h^s$ has poles at $h = 1 + 2n$, but will be cancelled by $1/\tan(\pi h/2)$. In the $\xi > 1$ case, we can push the contour from the s axis rightward to infinity. In the process, we cancel the sum over residues while picking up poles at $\tilde{k}_c^+ = 1$, notice that $h = 2$ is a double pole.

$$\mathcal{F}^{\chi_\chi_\chi_\chi}_{h \neq 2}(\xi) = -\sum_{n=2}^{\infty} \text{Res}_{h=h_m} \left[ \frac{2h - 1}{\pi \tan(\pi h/2)} f(h)\Phi_h^s(\xi) \right]_{h=h_m}, \quad \xi > 1$$

(5.91)

In the region $\xi < 1$, more subtle analysis shows

$$\mathcal{F}^{\chi_\chi_\chi_\chi}_{h \neq 2}(\xi) = -\sum_{n=2}^{\infty} \text{Res}_{h=h_m} \left[ \frac{2h - 1}{\pi \tan(\pi h/2)} f(h)\frac{\Gamma(h)^2\xi^h}{\Gamma(2h)} {}_2F_1(h, h; 2h; \xi) \right]_{h=h_m}, \quad \xi < 1$$

(5.92)

In the OPE limit, we should use the expression in $\xi < 1$, the operators running in the OPE channel of the kernel is determined by $k(h_m) = 1$. In the case $\alpha = 1$, this corresponds to the
supersymmetric SYK and this gives

$$h_m = 2, 3.066, 3.821, \ldots$$  \hspace{1cm} (5.93)

Consider for example $\alpha = 0.5$, we find that

$$h_m = 2, 3.058, 3.855, \ldots$$  \hspace{1cm} (5.94)

In the chaos region, we need to use the expression in $\xi > 1$. The out of time ordered correlation can be conveniently expressed by the configuration

$$\text{Tr} [y \chi_i(t)y \chi_j(0)y \chi_i(t)y \chi_j(0)], \quad y \equiv \rho(\beta)^{1/4}$$  \hspace{1cm} (5.95)

This means the cross ratio

$$\xi = \frac{2}{1 - i \sinh \frac{2\pi t}{\beta}}$$  \hspace{1cm} (5.96)

Because $t = 0$ corresponds to $\xi = 2$, and $t = \infty$ gives $\xi = 0$, we should continue the $\Phi_{\xi>1}^s$ function to $\xi = 0$. For small $\chi$, the continuation gives

$$\Phi_h^s(\xi) \sim \frac{\Gamma\left(\frac{1}{2} - \frac{h}{2}\right)\Gamma(h - \frac{1}{2})}{2^{1-h}\Gamma\left(\frac{h}{2}\right)}(-i\xi)^{1-h} + (h \to 1 - h)$$  \hspace{1cm} (5.97)

With this form, we find that the sum in (5.89) or (5.91) does not converge after the continuation. So we need to manipulate the expression in a form that is safe to continue and sum. The strategy is to find another expression that coincides with $f(h)$ with $h = 2n$ but there are only finite number of poles left. A natural choice is considering the retarded kernel. To be more explicit we have

$$\frac{k_{111}(1-h)}{k_{11c}(h)} = \frac{k_{122}(1-h)}{k_{12c}(h)} = \frac{\cos \frac{\pi}{2}(-h + 2\Delta_f)}{\cos \frac{\pi}{2}(h + 2\Delta_f)}$$  \hspace{1cm} (5.98)

$$\frac{k_{211}(1-h)}{k_{21c}(h)} = \frac{\sin \frac{\pi}{2}(h + 4\Delta_f)}{\sin \frac{\pi}{2}(-h + 4\Delta_f)}$$  \hspace{1cm} (5.99)
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We can see that for \( h = 2n \), \( K_R(h) = K_c(h) \). So we can replace the sum in (5.89) with eigenvalues of the retarded kernel. However, for the retarded kernel \( k_R(1 - h) = 1 \) happens only when \( h = 2 \). We conclude that

\[
\mathcal{F}_{h \neq 2}^{\text{XXX}}(\xi) = \int \frac{ds}{2\pi} \frac{2h - 1}{2\pi \tan(\pi h/2)} [f(h) - f_R(1 - h)] \Phi_h^*(\xi) - \text{Res} \left[ \frac{2h - 1}{\pi \tan(\pi h/2)} f_R(1 - h) \Phi_h^*(\xi) \right]_{h = -1}
\]

(5.100)

where

\[
f_R(1 - h) = \frac{f_0(h)}{(1 - k_R^{-1}(1 - h))(1 - k_R^+(1 - h))}
\]

(5.101)

The integral will vanish as \( \xi \to 0 \), so the growing piece comes from the residue at \( h = -1 \). Because this is a double pole, we also have a contribution from \( \partial_h \Phi_h^*(\xi)|_{h=2} \) and this corresponds to a term proportional to \( te^{2\pi t/\beta} \). And this finite shift in the Lyapunov exponent (decrease) is proportional to \( \frac{1}{k_R(-1)} \). Other factors are finite in this limit: \( 1 - k_R^{-1}(-1) = 2 + \mathcal{O}(\Delta_f) \) and \( f_0(h = 2) = 2 + \mathcal{O}(\Delta_f) \). This verifies our argument about \( h \neq 2 \) modes in Sec 5.3.2.

Plugging the number, we find that the term proportional to \( te^{2\pi t/\beta} \) is

\[
\frac{F_{\text{sub}}(t)}{G(f)G(\pi)} = \frac{3}{2\pi k_R(-1)} \frac{2\pi t}{\beta} e^{2\pi t/\beta}
\]

(5.102)

\( h = 2 \) MODE

This mode will give a maximal chaotic OTOC using the Schwarzian action (5.67)\([119, 135]\)

\[
\frac{F_{\text{big}}(t)}{G(f)G(\pi)} \approx -\frac{\Delta_f^2 \beta J}{4\pi (\alpha_f^S + \alpha_b^S)} e^{2\pi t/\beta}
\]

(5.103)

with \( \alpha_f^S, \alpha_b^S \) in (5.68). In the small \( \Delta_f \) limit, we have

\[
\frac{F_{\text{big}}(t)}{G(f)G(\pi)} \approx \frac{24\Delta_f^2 \beta J}{\pi \alpha_f^G} e^{2\pi t/\beta}
\]

(5.104)

111
We assume that combining with the \( h \neq 2 \) sectors, the full four-point function looks like

\[
\frac{F(t)}{G_f(\pi)G_f(\pi)} \approx \frac{24\Delta_f^2\beta J}{\pi \alpha_G^f} e^{\frac{2\pi t}{\pi^2} (1 - \delta \lambda_L)}
\]  

(5.105)

Comparing with (5.102), this means that

\[
\delta \lambda_L \approx \frac{-\alpha_G^f}{16\beta J \Delta_f^2 k_R^f(-1)} \approx \frac{\alpha_G^f}{16(1/2 - \pi^2/3)\beta J \Delta_f^3}
\]  

(5.106)

The correction will depend on \( \alpha_G^f(\Delta_f, \beta J) \).

Compared to the original large \( q \) SYK model, the functional form of the \( h \neq 2 \) and \( h = 2 \) contributions to the four-point function (5.102)(5.104) are quite similar, the difference comes from the fact that in the large \( q \) model, both \( k'_c(2) \) and \( k'_R(-1) \) scales as constant in the small scaling dimension limit while in our case we have \( k'_c(2) = O(\Delta_f^{-1}) \) and \( k'_R(-1) = O(\Delta_f) \). This will make the \( h \neq 2 \) part more important in the \( \Delta_f \to 0 \) limit.

5.3.5 Numerical imaginary time Green’s function correction

In this section, we numerically solve the SD equation (5.8) and fit the solution in the imaginary time domain \((0, \beta)\) with the form

\[
G_f(\tau) = F \left( \frac{\pi}{\beta \sin(\frac{\pi \tau}{\beta})} \right)^{2\Delta_f} (1 - \alpha_G^f l_0(\tau)) , \quad l_0 = 2 + \frac{\pi - \frac{2\pi \tau}{\beta}}{\tan \frac{\pi \tau}{\beta}}
\]  

(5.107)

\[
G_b(\tau) = B \left( \frac{\pi}{\beta \sin(\frac{\pi \tau}{\beta})} \right)^{2\Delta_b} (1 - \alpha_G^b l_0(\tau)) , \quad l_0 = 2 + \frac{\pi - \frac{2\pi \tau}{\beta}}{\tan \frac{\pi \tau}{\beta}}
\]  

(5.108)

With the fitted numerical data, we obtain \( \alpha_S^f \) and \( \alpha_S^b \) from (5.68). And use this we can get the \( h = 2 \) contribution, together with the \( h \neq 2 \) part (5.100), we compute the correction in Lyapunov exponent \( \delta \lambda_L \) as shown in Figure (5.3). We can see that the correction gets larger with smaller \( \beta J \) and smaller \( \alpha \).

We show the \( \alpha \) dependence of \( \delta \lambda_L \) in the large \( \beta J \) conformal limit with \( \frac{\beta J}{2\pi} = 100 \) as shown
Chapter 5. A continuous set of boson-fermion SYK models

Figure 5.3: $\delta \lambda_L$ as a function of $\frac{\beta J}{2\pi}$ and $\alpha$ in a log – log – log plot.

in Figure (5.4). When we fit the slope at small $\alpha$, we obtain approximately $-1/2$. This means that at small $\alpha$

$$
\delta \lambda_L \sim \alpha^{-1/2} \sim \Delta_f^{-1}
$$

(5.109)

We can also find the $\beta J$ dependence of $\delta \lambda_L$ and it’s approximately $\delta \lambda_L \sim \frac{1}{\beta J}$. Together we have $\delta \lambda_L \sim \frac{1}{\beta J \Delta_f}$. So this means that different from the large $q$ normal SYK case, we get a bigger correction when we decrease $\Delta_f$ while keeping everything else fixed.

But it may be more physical to keep some physical quantify fixed while approaching the free fermion limit. There are two choices: energy density or specific heat density. We can estimate the energy density of the model in the small $\alpha$ limit. From (5.6) that for each fermion, its self-energy scales as $\sqrt{\alpha} J$ and for boson, it scales as $\frac{1}{\sqrt{\alpha}} J$. So the total energy scales as $(\sqrt{\alpha}N + \frac{1}{\sqrt{\alpha}} M)J \sim \sqrt{MN}J$. Then the energy density scales as $\sqrt{\alpha} J \sim \Delta_f J$. If we fix this energy while reducing $\Delta_f$ the $\delta \lambda_L$ will remain the same. But if we are keeping the specific heat density as constant, then we are keeping $\frac{\alpha_f}{\beta J}$ constant. Because from numerics, $\alpha_f \sim \Delta_f^2$, when we reduce $\Delta_f$ while keeping $\frac{\Delta_f^2}{\beta J}$ constant, $\delta \lambda_L \sim \frac{1}{\beta J \Delta_f}$ will become larger.

So it seems in our case when we keep the spectral density constant, we can get a smaller Lyapunov exponent in the free fermion limit. This can be compared with the large $q$ normal
Figure 5.4: $\delta \lambda_L$ as a function of $\alpha$ in a log -- log plot at $\frac{\beta J}{2\pi} = 100$. Blue line is achieved by the full expression while orange line is obtained by the asymptotic form (5.106).

SYK case, where the correction is found to be $\delta \lambda_L \sim \frac{1}{\beta J}$. The energy density scales as $\frac{J}{q^2}$, if we fix this energy while increasing $q$, the correction reduces. The specific heat density scales as $\frac{1}{q^2 J}$, if we fix this scale, the correction increases as we increase $q$. So we have a similar conclusion in this case.

There is another example in Append.H of Ref. [135], where one can consider a time-dependent coupling SYK model. The fermion scaling dimension can be tuned by tuning the time correlation of the coupling. And one indeed finds a non-maximal Lyapunov exponent with a smaller fermion scaling dimension.

5.3.6 Dephasing time

We adapt the definition from (10.100) in Ref. [170] for the dephasing or relaxation rate:

$$\Gamma_\phi^{-1} \equiv iG^R(0)\left.\frac{\partial G^R(\omega)^{-1}}{\partial \omega}\right|_{\omega=0}$$

(5.110)

where $G^R(\omega)$ is the retarded Green’s function at finite temperature. The dephasing time is defined as the inverse of the relaxation rate. In our model, the fermion and boson Green’s
function will be

$$G^R_f(\omega) \sim i \frac{\Gamma(\frac{i\beta \omega}{2\pi} + \Delta_f)}{\Gamma(\frac{i\beta \omega}{2\pi} - \Delta_f + 1)}$$

$$G^R_b(\omega) \sim \frac{\Gamma(\frac{i\beta \omega}{2\pi} + \Delta_b)}{\Gamma(\frac{i\beta \omega}{2\pi} - \Delta_b + 1)}$$

(5.111)

Plugging this in (5.110), and in the small $\Delta_f$ limit

$$\tau^f_\phi \sim \frac{\beta}{\Delta_f}$$

$$\tau^b_\phi \sim \frac{\beta}{\Delta_f}$$

(5.112)

Indeed, the dephasing time becomes larger in the free fermion limit, however, we do not see a clear relation between the decoherence time and the Lyapunov exponent except for the $\beta$ dependence.

### 5.4 Discussion

In this chapter, we defined an SYK-like model with $N$ fermions and $M$ bosons. By tuning the number ratio $\alpha = \frac{M}{N}$, we get a continuous set of conformal theories in the low temperature limit and the scaling dimension depends on the ratio $\alpha$. The free fermion limit can be approached in the small $\alpha$ limit.

By computing the four-point function from the $h = 2$ mode and $h \neq 2$ modes, we obtained the out of time ordered correlators and thus exacted out the Lyapunov exponent. The correction to the maximal Lyapunov exponent became more significant in the free fermion limit. Physically this means a nearly free system is less chaotic.

The relation to the dephasing time is less clear, although we have computed a dephasing time proportional to $\beta$ and becomes larger in the free fermion limit, the coefficient still can not match the Lyapunov exponent. Actually, there is no unique definition of dephasing time: there is a different definition of dephasing time from (13.67) in Ref. [170] in the computation of $O(N)$ model, this results in a relaxation rate with a $\frac{1}{N}$ prefactor. In Ref. [35], the Lyapunov exponent is also computed with a $\frac{1}{N}$ prefactor. However, it is argued that there should be
O(1) piece in the Lyapunov exponent at low temperature when considering Green’s function with anomalous dimension\(^3\). There are also other definitions of dephasing time using dc conductivity or optical conductivity\([36, 189]\), and one obtain the dephasing rate to scale as \(T\) with \(O(1)\) prefactor or \(T^\alpha\) for \(\alpha < 1\), in different models or different regimes. So maybe Lyapunov exponent is a better scale to characterize the non-fermi liquid nature of phases.

There is another definition of decoherence in Ref.\([118]\), where Kitaev defined a decoherence factor in a general system which characterizes the difference between the normal OTOC and the commutator OTOC. There the decoherence factor is related to the correction in the Lyapunov exponent and can be expressed in terms of the retarded ladder kernel \(k_R\), which is quite like what we have obtained here. The \(k'_R\) is interpreted as the branching time which characterizes the branching process in the operator evolution under the Hamiltonian. But it is the branching for ladders while the traditional definition of dephasing time is more like the branching time for single particle (self-energy). We are not sure whether these two physical processes can be related.

\(^3\)Private communication with Aavishkar A. Patel
fractionalized phases of a generalized SYK model

6.1 Introduction

One of the main outstanding puzzles in the study of the cuprate superconductors is the nature of the transformation in the electronic state near optimal doping. There are numerous experimental indications that the underlying electronic state changes from a Mott-like state with a small density of carriers at low doping, to a Fermi liquid-like state with a large density of carriers at high doping. The most recent indication of this transformation is in the doping dependence of Hall coefficient [8]. It is also becoming clear that this phenomenology cannot be described solely in terms of a conventional symmetry-breaking phase transition in the Landau framework: despite much experimental effort, no suitable order parameter with sufficient strength has been found near optimal doping. Furthermore such order parameters are also sensitive to quenched disorder, while the cuprate transition appears quite robust to varying degrees of disorder e.g. the transformation in the electronic state is seen in STM experiments in both the ‘2212’ and ‘2201’ series of compounds [63, 97].
Chapter 6. $\mathbb{Z}_2$ fractionalized phases of a generalized SYK model

The most promising route therefore appears to lie in investigating non-Landau transitions which have a ‘topological’ character. Moreover, we need to understand such transitions in the presence of finite density fermionic matter, and also with quenched randomness. There are no known theories of quantum phase transitions under such conditions. Solvable examples in simple limits would clearly be valuable.

In this chapter we propose a solvable $t$-$J$ model of electrons which exhibits a phase transition under such conditions. Both phases of our model are deconfined, possessing gapless fermionic excitations, $f$, which carry $\mathbb{Z}_2$ gauge charges. Our model also possesses a bosonic “Higgs” field $\phi$, carrying $\mathbb{Z}_2$ gauge charges, and the electron is a composite of $f$ and $\phi$. The Higgs field is gapped in one of the phases, and so is the electron: this phase can be considered as a toy model for the underdoped cuprates. The other phase has power-law correlations of the Higgs field: so it is not quite a Higgs/confining phase of the $\mathbb{Z}_2$ gauge theory, but a novel ‘quasi-Higgs’ phase with slowly decaying correlations of the Higgs field. The electron operator in this quasi-Higgs phase has a leading $1/\tau$ decay in imaginary time, as in a Fermi liquid. We propose this phase as a toy model for the overdoped cuprates.

Our model is a 0+1 dimensional quantum theory, in the class of the Sachdev-Ye-Kitaev (SYK) models [116, 173]. Although these models do not have any spatial structure, they exhibit a ‘local criticality’ which is interesting for a number of physical questions:

- The SYK models are the simplest solvable models without quasiparticle excitations. So they can be used as fully quantum building blocks for theories of strange metals like in Chapter 3 and in [36, 81, 83, 151, 156, 179].

- The SYK models exhibit many-body chaos [116, 135], and saturate the lower bound on the Lyapunov time of large-$N$ model to reach chaos [136]. So they are “the most chaotic” quantum many-body systems. The presence of maximal chaos is linked to the absence of quasiparticle excitations, and the proposed [170] lower bound of order $\hbar/(k_B T)$ on a ‘dephasing time’.
• Related to their chaos, the SYK models exhibit [88, 149, 180] eigenstate thermalization (ETH) [50, 181], and yet many aspects are exactly solvable.

• The SYK models are dual to gravitational theories in $1 + 1$ dimensions which have a black hole horizon. The connection between the SYK models and black holes with a near-horizon AdS$_2$ geometry was proposed in Refs. [168, 169], and made much sharper in Refs. [116, 119, 137]. It has been used to examine aspects of the black hole information problem [138].

### 6.2 Model

#### 6.2.1 The SYK model from a $t$-$J$ model

We model the underdoped state of the cuprate superconductors as a deconfined phase of a $\mathbb{Z}_2$ gauge theory [172]. The case which we have found to be most amenable to a SYK-like description is to represent the deconfined phase as an ‘orthogonal metal’ [146, 167]. In this description, the electron operator $c_{i\alpha}$ ($i$ is a site index, and $\alpha$ is a spin label) fractionalizes into an ‘orthogonal fermion’, $f_{i\alpha}$, which carries both the spin and charge of the electron, and an Ising variable $\sigma^z_i$:

$$c_{i\alpha} = \sigma^z_i f_{i\alpha}.$$  \hspace{1cm} (6.1)

Note that this decomposition is invariant under the $\mathbb{Z}_2$ gauge transformation

$$\sigma^z_i \rightarrow \eta_i \sigma^z_i, \quad f_{i\alpha} \rightarrow \eta_i f_{i\alpha},$$  \hspace{1cm} (6.2)

where $\eta_i = \pm 1$. We can then set up a $t$-$J$ model for these degrees of freedom, with a Hamiltonian like

$$\mathcal{H}_\sigma = - \sum_{i,j} t_{ij} \sigma^z_i \sigma^z_j f_{i\alpha}^\dagger f_{j\alpha} + \sum_{i>j,\alpha\beta} J_{ij} f_{i\alpha}^\dagger f_{j\beta} f_{j\beta}^\dagger f_{i\alpha} - g \sum_i \sigma^x_i.$$  \hspace{1cm} (6.3)
At large $g$, the value of $\sigma^z$ will rapidly average to zero, and only the $J_{ij}$ term will be active: so we expect a fractionalized orthogonal metal state in which the $\sigma^z$ excitations are gapped, and the orthogonal fermions $f$ are deconfined. In contrast, at small $g$, the $\sigma^z$ can condense and then $\mathbb{Z}_2$ charges are confined: this would be a conventional state in which $c \sim f$. Indeed, a similar transition has appeared in a recent Monte Carlo study on the square lattice at half-filling, between an orthogonal semi-metal and a confining superconductor or a confining antiferromagnet [7, 71, 72]. However, as we noted above, the specific model we shall study here only has a gapless, ‘almost confining’, quasi-Higgs phase.

The model $\mathcal{H}_\sigma$ is not directly amenable to a SYK-like large $N$ limit. However, it does become so when we promote the $\mathbb{Z}_2$ Ising spin to an $O(M')$ quantum rotor [166, 193], $\phi_p$, $p = 1 \ldots M'$ which obeys the constraint

$$\sum_{p=1}^{M'} \phi_{ip}^2 = M'.$$

(6.4)

As in Ref. [166, 193], we expect that this promotion from Ising to large $M'$ rotors does not modify the universal critical properties. To obtain a suitable large $M$ limit, we also promote the spin index $\alpha = 1 \ldots M$ to a $SU(M)$ spin index (as in Ref. [173]). For this purpose, we introduce an orbital index, $p$, and fractionalize the electron as

$$c_{ip\alpha} = \phi_{ip} f_{i\alpha},$$

(6.5)

so that

$$\phi_{ip} \rightarrow \eta_i \phi_{ip}$$

(6.6)

under the $\mathbb{Z}_2$ gauge transformation. Then we obtain the final Lagrangian of the $t$-$J$ model
for our future diagrammatic analysis, we represent the interaction vertices in $\mathcal{L}$ to be solved in this paper:

\[
\mathcal{L} = \frac{1}{2g} \sum_{i,p} (\partial_\tau \phi_{ip})^2 + \sum_{i,\alpha} f_{i\alpha}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) f_{i\alpha} \\
+ \frac{1}{\sqrt{NM}} \sum_{i,j,p,\alpha} t_{ij} \phi_{ip} \phi_{jp} f_{i\alpha}^\dagger f_{j\alpha} + \frac{1}{\sqrt{NM}} \sum_{i \geq j, \alpha, \beta} J_{ij} f_{i\alpha}^\dagger f_{i\beta} f_{j\beta}^\dagger f_{j\alpha},
\]

(6.7)

(6.8)

where the site indices $i, j = 1 \ldots N$. With $t_{ij}$ and $J_{ij}$ independent random numbers with zero mean, we will show that this Lagrangian is solvable in the limit of large number of sites, $N$, followed by the limit of large $M$ and $M'$ at fixed

\[
k \equiv \frac{M'}{M}.
\]

(6.9)

For our future diagrammatic analysis, we represent the interaction vertices in $\mathcal{L}$ in Fig. 6.1.

6.2.2 The large $N$ limit

To take the large $N$ limit, we average over $t_{ij}$ and $J_{ij}$, with $|t_{ij}|^2 = t^2/2$ and $J_{ij}^2 = J^2$. As usual, everything reduces to a single site problem, with the fields carrying replica indices.
However, for simplicity, we drop the replica indices. Then the single-site Lagrangian is

\[
\mathcal{L} = \frac{1}{2g} \sum_p \left( \partial_\tau \phi_p \right)^2 + i\lambda \left( \sum_p \phi_p^2 - M' \right) + \sum_\alpha f^\dagger_\alpha \left( \frac{\partial}{\partial \tau} - \mu \right) f_\alpha
\]

(6.10)

\[- \frac{t^2}{M} \sum_{p,\alpha} \int_0^{1/T} d\tau d\tau' R^*(\tau - \tau')\phi_p(\tau)\phi_p(\tau')f^\dagger_\alpha(\tau)f_\alpha(\tau') \]

(6.11)

\[- \frac{J^2}{2M} \sum_{\alpha,\beta} \int_0^{1/T} d\tau d\tau' Q(\tau - \tau')f^\dagger_\alpha(\tau)f_\beta(\tau)f^\dagger_\beta(\tau')f_\alpha(\tau') , \]

(6.12)

where \( T \) is the temperature and \( \lambda \) is the Lagrange multiplier imposing Eq. (6.4). More precisely, as in Ref. [173], decoupling the large \( N \) path integral introduces fields analogous to \( R \) and \( Q \) which are off-diagonal in the \( \text{SU}(M) \) and \( \text{O}(M') \) indices. We have assumed above that the large \( N \) limit is dominated by the saddle point in which these fields are \( \text{SU}(M) \) and \( \text{O}(M') \) diagonal. This requires that the large \( N \) limit is taken before the large \( M \) and \( M' \) limits. This procedure supplements the Lagrangian with the self-consistency conditions

\[
R(\tau - \tau') = -\frac{1}{MM'} \sum_{p,\alpha} \langle \phi_p(\tau)\phi_p(\tau')f^\dagger_\alpha(\tau)f_\alpha(\tau') \rangle
\]

(6.13)

\[
Q(\tau - \tau') = \frac{1}{M^2} \sum_{\alpha,\beta} \langle f^\dagger_\alpha(\tau)f_\beta(\tau)f^\dagger_\beta(\tau')f_\alpha(\tau') \rangle . \]

(6.14)
It is convenient to rescale $\phi_p \to \sqrt{g} \phi_p$ so that the Lagrangian becomes

$$
\mathcal{L} = \frac{1}{2} \sum_p (\partial_\tau \phi_p)^2 + \imath \lambda \left( \sum_p \phi_p^2 - \frac{M'}{g} \right) + \sum_\alpha f^{\dagger}_\alpha \left( \frac{\partial}{\partial \tau} - \mu \right) f_\alpha
$$

(6.15)

$$
- \frac{\tilde{t}^2}{M} \sum_{p,\alpha} \int_0^{1/T} d\tau d\tau' R^*(\tau - \tau') \phi_p(\tau) \phi_p(\tau') f^{\dagger}_\alpha(\tau) f_\alpha(\tau')
$$

(6.16)

$$
- \frac{J^2}{2M} \sum_{\alpha,\beta} \int_0^{1/T} d\tau d\tau' Q(\tau - \tau') f^{\dagger}_\alpha(\tau) f_\beta(\tau) f^{\dagger}_\beta(\tau') f_\alpha(\tau')
$$

(6.17)

where $\tilde{t} = t g$.

Next we take the large $M$ and $M'$ limit at fixed $k = M/M'$. Note that the large $N$ limit has already been taken. By this sequence of limits we obtain for the fermion Green's function, $G$, and the $\phi$ correlator $\chi$

$$
G(\imath \omega_n) = \frac{1}{\imath \omega_n + \mu - \Sigma(\imath \omega_n)} \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau) + k \tilde{t}^2 G(\tau) \chi^2(\tau)
$$

(6.18)

$$
\chi(\imath \omega_n) = \frac{1}{\omega_n^2 + \chi_0^{-1} - P(\imath \omega_n) + P(\imath \omega_n = 0)} \quad P(\tau) = -2 \tilde{t}^2 G(\tau) G(-\tau) \chi(\tau)
$$

(6.19)

where

$$
\imath \lambda = \chi_0^{-1} + P(\imath \omega_n = 0)
$$

(6.20)

is the saddle point value of $i \lambda$. Note that we have introduced notation so that

$$
\chi(\imath \omega_n = 0) \equiv \chi_0
$$

(6.21)

is the static $\phi$ susceptibility. Formally, the value of $\chi_0$ is to be determined by solving the constraint equation Eq. (6.4):

$$
T \sum_{\omega_n} \chi(\imath \omega_n) = \frac{1}{g}
$$

(6.22)

In practice, we will treat the value of $\chi_0$ as a parameter that can be tuned to access all
the regions of the phase diagram, and use Eq. (6.22) to determine the value of $g$. This is convenient because the coupling $g$ does not appear in any of the other saddle-point equations (after our definition of $\tilde{t}$). Finally, the results as a function of $\chi_0$ will be recast as functions of $g$. We note that the large $N$ equations in Eqs. (6.18) and (6.19) can also be derived diagrammatically, as illustrated in Fig. 6.2.

Coupled equations of Green's functions of bosons and fermions have been considered previously in a supersymmetric model as in Chapter 4, but the present equations have a different structure. The supersymmetric model has a single boson field coupling to fermion composites, while $\mathbb{Z}_2$ gauge invariance of our model requires that pairs of bosons couple to fermions.

6.2.3 Gapless solutions

First, we search for solutions of Eqs. (6.18), (6.19), and (6.22) in which both the fermions and the bosons are gapless. In our initial analysis, we will work on the imaginary frequency
axis at $T = 0$. The extension to $T > 0$ appears in Section 6.2.4.

For the gapless solutions, we make the ansatzes valid as $\tau \to \infty$ at $T = 0$

$$G(\tau) = -\text{sgn}(\tau) \frac{F}{(J|\tau|)^{2\Delta_f}}$$

$$\chi(\tau) = \frac{C/J}{(J|\tau|)^{2\Delta_b}},$$

where $F > 0$ and $C > 0$, and they are both dimensionless. The Fourier transforms at $T = 0$ are

$$G(i\omega) = -2i \text{sgn}(\omega) \frac{F/J^{2\Delta_f}}{\omega^{1-2\Delta_f}} \cos(\pi \Delta_f) \Gamma(1 - 2\Delta_f)$$

$$\chi(i\omega) = 2 \frac{C/J^{2\Delta_b+1}}{|\omega|^{1-2\Delta_b}} \sin(\pi \Delta_b) \Gamma(1 - 2\Delta_b).$$

From Eq. (6.18) and (6.19), the self energies are

$$\Sigma(\tau) = -\text{sgn}(\tau) \left( \frac{J^2 F^3}{(J|\tau|)^{6\Delta_f}} + \frac{k t^2 F C^2 / J^2}{(J|\tau|)^{2\Delta_f+4\Delta_b}} \right)$$

$$P(\tau) = \frac{2 \tilde t^2 F^2 C / J}{(J|\tau|)^{4\Delta_f+2\Delta_b}},$$

and their Fourier transforms are

$$\Sigma(i\omega) = -2i \text{sgn}(\omega) \left( \frac{J^{2-6\Delta_f} F^3}{|\omega|^{1-6\Delta_f}} \cos(3\pi \Delta_f) \Gamma(1 - 6\Delta_f)$$

$$+ \frac{k t J^2 J^{2-2\Delta_f-4\Delta_b} F C^2}{|\omega|^{1-2\Delta_f-4\Delta_b}} \cos(\pi(\Delta_f + 2\Delta_b)) \Gamma(1 - 2\Delta_f - 4\Delta_b) \right)$$

$$P(i\omega) = 4 \frac{(t/J^2)^2 J^{3-4\Delta_f-2\Delta_b} F^2 C}{|\omega|^{1-4\Delta_f-2\Delta_b}} \sin(\pi(2\Delta_f + \Delta_b)) \Gamma(1 - 4\Delta_f - 2\Delta_b).$$
From Eqns (6.26) and (6.31), and using \(G(i\omega)\Sigma(i\omega) = -1\) and \(\chi(i\omega)P(i\omega) = -1\) in the limit of low \(\omega\), we see that solutions are only possible when

\[
\Delta_f + \Delta_b = 1/2. 
\] (6.32)

Further examination of the saddle point equations shows that two classes of solutions are possible, depending upon whether \(\Delta_f > 1/4\) or \(\Delta_f = 1/4\). We will examine these solutions in the following subsections.

\(\Delta_f > 1/4\)

In this case, the first term in \(\Sigma(i\omega)\) in Eq. (6.31) is subdominant and can be ignored. Then the Schwinger-Dyson equations simplify

\[
k(\bar{t}/J^2)^2F^2C^2 2\pi \cot(\pi\Delta_f) = 1
\] (6.33)

\[
(\bar{t}/J^2)^2F^2C^2 \frac{2\pi \cot(\pi\Delta_f)}{\Delta_f} = 1
\] (6.34)

These equations are consistent only if we choose the scaling dimensions

\[
\Delta_f = \frac{1}{k + 2} \quad \Delta_b = \frac{k}{2(k + 2)}.
\] (6.35)

Note that \(\Delta_f > 1/4\) requires \(k < 2\). So the exponents are limited to the ranges

\[
\frac{1}{4} < \Delta_f < \frac{1}{2}, \quad 0 < \Delta_b < \frac{1}{4}.
\] (6.36)

The above analysis of the low \(\omega\) limit of the saddle point equations does not determine the values of \(F\) and \(C\) separately, only the value of their product \(CF\). So we expect that the \(\Delta_f > 1/4\) solution defines a phase which extends over a range of value of \(g\).
Now both terms in $\Sigma$ in Eq. (6.31) have the same frequency dependence, and so both contribute to the low $\omega$ limit. The Schwinger-Dyson equations now become

$$F^4 + k(\tilde{t}/J^2)^2C^2F^2 = \frac{1}{4\pi} \quad (6.37)$$

$$\tilde{t}/J^2C^2F^2 = \frac{1}{8\pi} \quad (6.38)$$

These can be solved uniquely for both $F > 0$ and $C > 0$ provided again $k < 2$. The existence of unique low $\omega$ solution with these exponents indicates that Eq. (6.22) will yield only a particular value of $g$. We will find that is the case in our numerics, and this solution appears to describe a critical point between our $\Delta_f > 1/4$ gapless and gapped phases.

### 6.2.4 Non-Zero Temperatures

It turns out that a $T > 0$ conformal extension of the above gapless solutions satisfies the saddle point equations in Eqs. (6.18) and (6.19) at $T > 0$, just as was noted in Refs. [151, 169]. From Eq. (6.24), the conformal extension is

$$G(\tau) = -\text{sgn}(\tau) \frac{F}{J^{2\Delta_f}} \left( \frac{\pi T}{\sin(\pi T \tau)} \right)^{2\Delta_f}$$

$$\chi(\tau) = \frac{C}{J^{2\Delta_b+1}} \left( \frac{\pi T}{\sin(\pi T \tau)} \right)^{2\Delta_b} \quad (6.39)$$

But, we also have to verify that the Eq. (6.22) yields the same value of $g$ as at $T = 0$. The frequency summation in Eq. (6.22) is dominated by high energies, and we don’t expect significant change in the spectral weight at such frequencies at a small $T > 0$. So we need only examine the low frequencies in Eq. (6.22), in which case we can use the conformal solution. To focus on low frequencies, we subtract Eq. (6.22) between its $T = 0$ and $T > 0$ values, and
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regulate the higher frequencies by inserting a point-splitting $\tau$. Then the requirement that the value of $g$ is the same at $T = 0$ and in the conformal solution is

$$\lim_{\tau \to 0} [\chi(\tau, T) - \chi(\tau, T = 0)] = 0 \quad (6.41)$$

It is now easy to verify that Eq. (6.40) does indeed satisfy Eq. (6.41).

Taking the Fourier transform of Eq. (6.40), we have the low $\omega$ gapless solution as a function of $\omega$ and $T$

$$G(i\omega_n) = \left[-i \frac{F\Pi(2\Delta_f)}{J^{2\Delta_f}}\right] \frac{T^{2\Delta_f-1} \Gamma(\Delta_f + \frac{\omega_n}{2\pi T})}{\Gamma \left(1 - \Delta_f + \frac{\omega_n}{2\pi T}\right)}$$

$$\chi(i\omega_n) = \left[C\Pi(2\Delta_b)\right] \frac{T^{2\Delta_b-1} \Gamma(\Delta_b + \frac{\omega_n}{2\pi T})}{\Gamma \left(1 - \Delta_b + \frac{\omega_n}{2\pi T}\right)} \quad (6.42)$$

where

$$\Pi(s) \equiv \pi^{s-1} 2^s \cos \left(\frac{\pi^s}{2}\right) \Gamma(1 - s) \quad , \quad \Pi(s) \equiv \pi^{s-1} 2^s \sin \left(\frac{\pi^s}{2}\right) \Gamma(1 - s) \quad (6.43)$$

From Eq. (6.21), we therefore obtain the $T$-dependence of the static susceptibility

$$\chi_0 = \left[C\Pi(2\Delta_b)\Gamma(\Delta_b)\right] \frac{T^{2\Delta_b-1}}{J^{2\Delta_b+1}\Gamma(1 - \Delta_b)} \quad (6.44)$$

The susceptibility must have this $T$ dependence at low $T$ to keep $g$ fixed while $T$ varies.

6.2.5 Gapped boson solution

Now we search for a possible solution of Eqs. (6.18) and (6.19) with a gap in the boson spectrum at $T = 0$. With an energy gap, $m$, from Eq. (E.10) we can conclude that the boson
Green’s function decays exponentially at long times. So we write

\[ \chi(\tau) = \frac{B}{(J|\tau|)} e^{-m|\tau|}, \quad |\tau| \gg 1/m, \quad T = 0, \tag{6.45} \]

parameterized by the gap \( m \), the exponent \( \gamma \) and the dimensionless prefactor \( B \). From the spectral analysis in Appendix E.1 we conclude that the boson Green’s function \( \chi(z) \) has branch cuts in the complex frequency plane at \( z = \pm m \). At \( z = m \), the singular (non-analytic) part of \( \chi(z) \) is

\[ \chi_{\text{sing}}(z) = \frac{\pi B}{J^2 \Gamma(\gamma)} \frac{iJ^{1-\gamma}}{\Gamma(\gamma)(z - m)^{1-\gamma}} , \quad z \sim m \tag{6.46} \]

With a gap in the boson spectrum, Eq. (6.18) imply that we can ignore the boson correlator in the determination of the fermion spectrum at small \( \omega \). Indeed, the fermionic component of the equations are the same as those in Ref. [173], and so we have the same gapless solution \( i.e. \)

\[ G(\tau) = -\text{sgn}(\tau) \frac{A}{J^{1/2}} \left( \frac{\pi T}{\sin(\pi T\tau)} \right)^{1/2} , \quad A = \frac{1}{(4\pi)^{1/4}} \tag{6.47} \]

From Eq. (6.19) we can then obtain the long time behavior of the boson self energy

\[ P(\tau) = \frac{2A^2 B^2 J}{J} \frac{1}{(J|\tau|)^{1+\gamma}} e^{-m|\tau|} , \quad |\tau| \gg 1/m, \quad T = 0. \tag{6.48} \]

From the analysis in Appendix E.1, as for Eq. (6.46), we conclude that \( P(z) \) has a branch cut in the complex frequency plane at \( z = \pm m \) with singular part

\[ P_{\text{sing}}(z) = \frac{2\pi A^2 B^2 J}{J^2 \Gamma(1+\gamma)} \frac{(z - m)^\gamma}{J^\gamma} , \quad z \sim m \tag{6.49} \]

Comparing Eqs. (6.46) and (6.49) with the Dyson equation in Eq. (6.19), it is not difficult
to see that a consistent solution is only possible if
\[ -m^2 + \chi_0^{-1} - P(z = m) + P(z = 0) = 0 \] (6.50)
and the exponent
\[ \gamma = \frac{1}{2}. \] (6.51)

The dimensionless pre-factor \(B\) is also determined to be
\[ \left(\frac{i}{J^2}\right)^2 A^2 B^2 = \frac{1}{4\pi}. \] (6.52)

## 6.3 Results

### 6.3.1 Composite Operators

\(\lambda\) Operator

Now we consider the structure of fluctuations about the saddle point solutions described in the previous sections. First, we focus only on the fluctuations of the Lagrange multiplier field \(\lambda_i\) about the saddle point value in Eq. (6.20). This field represents the \(\phi^2\) operator [161], and so its scaling properties are important in determining the manner in which the gap in the \(\phi\) spectrum opens up [161], as we will discuss in Section 6.3.2.

We write
\[ i\lambda_i = \chi_0^{-1} + P(i\omega_n = 0) + i\overline{\lambda}_i, \] (6.53)
and then determine the effective action for \(\overline{\lambda}_i\) fluctuations to leading order in large \(N\) and large \(M\), after integrating out the \(f\) and the \(\phi\) fields. The diagrams that contribute to this effective action are quite complicated and readers may refer to the original paper for more
details. In summary, the diagrams lead to an action of the form

$$S_\lambda = \frac{T M'}{2} \sum_{\omega_n, i,j} \bar{\lambda}_i(\omega_n) \left( \Pi^i_0(\omega_n) + \Pi^i_1(\omega_n) \right) \bar{\lambda}_j(-\omega_n).$$ \hfill (6.54)

Where we denote the bubble diagrams by $\Pi^i_0$, which is diagonal in site index (i.e. $\Pi^i_0 = \Pi_0 \delta_{ij}$) and yields (in time domain):

$$\Pi_0(\tau) = i \cdots \cdots \cdots \cdots i = [\chi(\tau)]^2,$$ \hfill (6.55)

where $\chi(\tau)$ is given by Eq. (6.24). We use $\Pi^i_1$ to represent ladder diagrams with external indices $ij$. In general, we expect the matrix $\Pi^i_1$ has permutation symmetry of the indices, which constrains the form of $\Pi^i_1$ to be a matrix with identical diagonal elements and identical off-diagonal elements, i.e. there are only two free parameters. Such matrix admits one eigenvector that is uniform in site index with eigenvalue ($\Pi^i_1 + (N - 1)\Pi^i_1$) and $(N - 1)$ non-uniform eigenvectors with eigenvalue ($\Pi^i_1 - \Pi^i_1$). We are interested in the site-uniform mode, whose eigenvalue for the whole kernel including the bubble term can be written in the following symmetric way:

$$\Pi^i_0 + \Pi^i_1 + (N - 1)\Pi^i_1 = \Pi_0 + \frac{1}{N} \sum_{ij} \Pi^i_1$$ \hfill (6.56)

We denote the second term by $\Pi_1 := \frac{1}{N} \sum_{ij} \Pi^i_1$, which requires evaluation of multiple infinite series of diagrams; they yield the result:

$$\Pi_1(\tau) \approx \frac{1}{2} \left( 1 - \frac{2\pi}{(\log A|\tau|)^2} \right) \Pi_0(\tau).$$ \hfill (6.57)

which is proportional to $\Pi_0(\tau)$ with a small $(\log A|\tau|)^{-2}$ correction. Therefore we have the
correlator for the site-uniform $\bar{\lambda}$ fluctuation

$$\langle \bar{\lambda}(\omega_n)\bar{\lambda}(-\omega_n) \rangle = \frac{1}{M'(\Pi_0(\omega_n) + \Pi_1(\omega_n))}. \quad (6.58)$$

Limiting ourselves to the $\Delta_b = 1/4$ critical state, to leading log accuracy at low frequency, this propagator is dominated by the Fourier transform of $\Pi_0(\tau) \sim 1/|\tau|$, which yields

$$\langle \bar{\lambda}(\tau)\bar{\lambda}(0) \rangle \sim \int d\omega \frac{e^{i\omega\tau}}{\ln(\Lambda/|\omega|)} \sim \frac{1}{|\tau|^{1+\epsilon}}, \quad (6.59)$$

with $\epsilon = 1/(\ln(\Lambda|\tau|))$. So we can write the scaling dimension $[\bar{\lambda}] = (1 + \epsilon)/2$, with $\epsilon$ representing logarithmic corrections to scaling.

**Electron operator**

From the definition of the electron operator in Eq. (6.5), we have to leading order in $1/N$ for the electron Green’s function, $G_e$,

$$G_e(\tau) = G(\tau)\chi(\tau)$$

$$= -\text{sgn}(\tau)\frac{FC}{J^2|\tau|} \quad (6.60)$$

$$= -\text{sgn}(\tau)\frac{FC}{J^2|\tau|} \quad (6.61)$$

where we have used Eq. (6.24) and the exponent relation in Eq. (6.32). Note that Eq. (6.32), and hence Eq. (6.61), hold for the both the gapless solutions in Sections 6.2.3 and 6.2.3. As was the case for the $\lambda$ fluctuations discussed above, additional contributions to Eq. (6.61) from ladder diagrams only yield off-site terms which are suppressed by $1/N$. So Eq. (6.61) is exact to leading order in the large $N$ limit of this paper.

It is remarkable that $G_e(\tau)$ has the same form as that in a Fermi liquid state. This can be seen to be a consequence of the relevance of the hopping, $t$, which moves single electrons between sites. However, it is important to note that despite the Fermi liquid
form in Eq. (6.61), the states under considerations are not Fermi liquids: their elementary excitations are the fractionalized $f$ and $\phi$ excitations, which carry anomalous exponents.

### 6.3.2 Numerical Results

We now present numerical tests of the solutions of Eqs. (6.18) and (6.19). These go beyond the low frequency analytical analyses of Sections 6.2.3 and 6.2.5, and include all frequencies. There are no ultraviolet divergencies, and so the solutions depend only upon the parameters in the Lagrangian.

Our numerical strategy was to pick at first the values of the parameters $\tilde{t}$ and $J$, and then make a choice for the boson susceptibility at zero frequency, $\chi_0$. Then we iterate Eqs. (6.18) and (6.19) until the solution converges. Finally, we insert the solution in Eq. (6.22) and determine the value of $g$. So we determine $g$ as a function of $\chi_0$, rather than the other way around.

![Figure 6.3](image.png)

Figure 6.3: The numerical results for $G(\tau)$ and $\chi(\tau)$ at the gapless phase are shown in solid lines for $J = 1$, $\tilde{t} = 1$, $k = 1$, $T = 0.005$ and $g \approx 0.8077$ (the input $\chi_0 \approx 53.6$). The gapless conformal answers for $G$ and $\chi$ are plotted in dashed lines with the values from Eq. (6.35), $\Delta_f = 1/3$, $\Delta_b = 1/6$. The prefactor $C$ is determined by (6.44) with $\Delta_b = 1/6$, then $F$ is determined by (6.34).

First, we examined the gapless solutions, with the input $\chi_0$, the conformal solution prefactor is determined by (6.44). A solution with $\Delta_f > 1/4$ is shown in Fig. 6.3. In this case, at any finite temperature, although the prefactor equations (6.34) from the saddle point
equations do not determine the prefactors $F$ and $C$ separately, but the matching condition (6.44) determines them. We can think about it this way: different $\chi_0$ results from different $g$ and it determines different $F$ and $C$. So such a gapless solution can be obtained for a range of values of $g$ at a fixed $\Delta_f$. And at zero temperature, when $\chi_0$ diverges, we cannot determine $F$ and $C$ separately. Thus this gapless solution defines a critical phase.

Next, we examined the gapless solution with $\Delta_f = 1/4$ in Fig. 6.4. In this case, the saddle point equations determine $F$ and $C$ separately in the prefactor equations Eq. (6.38). For each value of $\tilde{t}$, $J$, $k$ and $T$, the critical susceptibility is determined by (6.44), thus it determines an unique $g_c$.

We also examined the $T$ dependence of $\chi_0^{-1}$ predicted by Eq. (6.44). We choose different values of $T$ with other parameters fixed, and found a $T$-independent value of $g$. This confirms the analysis in Section 6.2.4 on extending the $T = 0$ gapless solution to nonzero $T$.

Finally, we examined the gapped boson solution of Section 6.2.5 in Fig. 6.5. The normalization constant for the fermion conformal answer is defined in Eq. (6.47). Again we find good agreement between the numerical solution and our analytic form. We also computed the high temperature expansion in Appendix. E.2 and find qualitative agreements when
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Figure 6.5: The numerical results for $G(\tau)$ and $\chi(\tau)$ at the gapped boson phase are shown in solid lines for $J = 1, \tilde{t} = 1, k = 1, T = 0.005$ and $g \approx 1.149$ (The input $\chi_0 \approx 6.6$). The critical conformal answers for $G$ from (6.47) is plotted in dashed line.

computing $g$.

Figure 6.6: The numerical results for $g$ as function of $\chi_0^{-1}$ for various values of the parameter $k$ at $J = 1, \tilde{t} = 1$ and $T = 0.005$. The critical $g_c$ is shown in dashed line. For $k < k_c \approx 0.9$ the critical point is located inside an unphysical domain, which means that the critical phase is absent and there is first-order phase transition.

Now we turn to determining how the various solutions fit together in a phase diagram as a function of $k$ and $g$. We set $\tilde{t} = J = 1$ in this analysis. As our independent parameter is $\chi_0$, and not $g$, we show in Fig. 6.6 the value of $g$ determined from Eq. (6.22) as a function of $\chi_0^{-1}$ for various values of $k$. The values of $\chi_0^{-1}$ corresponding to Eq. (6.44) at $\Delta_b = 1/4$ yield
the value of $g_c$ for each $k$: this is plotted as the dashed line. The most notable feature of Fig. 6.6 is the non-monotonic dependence of $g$ on $\chi_0^{-1}$ for certain $k$. This implies that for a given $g$ there are multiple solutions of the saddle point equations in Eqs. (6.18), (6.19), and (6.22) corresponding to the different solutions for the value of $\chi_0$. To distinguish between the solutions, we have to evaluate the free energy of each solution and pick the one with the lowest free energy. We have not carried out this evaluation, and so are unable to determine the precise location of the transition between the gapless and gapped solutions. In any case, we can conclude that there is a first-order transition from the gapless to the gapped solution when $g$ is a decreasing function of $\chi_0^{-1}$ near $g_c$.

On the basis of the above analysis and Fig. 6.6, we assemble the schematic phase diagram in Fig. 6.7. The gapless phase with $\Delta_f > 1/4$ is separated from the gapped boson phase by either a first-order or a second-order phase transition. For the latter case, the critical state is described by the $\Delta_f = 1/4$ solution described in Section 6.2.3.

![Schematic phase diagram](image)

Figure 6.7: Schematic phase diagram. The fermions are gapless in all phases. The thick line indicates a first-order transition, while the thin line is a $\Delta_b = \Delta_f = 1/4$ critical state.
6.3.3 Small gap scaling

We now examine the nature of the scaling properties of the gapped side of the second-order transition in Fig. 6.7. On general grounds, we introduce the exponent $z$ by assuming that the boson energy gap, $m$, vanishes as

$$m \sim (g - g_c)^z, \quad T = 0,$$  \hspace{1cm} (6.62)

As the energy gap appears from a $\sim \phi^2$ perturbation of the critical theory, we expect that the scaling dimension of $\phi^2$ is related to $z$ via $[\phi^2] = 1 - 1/z$. On the other hand, as explained in the context of the Wilson-Fisher CFT in Ref. [161], in the large $M$ expansion $[\lambda] = [\phi^2]$ and so

$$[\lambda] = 1 - \frac{1}{z}$$  \hspace{1cm} (6.63)

We examined the scaling dimension of $\lambda$ in Section 6.3.1, and found that $[\lambda] = (1 + \epsilon)/2$, with $\epsilon$ representing logarithmic corrections to scaling. So $z = 2(1 + \epsilon)$.

From our numerical solutions, it turned to be difficult to obtain accurate values of the boson gap, $m$, to test the above scaling predictions. So we employed an alternative method, which examined the full functional form of the boson susceptibility $\chi(\tau)$. From the structure of the gapped solution in Eq. (6.45) we can expect a scaling solution for the $T = 0$ susceptibility of the form

$$\chi(\tau) = \left(\frac{m}{f^3}\right)^{1/2} \Phi_1(m \tau)$$  \hspace{1cm} (6.64)

for some scaling function $\Phi_1$. Clearly, Eq. (6.64) is compatible with the long-time limit in Eq. (6.45). Then integrating Eq. (6.64) over $\tau$, we obtain the divergence of the static susceptibility as the gap, $m$, vanishes

$$\chi_0 \sim m^{-1/2}, \quad T = 0.$$  \hspace{1cm} (6.65)
For a second-order transition to a gapless phase, with the critical point described by the $\Delta_b = 1/4$ solution in Section 6.2.3, Eq. (6.44) implies that the static susceptibility behaves as

$$\chi_0 \sim T^{-1/2}, \quad m \to 0.$$  

(6.66)

Combining Eqs. (6.65) and (6.66), we propose the scaling form

$$\chi_0 = T^{-1/2} \Phi_2(m/T).$$  

(6.67)

In our numerical solution, Eq. (6.67) is difficult to test directly because we treat $\chi_0$ as an independent parameter and compute $m$ and $g$, and also $m$ is only defined at $T = 0$. As we can also measure the deviation from criticality by $\chi_0^{-1}$, we can combine the scaling in Eqs. (6.62) and (6.67) to write

$$g - g_c = T^{1/2} \Phi_3(\chi_0 T^{1/2}), \quad g > g_c,$$  

(6.68)

where $\Phi_3$ is another scaling function. Eq. (6.68) is now expressed in a form which is adapted to our numerical approach: we pick the values of $\chi_0$ and $T$, and compute $g$. Also, we can compute the value $g_c$ by requiring that Eq. (6.68) be compatible with Eq. (6.44) i.e.

$$\Phi_3(x) = 0 \text{ at } x = \frac{C\Pi(1/2)\Gamma(1/4)}{J^{3/2}\Gamma(3/4)}$$  

(6.69)

We show tests of the scaling in Eq. (6.68) in Figs. 6.8 and 6.9. We find that scaling as a function of $\chi_0 T^{1/2}$ is extremely well obeyed, confirming that the critical state is described by $\Delta_f = \Delta_b = 1/4$. Specifically, we verified that at $g = g_c$, the right hand side of Eq. (6.68) was $T$ independent as $T$ was varied while keeping $\chi_0 T^{1/2}$ fixed.

On the other hand, scaling with $(g - g_c)/T^{1/2}$ yields variable values of $z$ depending upon

\[\text{Footnote 1: We also examined the first order transition region and did not find scaling behavior, as expected}\]
Figure 6.8: Tests of the scaling in Eq. (6.68): numerical plots for \((g - g_c)/T^{1/z}\) as a function of \(\chi_0 T^{1/2}\) at \(J = 1, \hat{t} = 1\) and \(k = 1, 1.5\) and \(k = 1.7\). We see that at \(z = 0.75, z = 3.1\) and \(z = 5.6\) all lines almost exactly overlap, as expected in the critical region.

The value of \(k\), and of the window of parameters used for the scaling plots, as is apparent from Fig. 6.8. We generally obtained values of \(z > 2\), except at values of \(k\) near the onset of the first order transition. Fig. 6.9 shows that scaling with \(z = 2\) yields reasonable data collapse, with deviations which appear to be within the range of logarithmic corrections described in Section 6.3.1. However, data collapse could be improved with larger values of \(z\) especially by focusing on values of \(g\) very close to \(g_c\): it does not appear these large and variable values of \(z\) are meaningful. More precise tests of the nature of the phase transitions requires a detailed knowledge of the structure of the logarithmic corrections, which we have not computed here.
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1.304 1.306 1.308 1.31 1.312 1.314 1.316 1.318 1.32 1.322
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
1.38 1.4 1.42 1.44 1.46 1.48 1.5 1.52 1.54 1.56 1.58
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
1.66 1.68 1.7 1.72 1.74 1.76 1.78 1.8
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Figure 6.9: The same plots as in Fig. 6.8, but now scaled with $z = 2$.

6.4 Discussion

This chapter has presented an exactly solvable model of fractionalization in metallic states in the presence of disorder and interactions. We considered a $t$-$J$ model in which the electron, $c$, fractionalizes into a fermion $f$ and a boson $\phi$ both carrying $\mathbb{Z}_2$ gauge charges. As the fermions $f$ carries both the global U(1) charge and SU(2) spin of the electron, these fractionalized phases can be considered as realizations of the ‘orthogonal metal’ of Ref. [146].

The phase diagram of our model is schematically presented in Fig. 6.7. There are two extended phases, separated either by a first order transition, or a critical line.

In one phase, the boson $\phi$ is gapped, while the fermion $f$ is gapless. This implies that the electron $c \sim f\phi$ also has a gapped spectral function. On the other hand, thermodynamic properties are largely controlled by the gapless fermions. We propose this gapped boson phase as a toy model for the pseudogap regime of the cuprates.

In the other phase, and also on the critical line, both the fermions and bosons are gapless,
and decay with time as $|\tau|^{-2\Delta_{f,b}}$, where the values of the exponents are specified in Fig. 6.7. In a Higgs phase, in which the $\mathbb{Z}_2$ charges are confined, the boson correlator would decay to a non-zero constant. As the boson decay here is a power-law in time, we labeled this phase as a quasi-Higgs phase.

One of the most interesting properties of the quasi-Higgs phase follows from the exponent identity in Eq. (6.32). The Green’s function of the electron operator, $c$, decays with time as $1/\tau$ (Eq. (6.61)), which is the form of the local Green’s function in a Fermi liquid. This result is a consequence of the relevance of the hopping term, $t$, in the Hamiltonian which transfers single electrons between sites. Unlike previous SYK models, the present model balances the hopping ($t$) and interaction ($J$) terms against each other, rather than one of them dominating; this leads to the Fermi liquid form of the one-electron Green’s function. However, despite this form, most other properties are not Fermi liquid-like e.g. the spin susceptibility is dominated by the response of the $f$ fermions which have an anomalous scaling dimension $\Delta_f$. These intriguing properties are suggestive of the overdoped regime of the cuprates, where there are indications of an extended non-Fermi liquid regime, although photoemission indicates a well-formed large Fermi surface [23, 38, 134, 160].

Extending our toy model to a more realistic model of the cuprates requires introducing spatial structure and examining transport properties. A number of methods of doing so have been introducing recently, like in Chapter 3 and in [36, 81, 83, 156, 179] for the SYK model, and it would be interesting to apply these, or others, to models similar to the one presented here.
Appendices
Appendix for chapter 2

A.1 SYK MODEL FOR BOSONS

Now we consider a ‘cousin’ of the present model: SYK model for hardcore bosons. The
bosonic case was also considered in the early work [73, 74, 151, 173] but with a large number
of bosons on each site. It was found that over most of the parameter regime the ground
state had spin glass order. We will find evidence of similar behavior here.

The Hamiltonian will be quite similar as Eq. (2.1), except that because of the Bose statis-
tics now the coefficients obey

\[ J_{ji;kl} = J_{ij;kl}, \quad J_{ij;ik} = J_{ij;kl}, \quad J_{kl;ij} = J_{ij;kl} \]  (A.1)

Hardcore boson satisfies \([b_i, b_j] = 0\) for \(i \neq j\) and \(\{b_i, b_i^\dagger\} = 1\). Also to make particle-hole
symmetry (2.39) hold, we only consider pair hoping between different sites, i.e. site indices
\(i, j, k, l\) are all different, and we drop the normal order correction terms. The spin formalism
We can define a similar Green’s function for bosons:

\[ G_B(t) = -i\theta(t)\langle \{ b(t), b^\dagger(0) \} \rangle \]  

We identify the infinite time limit of \( G_B \) as the Edward-Anderson order parameter \( q_{EA} \), which can characterize long-time memory of spin-glass:

\[ q_{EA} = \lim_{t \to \infty} G_B(t) \]  

Then \( q_{EA} \neq 0 \) indicates that \( G_B(\omega) \sim \delta(\omega) \). This is quite different from the fermionic case, where we have \( G_F(z) \sim 1/\sqrt{z} \); this inverse square-root behavior also holds in the bosonic case without spin glass order \([173]\). Fig. A.1 is our result from ED, with a comparison between \( G_B \) with \( G_F \). It is evident that the behavior of \( G_B \) is qualitatively different from \( G_F \), and so an inverse square-root behavior is ruled out. Instead, we can clearly see that, as system size gets larger, \( G_B \)'s peak value increases much faster than the \( G_F \)'s peak value. This supports the presence of spin glass order.

Unlike the fermionic case, \( P^2 = 1 \) for all \( N \) in the bosonic model. We can apply similar symmetry argument as in Ref. \([196]\): for the half-filled sector (only in even \( N \) cases), the level statistics obeys the Wigner-Dyson distribution of Gaussian orthogonal random matrix ensembles, while in other filling sectors, it obeys distribution of Gaussian unitary random matrix ensembles.

Our thermal entropy results for bosons are similar to the fermionic results: although the entropy eventually approaches 0 at zero temperature, there is still a trend of a larger low temperature entropy residue as the system size gets larger.
Figure A.1: Imaginary part of Green’s function for hardcore boson and fermion model. The peak near the center gets much higher in the boson model when system size gets larger. The inset figure is zoomed in near $\omega = 0$.

We have also computed the entanglement entropy for the ground state of the hardcore boson SYK model. It still satisfies volume law, and the entanglement entropy density is still quite close to $\ln 2$. Finally results for the OTOC are qualitatively similar to the fermionic results.
Appendix for chapter 3

B.1 Saddle point solution of the SYK model

We follow the condensed matter notation for Green’s functions in which

\[ G(\tau) = -\langle T_\tau (f(\tau)f^\dagger(0)) \rangle. \]  

(B.1)

It is useful to make ansatzes for the retarded Green’s functions in the complex frequency plane, because then the constraints from the positivity of the spectral weight are clear. At the Matsubara frequencies, the Green’s function is defined by

\[ G(i\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n \tau} G(\tau). \]  

(B.2)
So the bare Green’s function is

\[ G_0(i\omega_n) = \frac{1}{i\omega_n + \mu}. \]  

(B.3)

The Green’s functions are continued to all complex frequencies \( z \) via the spectral representation

\[ G(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\rho(\Omega)}{z - \Omega}. \]  

(B.4)

For fermions, the spectral density obeys

\[ \rho(\Omega) > 0, \]  

(B.5)

for all real \( \Omega \) and \( T \). The retarded Green’s function is \( G^R(\omega) = G(\omega + i\eta) \) with \( \eta \) a positive infinitesimal, while the advanced Green’s function is \( G^A(\omega) = G(\omega - i\eta) \). It is also useful to tabulate the inverse Fourier transforms at \( T = 0 \)

\[ G(\tau) = \begin{cases} 
- \int_{0}^{\infty} \frac{d\Omega}{\pi} \rho(\Omega)e^{-\Omega\tau}, & \text{for } \tau > 0 \text{ and } T = 0, \\
\int_{0}^{\infty} \frac{d\Omega}{\pi} \rho(-\Omega)e^{\Omega\tau}, & \text{for } \tau < 0 \text{ and } T = 0.
\end{cases} \]  

(B.6)

Using (B.6) we obtain in \( \tau \) space

\[ G(\tau) = \begin{cases} 
- \frac{C\Gamma(2\Delta) \sin(\pi\Delta + \theta)}{\pi|\tau|^{2\Delta}}, & \text{for } \tau > 0 \text{ and } T = 0, \\
\frac{C\Gamma(2\Delta) \sin(\pi\Delta - \theta)}{\pi|\tau|^{2\Delta}}, & \text{for } \tau < 0 \text{ and } T = 0.
\end{cases} \]  

(B.7)

We also use the spectral representations for the self energies

\[ \Sigma(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\sigma(\Omega)}{z - \Omega}. \]  

(B.8)
Using (B.7) and (6.38) we obtain

\[
\sigma(\Omega) = \begin{cases} 
\frac{\pi J^2 q}{\Gamma(2(q-1)\Delta)} \left[ \frac{C \Gamma(2\Delta)}{\pi} \right]^{q-1} [\sin(\pi \Delta + \theta)]^{q/2} [\sin(\pi \Delta - \theta)]^{q/2-1} |\Omega|^{2\Delta(q-1)-1}, & \text{for } \Omega > 0 \\
\frac{\pi J^2 q}{\Gamma(2(q-1)\Delta)} \left[ \frac{C \Gamma(2\Delta)}{\pi} \right]^{q-1} [\sin(\pi \Delta + \theta)]^{q/2-1} [\sin(\pi \Delta - \theta)]^{q/2} |\Omega|^{2\Delta(q-1)-1}, & \text{for } \Omega < 0
\end{cases}
\]  

(B.9)

Now from (3.6) we have in the IR limit

\[
\Sigma(z) - \mu = -\frac{1}{C} e^{i(\pi \Delta + \theta)z/(1-2\Delta)}. 
\]  

(B.10)

So comparing (B.9) and (B.10), we have the solutions in Eqs. (3.4) and (3.9), provided \( \Sigma(z = 0) = \mu \) at \( T = 0 \) [173].

**B.2 Kernel and spectrum**

We will work at zero temperature, then the conformal Green’s function is

\[
G_c(\tau) = \frac{-\text{sgn}(\tau)(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \frac{C \Gamma(2\Delta)}{\pi}}{2} \frac{1}{|\tau|^{2\Delta}}
\]  

(B.11)

with

\[
\alpha_1 = \sin (\pi \Delta + \theta), \quad \alpha_2 = \sin (\pi \Delta - \theta)
\]  

(B.12)

and the coefficient \( C \) is specified in Eq. (3.9).

We now consider the fluctuations above the saddle point where we can find the form of the kernel. We make the expansion

\[
G(\tau_{12}) = G_s(\tau_{12}) + \delta G(\tau_{12}), \quad \delta G(\tau_{12}) = [-G_s(\tau_{12})G_s(\tau_{21})]^{-q/4} G_s(\tau_{12}) g(\tau_{12})
\]  

(B.13)

\[
\Sigma(\tau_{12}) = \Sigma_s(\tau_{12}) + \delta \Sigma(\tau_{12}), \quad \delta \Sigma(\tau_{12}) = [-G_s(\tau_{12})G_s(\tau_{21})]^{q/4} G_s(\tau_{21})^{-1} \sigma(\tau_{12})
\]  

(B.14)
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Then the effective action can be expanded to the second order in $g$ and $\sigma$

$$\frac{S_{\text{eff}}[g, \sigma]}{N} = \frac{1}{2} \sum_n (G_c \delta \Sigma(i \omega_n))^2$$

$$- \int d\tau_1 d\tau_2 \left[ g(\tau_{12})\sigma(\tau_{21}) + \frac{qJ^2}{2} (\frac{q}{2} - 1) g(\tau_{12})^2 + \frac{qJ^2}{2} g(\tau_{12}) g(\tau_{21}) \right]$$

Integrate out $\sigma$ field, we obtain

$$\frac{S_{\text{eff}}[g]}{N} = \frac{qJ^2}{2} \int d^4 \tau g(\tau_{12}) \left[ \tilde{K}(\tau_1, \tau_2; \tau_3, \tau_4)^{-1} - (\frac{q}{2} - 1) \delta(\tau_{13}) \delta(\tau_{24}) - \frac{q}{2} \delta(\tau_{14}) \delta(\tau_{23}) \right] g(\tau_{34})$$

where

$$\tilde{K}(\tau_1, \tau_2; \tau_3, \tau_4) = -qJ^2 \left[ -G_s(\tau_{12})G_s(\tau_{21}) \right]^{9/4-1} G_s(\tau_{21})G_s(\tau_{32})G_s(\tau_{14})G_s(\tau_{43}) \left[ -G_s(\tau_{34})G_s(\tau_{43}) \right]^{9/4-1}$$

Now we want to diagonalize the kernel matrix in the conformal limit with symmetric and anti-symmetric functions, so we divide $g$ into two parts:

$$g(\tau_{12}) = g_s(\tau_{12}) + g_A(\tau_{12})$$

where $g_s(\tau_{12}) = g_s(\tau_{21})$ is the symmetric part and $g_a(\tau_{12}) = -g_A(\tau_{21})$ is the anti-symmetric part. Then the $\delta$ function part contribution become

$$\left( \frac{q}{2} - 1 \right) g(\tau_{12})^2 + \frac{q}{2} g(\tau_{12}) g(\tau_{21}) = (q - 1) g_s(\tau_{12})^2 - g_A(\tau_{12})^2$$
The kernel can be expressed as

\[
\tilde{K}(\tau_1, \tau_2; \tau_3, \tau_4) = -q_j^2 \left( \frac{C_2 \Gamma(2\Delta)^2}{\pi^2} \alpha_1 \alpha_2 \right)^{\frac{2}{2 - 2}} \frac{1}{|\tau_{12}|^{1 - 4\Delta}} \frac{1}{|\tau_{34}|^{1 - 4\Delta}} \left( \frac{C \Gamma(2\Delta)}{2\pi} \right)^{4} \left[ -\text{sgn}(\tau_{21})(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \right] \frac{1}{|\tau_{12}|^{2\Delta}} \\
-\text{sgn}(\tau_{32})(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \frac{1}{|\tau_{32}|^{2\Delta}} \\
-\text{sgn}(\tau_{14})(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \frac{1}{|\tau_{14}|^{2\Delta}} \\
-\text{sgn}(\tau_{34})(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \frac{1}{|\tau_{34}|^{2\Delta}} \right] \quad (B.21)
\]

We consider the action of the kernel matrix on symmetric function \[ \frac{1}{|\tau_{34}|^{1 - \kappa}} \] and anti-symmetric function \[ \frac{\text{sgn}(\tau_{34})}{|\tau_{34}|^{1 - \kappa}} \]. The overall pre-factor is

\[
- \frac{\Gamma(2\Delta) \Gamma(2 - 2\Delta)}{16\pi^2} \frac{1}{\alpha_1 \alpha_2} \quad (B.22)
\]

To perform the matrix action, we need to introduce the functions

\[
c_{f}(\Delta) = 2i \cos \pi \Delta \Gamma(1 - 2\Delta), \quad c_{b}(\Delta) = 2 \sin (\pi \Delta) \Gamma(1 - 2\Delta) \quad (B.23)
\]

It appears in the Fourier transform of a symmetric/anti-symmetric scaling function:

\[
\int d\tau e^{i\omega \tau} \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta}} = c_{f}(\Delta) \text{sgn}(\omega)|\omega|^{2\Delta - 1}, \quad \int d\tau e^{i\omega \tau} \frac{1}{|\tau|^{2\Delta}} = c_{b}(\Delta)|\omega|^{2\Delta - 1} \quad (B.24)
\]

For example, if we consider choose the \((\alpha_1 - \alpha_2)\) part from the last three lines of (B.21) and use symmetric base, then we need to evaluate an integral

\[
-(\alpha_1 - \alpha_2)^3 \int d\tau_3 d\tau_4 \frac{1}{|\tau_{32}|^{2\Delta}} \frac{1}{|\tau_{14}|^{2\Delta}} \frac{1}{|\tau_{34}|^{2 - 2\Delta - k}} = -(\alpha_1 - \alpha_2)^3 \frac{c_{b}(\Delta)^2 c_{b}(1 - \Delta - \frac{h}{2})}{c_{b}(\Delta - \frac{h}{2})} \frac{1}{|\tau_{12}|^{2\Delta - k}} \quad (B.25)
\]
Then the kernel matrix under the base \( (\frac{1}{|\tau_4|^{1-\pi}}, \text{sgn}(\tau_4))_T \) can be expressed as

\[
\begin{pmatrix}
-A(\alpha_1 - \alpha_2) + B(\alpha_1 + \alpha_2) & -X(\alpha_1 - \alpha_2) + Y(\alpha_1 + \alpha_2) \\
A(\alpha_1 + \alpha_2) - B(\alpha_1 - \alpha_2) & X(\alpha_1 + \alpha_2) - Y(\alpha_1 - \alpha_2)
\end{pmatrix}
\] (B.26)

where

\[
A = \frac{cf(\Delta)^2cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^2\right] + 2\frac{cb(\Delta)cf(\Delta)cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^2\right] + \frac{cb(\Delta)^2cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)^3\right] (B.27)
\]

\[
B = \frac{cf(\Delta)^2cf(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)^3\right] + 2\frac{cf(\Delta)cb(\Delta)cb(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)^2\right] + \frac{cb(\Delta)^2cf(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)^2\right] (B.28)
\]

\[
X = -\frac{cf(\Delta)^2cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)^3\right] - 2\frac{cb(\Delta)cf(\Delta)cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)^2\right] - \frac{cb(\Delta)^2cb(1 - \Delta - \frac{h}{2})}{cb(\Delta - \frac{h}{2})} \left[-(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)^2\right] (B.29)
\]

\[
Y = -\frac{cf(\Delta)^2cf(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^2\right] - 2\frac{cb(\Delta)cf(\Delta)cb(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^2\right] - \frac{cb(\Delta)^2cf(1 - \Delta - \frac{h}{2})}{cf(\Delta - \frac{h}{2})} \left[-(\alpha_1 - \alpha_2)^3\right] (B.30)
\]

In the simple case, when \( \theta = 0 \) as in Ref.[25], the matrix becomes diagonal, and it reduces
to

\[
\begin{pmatrix}
\frac{\Delta}{1-\Delta} k_A(h, \Delta) & 0 \\
0 & -k_S(h, \Delta)
\end{pmatrix}
\]  \hspace{1cm} \text{(B.31)}

where

\[
k_A(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(-2\Delta)}{\Gamma(2\Delta - 2)} \Gamma(2\Delta - h) \Gamma(2\Delta + h - 1)(\sin \pi h - \sin 2\pi \Delta)
\]  \hspace{1cm} \text{(B.32)}

\[
k_S(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(1-2\Delta)}{\Gamma(2\Delta - 1)} \Gamma(2\Delta - h) \Gamma(2\Delta + h - 1)(\sin \pi h + \sin 2\pi \Delta)
\]  \hspace{1cm} \text{(B.33)}

The subscript "A" and "S" correspond to anti-symmetric (symmetric) $\delta G (g_S)$ and symmetric (anti-symmetric) $\delta G (g_A)$. In this case, the symmetric and anti-symmetric fluctuations decouple:

\[
\frac{S_{\text{eff}}[g]}{N} = \frac{J^2}{2} \int d^2 \tau g_S(\tau_{12})^2(q - 1) \left[ k_A(h, \Delta)^{-1} - 1 \right] + g_A(\tau_{12})^2 \left[ -k_S(h, \Delta)^{-1} + 1 \right]
\]  \hspace{1cm} \text{(B.34)}

One can show that the corresponding zero mode is $h = 2$ for $k_A$ and $h = 1$ for $k_S$. In the more general case, when $\theta \neq 0$, we have for $h = 2$, the kernel reduces to

\[
\tilde{K}(h = 2, \Delta) = \frac{\Delta}{1-\Delta} \begin{pmatrix}
1 & -2 \csc (2\pi \Delta) \sin (2\theta) \\
0 & 1
\end{pmatrix} \Rightarrow \tilde{K}(h = 2, \Delta)^{-1} = (q - 1) \begin{pmatrix}
1 & 2 \csc (2\pi \Delta) \sin (2\theta) \\
0 & 1
\end{pmatrix}
\]  \hspace{1cm} \text{(B.35)}

For $h = -1$,

\[
\tilde{K}(h = -1, \Delta) = \frac{\Delta}{1-\Delta} \begin{pmatrix}
1 & 0 \\
-2 \csc (2\pi \Delta) \sin (2\theta) & 1
\end{pmatrix} \Rightarrow \tilde{K}(h = -1, \Delta)^{-1} = (q - 1) \begin{pmatrix}
1 & 0 \\
2 \csc (2\pi \Delta) \sin (2\theta) & 1
\end{pmatrix}
\]  \hspace{1cm} \text{(B.36)}
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For $h = 1$,

$$
\tilde{K}(h = 1, \Delta) = - \begin{pmatrix} 1 & 0 \\ -2 \csc(2\pi \Delta \sin(2\theta)) & 1 \end{pmatrix} \Rightarrow \tilde{K}(h = 1, \Delta)^{-1} = - \begin{pmatrix} 1 & 0 \\ 2 \csc(2\pi \Delta \sin(2\theta)) & 1 \end{pmatrix}
$$

(B.37)

For $h = 0$,

$$
\tilde{K}(h = 0, \Delta) = - \begin{pmatrix} 1 & -2 \csc(2\pi \Delta \sin(2\theta)) \\ 0 & 1 \end{pmatrix} \Rightarrow \tilde{K}(h = 0, \Delta)^{-1} = - \begin{pmatrix} 1 & 2 \csc(2\pi \Delta \sin(2\theta)) \\ 0 & 1 \end{pmatrix}
$$

(B.38)

We conclude that the zero modes are symmetric $g$ with $h = 2$ or anti-symmetric $g$ with $h = 1$:

$$
g_s(\tau) = \frac{1}{|\tau|^{1-2}}, \quad g_A(\tau) = \frac{\text{sgn}(\tau)}{|\tau|^{1-1}}
$$

(B.39)

The effective action (B.17) can be rewritten as

$$
\frac{S_{\text{eff}}}{N} = \frac{J^2}{2} g^T \tilde{K}^{-1}(I - K') g
$$

(B.40)

where

$$
K' = \tilde{K} \begin{pmatrix} q - 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(B.41)
with the matrix elements:

\[
K'_{11} = \left(\frac{1}{\Delta} - 1\right) \frac{(2\Delta - 1) \csc(\pi(h - 2\Delta)) \Gamma(1 - 2\Delta) \Gamma(h + 2\Delta - 1) \sin(2\pi \Delta)(\sin(2\pi \Delta) - \cos(2\theta) \sin(h \pi))}{\pi \Gamma(h - 2\Delta + 1)}
\]

\[
K'_{12} = -\frac{2(1 - 2\Delta) \cos^2 \left(\frac{h \pi}{2}\right) \csc(\pi(h - 2\Delta)) \Gamma(1 - 2\Delta) \Gamma(h + 2\Delta - 1) \sin(2\pi \Delta) \sin(2\theta)}{\pi \Gamma(h - 2\Delta + 1)}
\]

\[
K'_{21} = \frac{2 \left(\frac{1}{\Delta} - 1\right) (1 - 2\Delta) \csc(\pi(h - 2\Delta)) \Gamma(1 - 2\Delta) \Gamma(h + 2\Delta - 1) \sin^2 \left(\frac{h \pi}{2}\right) \sin(2\pi \Delta) \sin(2\theta)}{\pi \Gamma(h - 2\Delta + 1)}
\]

\[
K'_{22} = -\frac{(2\Delta - 1) \csc(\pi(h - 2\Delta)) \Gamma(1 - 2\Delta) \Gamma(h + 2\Delta - 1) \sin(2\pi \Delta)(\cos(2\theta) \sin(h \pi) + \sin(2\pi \Delta))}{\pi \Gamma(h - 2\Delta + 1)}
\]

Interestingly, even in the particle-hole asymmetric case, the eigenvalue of \(K'\) won’t depend on \(\theta\). The spectrum from the OPE of two fermions is still determined by

\[
k_A(h_m, \Delta) = 1 \quad \text{or} \quad k_S(h_m, \Delta) = 1
\]

The corresponding solutions are

\[
h_m = 2\Delta + 1 + 2m + \mathcal{O}\left(\frac{1}{m}\right) \quad \text{or} \quad h_m = 2\Delta + 2m + \mathcal{O}\left(\frac{1}{m}\right)
\]

**B.3 LARGE \(q\) EXPANSION OF THE COMPLEX SYK MODEL**

Section 3.3.1 obtained exact results for the universal parts of the thermodynamic observables. However, no explicit results for the non-universal parts dependent upon \(J\). In this appendix we will present the large \(q\) expansion of the Hamiltonian in Eq. (3.1): the results contain both the universal and non-universal parts.

We begin by recalling the universal results of Section 3.3.1 in the limit of small \(\Delta = 1/q\). At low \(T\), the thermodynamics contains 3 universal quantities which do not undergo any UV renormalization: they are the density, \(Q\), the entropy \(S\), and the ‘electric field’ \(E\). All 3 quantities can be expressed in terms of universal expressions of each other. First, we treat
\( Q \) as the independent quantity. Then, the \( T \to 0 \) limit of the entropy is from (3.20), (3.27), (3.28),

\[
S(Q) = Q \ln \left( \frac{1 - 2Q}{1 + 2Q} \right) + \frac{1}{2} \ln \left( \frac{4}{1 - 4Q^2} \right) - \frac{\pi^2}{2} (1 - 4Q^2) \Delta^2 + \mathcal{O}(\Delta^3). \tag{B.45}
\]

By taking a \( Q \) derivative, we have immediately

\[
\mathcal{E}(Q) = \frac{1}{2\pi} \ln \left( \frac{1 - 2Q}{1 + 2Q} \right) + 2\pi Q \Delta^2 + \mathcal{O}(\Delta^3). \tag{B.46}
\]

Next, we take \( \mathcal{E} \) as the independent variable. Then the inverse function (B.46) is

\[
Q(\mathcal{E}) = -\frac{1}{2} \tanh(\pi\mathcal{E}) - \frac{\pi^2 \sinh(\pi\mathcal{E})}{2 \cosh^3(\pi\mathcal{E})} \Delta^2 + \mathcal{O}(\Delta^3). \tag{B.47}
\]

The entropy is given by (3.28), \( S(\mathcal{E}) = \mathcal{G}(\mathcal{E}) + 2\pi \mathcal{E} Q(\mathcal{E}) \), where

\[
\mathcal{G}(\mathcal{E}) = \ln(2 \cosh(\pi\mathcal{E})) - \frac{\pi^2}{2 \cosh^2(\pi\mathcal{E})} \Delta^2 + \mathcal{O}(\Delta^3). \tag{B.48}
\]

Now we turn to the explicit large \( q \) expansion to the compute the thermodynamics in terms of microscopic parameters. The expressions here depend upon the underlying \( J \), and the specific form of the Hamiltonian in Eq. (3.1). We will verify that they are compatible with the universal results presented above.

The large \( q \) expansion was presented by Ref. [135] at \( \mu = 0 \), and we follow their analysis here. At \( q = \infty \) they showed that the Green’s function was that of a dispersionless free fermion. So, we write

\[
G_s(\tau) = G_0(\tau) \left[ 1 + \frac{1}{q} G_1(\tau) \right] \tag{B.49}
\]
where the dispersionless free fermion Green’s function is

\[ G_0(\tau) = \begin{cases} 
- e^{\mu \tau} (e^{\mu/T} + 1)^{-1}, & 0 < \tau < 1/T \\
 e^{\mu \tau} (e^{-\mu/T} + 1)^{-1}, & -1/T < \tau < 0
\end{cases} \]  

(B.50)

Then from (6.38) we have the self energy

\[ \Sigma_s(\tau) = -\frac{q J^2 e^{\mu \tau}}{(e^{\mu/T} + 1)(2 + 2 \cosh(\mu/T))^{q/2-1}} \left[ 1 + \frac{1}{q} G_1(\tau) \right]^{q/2} \left[ 1 + \frac{1}{q} G_1(-\tau) \right]^{q/2-1} \]  

(B.51)

Now we define

\[ J^2 = \frac{q^2 J^2}{2(2 + 2 \cosh(\mu/T))^{q/2-1}}. \]  

(B.52)

The large \( q \) can only be taken if we adjust the bare \( J \) so that \( J \) is \( q \) independent. To the order we shall work, it is legitimate to use the \( \mathcal{O}(\Delta^0) \) result above, in which case we will find

\[ J^2 = \frac{q^2 J^2}{2(2 + 2 \cosh(2\pi E))^{q/2-1}}. \]  

(B.53)

As \( E \) is only a function of \( Q \), we find that \( J \) remains finite as \( T \to 0 \). Then, in the large \( q \) limit

\[ \Sigma_s(\tau) = -\frac{2J^2}{q} G_0(\tau) \exp \left( \frac{1}{2} (G_1(\tau) + G_1(-\tau)) \right) \]  

(B.54)

In this form, the explicit \( \mu \) dependence has disappeared. Ref. [135] obtained a differential equation for \( G_1 \) at \( \mu = 0 \), and so this applies also here; the solution is

\[ G_1(\tau) = \ln \left( \frac{\cos^2(\pi v/2)}{\cos^2(\pi v/2|\tau|) - 1/2)} \right), \]  

(B.55)

where \( v \) is obtained by the solution of

\[ \frac{\pi v}{\cos(\pi v/2)} = \frac{J}{T}. \]  

(B.56)
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Assuming a fixed $J$, the low $T$ expansion of $v$ is

$$v = 1 - \frac{2T}{J} + \frac{4T^2}{J^2} + \ldots.$$  \hfill (B.57)

To compute the grand potential, $\Omega$, we use the effective action

$$S[G, \Sigma] = -\text{Tr} \ln \left[ \delta(\tau - \tau') \left( -\frac{\partial}{\partial \tau} + \mu \right) - \Sigma(\tau, \tau') \right] + \frac{\mu}{2T}$$

$$- \int_0^{1/T} d\tau d\tau' \left[ \Sigma(\tau, \tau')G(\tau', \tau) + (-1)^{q/2}J^2G(\tau, \tau')^{q/2} + [G(\tau', \tau)]^{q/2} \right].$$  \hfill (B.58)

The $G_s(\tau - \tau')$ and $\Sigma_s(\tau - \tau')$ above are the solutions to the saddle-point equations of $S$. It is simpler to evaluate $d\Omega/dJ$ because only the last term contributes

$$J \frac{d\Omega}{dJ} = -\frac{4J^2}{q^2(2 + 2 \cosh(\mu/T))} \int_0^{1/T} d\tau \exp(G_1(\tau))$$

$$= -\frac{4J^2 \sin(\pi v)}{q^2 \pi T v(2 + 2 \cosh(\mu/T))},$$  \hfill (B.59)

which implies

$$\frac{d\Omega}{dv} = -\frac{8\pi T}{q^2(2 + 2 \cosh(\mu/T))} \tan\left(\frac{\pi v}{2}\right) \left[1 + \frac{\pi v}{2} \tan\left(\frac{\pi v}{2}\right)\right] \Delta^2 + O(\Delta^3).$$  \hfill (B.60)

Integrating over $v$, we obtain the grand potential as a function of the bare $\mu$ and $T$

$$\Omega(\mu, T) = -T \ln(2 \cosh(\mu/(2T))) - \frac{2\pi v T}{\cosh^2(\mu/(2T))} \left[\tan\left(\frac{\pi v}{2}\right) - \frac{\pi v}{4}\right] \Delta^2 + O(\Delta^3).$$  \hfill (B.61)

This is the main result of the large $q$ expansion.

Now we can use thermodynamic relations to determine both universal and non-universal observables. From the grand potential in (B.61), we have the density

$$Q = \frac{1}{2} \tanh(\mu/(2T)) - \frac{2\pi v \sinh(\mu/(2T))}{\cosh^3(\mu/(2T))} \left[\tan\left(\frac{\pi v}{2}\right) - \frac{\pi v}{4}\right] \Delta^2 + O(\Delta^3).$$  \hfill (B.62)
Combining (B.61) and (B.62), we can obtain the free energy in the canonical ensemble

\[
F(Q; T) = \Omega(\mu, T) + \mu Q \\
= -T \left[ \frac{1}{2} \ln \left( \frac{4}{1 - 4Q^2} \right) + Q \ln \left( \frac{1 - 2Q}{1 + 2Q} \right) \right] - 2\pi T (1 - 4Q^2) \left[ \tan \left( \frac{\pi v}{2} \right) - \frac{\pi v}{4} \right] \Delta^2 + O(\Delta^3). \tag{B.63}
\]

It is more convenient to work with the canonical \( F(Q; T) \), rather than the grand canonical \( \Omega(\mu, T) \), because \( Q \) is universal, while \( \mu \) is not. We will use (B.63) to verify the universal expressions in Section 3.3.1, and also to obtain new non-universal results.

First, in the fixed \( Q \) ensemble, we can compute the chemical potential \( \mu(Q; T) \) needed to keep \( Q \) fixed. We find

\[
\mu(Q; T) = \left( \frac{\partial F}{\partial Q} \right)_T \\
= -T \ln \left( \frac{1 - 2Q}{1 + 2Q} \right) + 16\pi T Q v \left[ \tan \left( \frac{\pi v}{2} \right) - \frac{\pi v}{4} \right] \Delta^2 + O(\Delta^3) \tag{B.64}
\]

\[
= \mu_0 - 2\pi \mathcal{E}(Q) T + O(T^2)
\]

In the last line, we have taken the low \( T \) limit at fixed \( J \) using (B.57), and we find precisely the expression (3.21), with the universal function \( \mathcal{E}(Q) \) given by (B.46), and the non-universal bare chemical potential

\[
\mu_0 = 16J Q \Delta^2 + O(\Delta^3). \tag{B.65}
\]

Note that there is no \( O(\Delta^0) \) term in \( \mu_0 \); this has consequences for the compressibility. From (B.64) we can obtain the inverse compressibility, \( 1/K \), by taking a derivative w.r.t. \( Q \); at low \( T \) we have

\[
\frac{1}{K} = \left( \frac{\partial \mu}{\partial Q} \right)_T = \frac{4T}{1 - 4Q^2} + (16J - 4\pi^2 T) \Delta^2 + O(\Delta^3) \tag{B.66}
\]

So we now see that if take the limit \( \Delta \to 0 \) first, then the compressibility diverges as \( K \sim 1/T \).
in the low $T$ limit at fixed $J$. On the other hand, if we take the $T \to 0$ at non-zero \( \Delta \), then $K$ remains finite at \( K = q^2/(16J) \), as needed for the consistency of the analysis in Section 3.3.2. Note that the large $q$ expansion holds for $1/K$, and not for $K$, it is the expansion for $1/K$ which establishes the finiteness of $K$ as $T \to 0$.

We can also obtain the non-universal ground state energy

$$ E_0 = F(Q, T \to 0) = -2J(1 - 4Q^2)\Delta^2 + \mathcal{O}(\Delta^3). \quad (B.67) $$

This is compatible with (B.65) and (3.22).

Finally, we can compute the entropy, and perform its low $T$ expansion at fixed $J$; we find

$$ S(Q, T) = -\left( \frac{\partial F}{\partial T} \right)_Q = S(Q) + \gamma T + \mathcal{O}(T^2), \quad (B.68) $$

where the universal function $S(Q)$ agrees with (B.45), and the non-universal linear-in-$T$ coefficient of the specific heat at fixed $Q$ is

$$ \gamma = \frac{2\pi^2(1 - 4Q^2)}{J}\Delta^2 + \mathcal{O}(\Delta^3) \quad (B.69) $$

Again, note that there is no $\mathcal{O}(\Delta^0)$ term in $\gamma$.

### B.4 Numerical solution of the SYK model

This appendix describes our numerical solution of Eqs. (6.38) and (3.6) for the case $q = 4$.

We worked in the frequency domain by writing Eq. (6.38) as a convolution

$$ \Sigma(i\omega_n) = -\frac{J^2}{\beta^2} \sum_{\omega_n = \omega_1 + \omega_2 - \omega_3} G(i\omega_1)G(i\omega_2)G(i\omega_3) \quad (B.70) $$

We used the function package `conv_fft2` in Matlab to perform the convolution. We re-
Figure B.1: The entropy $S(Q)$ obtained from the exact results [74] in Section 3.3.1 (full line), and by the numerical solutions (stars).

...stricted the frequency argument in $G(i\omega_n)$ to be $2\pi T(n + \frac{1}{2})$ where $-N \leq n \leq N - 1$. After the convolution, we cut off the frequency argument in $\Sigma(i\omega_n)$ to be within the same regime. Finally, we updated the Green’s function in a weighted way:

$$G'_j(i\omega_n) = (1 - \alpha)G_j(i\omega_n) + \alpha \frac{1}{i\omega_n + \mu - \Sigma_{j-1}(i\omega_n)} \quad \text{(B.71)}$$

where we choose the weight $\alpha = 0.2$, and $j$ denotes the iteration step.

We also used a second numerical approach in which we directly evaluate Eqs. (6.38) and (3.6) in frequency space and time space separately, and then use fast Fourier transforms (FFT) between them. But there is a subtlety: when considering the transformation from $\tau$ space to $\omega_n$ space, we are doing a discrete sum to represent the numerical integral. For a sensible discrete sum, we do not want the exponential phase to vary too much between the two adjacent discrete points. So we want $\omega_n(\tau_j - \tau_{j-1}) \ll 1$. With $N_\tau$ and $N_\omega$ the number of points of $\tau$ and $\omega$, we need $N_\omega/N_\tau \ll 1$. We found $N_\omega = 2^{18}, N_\tau = 2^{20}$ gave accurate results.

From the numerical solution for Green’s function and self energy, we obtain the grand
potential by evaluating Eq. (B.58). In practice, we want to subtract the grand potential of a free theory and then add it back to obtain a convergent sum over frequencies. So we wrote the first term in Eq. (B.58) as

$$ T \sum_n \log \left[ G(i\omega_n)/G_0(i\omega_n) \right] + T \log \left[ 1 + e^{\mu/T} \right]. \quad (B.72) $$

By the equations of motion, the second integral can be written as

$$ -\frac{3}{4\beta} \sum_n \Sigma(i\omega_n)G(i\omega_n) \quad (B.73) $$

Then we put the solution into these two terms and obtained the grand potential $\Omega(\mu, T)$. The density, $Q$, the compressibility, $K$, and the entropy, $S$ were then obtained from suitable thermodynamic derivatives\(^1\). Our numerical results for $S(Q)$, obtained by both methods are shown in Fig. B.1, they are in excellent agreement with the exact analytic results [74]. In the frequency domain computation, we used the cutoff $N = 2 \times 10^6$. The points in Fig. B.1 are at moderate values of $Q$, and our numerics did not converge for $|Q|$ near 1/2.\(^2\)

For the compressibility, numerically near $\mu = 0$ and at $T = 0$, we find that $K = 1.04/J = 1.04/(\sqrt{2}J)$; With $q = 4$, this is of the same order of the large $q$ result: $K = q^2/(16J) = 1/J$.

\(^1\)Q can also be obtained from $G(\tau = 0^-)$, we have checked that it is consistent with the derivative method.

\(^2\)At large $\mu$, we always find the free Green’s function $G_0 = \frac{1}{i\omega_n + \mu}$ to be solution. The reason can be understood by the self-energy obtained from the free solution

$$ \Sigma_0(i\omega_n) = -\frac{J^2}{\beta^2} \sum_{\omega_n = \omega_1 + \omega_2 - \omega_3} G_0(i\omega_1)G_0(i\omega_2)G_0(i\omega_3) = -\frac{J^2}{i\omega_n + \mu} \left( \frac{1}{(2 \cosh \frac{\mu}{2})^2} \right). $$

Notice the exponential suppression at low temperature. This means at any finite $\mu$, at zero temperature, the free one is always a solution. Numerically we are always at small finite temperature to represent the zero temperature result, but when $\mu$ becomes large, the exponential suppression will make the free Green’s function converge well within the fixed tolerance.
B.5 Normal mode analysis of the SYK model

This appendix will generalize the analysis of Maldacena and Stanford [135], and describe the structure of the effective action for fluctuations directly from the action in Eq. (B.58). We will work here in an angular variable

$$\varphi = 2\pi T \tau$$  \hspace{1cm} (B.74)

which takes values on a temporal circle of unit radius. We also use the notation $\varphi_{12} \equiv \varphi_1 - \varphi_2$.

We begin with the saddle-point solution of Eq. (B.58), the Green’s function $G_s(\varphi)$. In the scaling limit, this is given by Eq. (3.13). We write this here as

$$G_s(\varphi) = b e^{-\frac{\varphi}{2\Delta}} \varphi, \quad \Delta = \frac{1}{q}, \quad \varphi \in [0, 2\pi)$$  \hspace{1cm} (B.75)

where the prefactor $b$ is specified in Eq. (3.13). We now expand the effective action Eq. (B.58) to quadratic order of the fluctuations $\delta G(\varphi_1, \varphi_2) = G(\varphi_1, \varphi_2) - G_s(\varphi_{12})$, $\delta \Sigma(\varphi_1, \varphi_2) = \Sigma(\varphi_1, \varphi_2) - \Sigma_s(\varphi_{12})$ and further integrate over $\delta \Sigma$. For convenience we use renormalized form of the fluctuation:

$$g(\varphi_1, \varphi_2) = [-G_s(\varphi_{12})G_s(\varphi_{21})]^{q/4} G_s(\varphi_{12})^{-1} \delta G(\varphi_1, \varphi_2)$$  \hspace{1cm} (B.76)

and obtaining the action (to quadratic order) in $g$:

$$S_{\text{eff}} = \frac{1}{N} \int d^4 \varphi g(\varphi_1, \varphi_2)Q(\varphi_1, \varphi_2, \varphi_3, \varphi_4)g(\varphi_3, \varphi_4),$$  \hspace{1cm} (B.77)

where $Q$ is a quadratic form on the space of functions with two time variables.

We now focus on just the zero mode fluctuations specified by the transformations in Eq. (3.35). First, examine the infinitesimal reparameterization mode, with an accompanying
U(1) transformation satisfying Eq. (3.39)

\[ f(\varphi) = \varphi + \epsilon(\varphi) , \quad \phi(\varphi) = -i\mathcal{E}\epsilon(\varphi). \]  

(B.78)

Notice that under this mode, \( \tilde{\varphi} = 0 \) in Eq. (3.44). Inserting Eq. (B.78) into Eq. (3.35), using Eq. (B.76) to get the normalized fluctuations for each Fourier mode \( \epsilon(\varphi) = \frac{1}{2\pi} \sum_n e^{-in\varphi} \epsilon_n \), we find that the linear order change in \( g \) is

\[ g_n^f(\varphi_1, \varphi_2) = \frac{i\Delta \theta/2e^{-2\pi q\mathcal{E}/4}}{\pi} \left( \frac{f_n(\varphi_{12})}{|\sin \frac{n\varphi_{12}}{2}|} \right) \epsilon_n e^{-in\frac{\varphi_1 + \varphi_2}{2}}. \]  

(B.79)

The functions \( f_n(\varphi_{12}) \) is a symmetric function of two variables \( \varphi_1, \varphi_2 \):

\[ f_n(\varphi) = \frac{\sin n\frac{\varphi}{2}}{\tan \frac{\varphi}{2}} - n \cos n\frac{\varphi}{2}, \quad \int_0^{2\pi} d\varphi \left( \frac{f_n(\varphi)}{\sin \frac{\varphi}{2}} \right)^2 = \frac{2\pi}{3} |n|(n^2 - 1). \]  

(B.80)

Similarly, we can examine the U(1) fluctuation mode, under which \( \epsilon \) is unchanged but \( \phi \) changes:

\[ g_n^\phi(\varphi_1, \varphi_2) = \frac{b\theta/2e^{-2\pi q\mathcal{E}/4}}{\pi} \left( \frac{\sin n\frac{\varphi_{12}}{2}}{|\sin \frac{\varphi_{12}}{2}|} \right) \phi_n e^{-in\frac{\varphi_1 + \varphi_2}{2}}, \]  

(B.81)

which implies that the phase fluctuation is anti-symmetric in two time variable. It is also useful to notice the following equation:

\[ \int_0^{2\pi} d\varphi \left( \frac{\sin n\frac{\varphi}{2}}{\sin \frac{\varphi}{2}} \right)^2 = 2\pi |n|. \]  

(B.82)

Turning to the structure of the quadratic form, \( Q \), we now make the key observation that evaluating \( Q \) from Eq. (B.58) and the conformal Green’s function in Eq. (B.75) leads to a vanishing action of \( Q \) on the normal modes described above. This is a direct consequence of the invariance of Eq. (3.35) under reparameterization and U(1) transformations. Ref. [135] argued that going beyond the conformal limit will lead to a shift in the eigenvalue of \( Q \) of order \( |n|T/J \) in the first order perturbation theory. Assuming this applies here to both
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modes discussed above,\(^3\) we have

\[
Q \cdot g^\phi_n = \alpha_\phi \frac{|n| T}{J} g^\phi_n, \quad Q \cdot g^\epsilon_n = \alpha_\epsilon \frac{|n| T}{J} g^\epsilon_n,
\]

(B.83)

where the numerical coefficients \(\alpha_\phi\) and \(\alpha_\epsilon\) cannot be obtained analytically, but can be computed in the large \(q\) expansion. Here, we can fix them by comparing with the large \(q\) results already obtained in Appendix B.3.

Inserting Eq. (B.83) into (B.77), and using the explicit form of the fluctuations in Eqs. (B.79) and (B.81), we obtain the effective action to quadratic order:

\[
\frac{S_{\text{eff.}}}{N} = \frac{1}{2} \sum_n \left\{ c_\phi n^2 |\tilde{\phi}_n|^2 + c_\epsilon n^2 (n^2 - 1) |\epsilon_n|^2 \right\}.
\]

(B.84)

where \(c_\phi\) and \(c_\epsilon\) are coefficients of order \(\frac{T}{J}\) and proportional to \(\alpha_\phi\) and \(\alpha_\epsilon\). We confirm that this is of the form in Eq. (3.45), and we can further express the ratio of \(K\) and \(\gamma\) in terms of the numerical coefficients here

\[
\frac{K}{\gamma} = \frac{c_\phi}{4\pi^2 c_\epsilon} = \frac{3\alpha_\phi}{4\pi^2 \Delta^2 \alpha_\epsilon}
\]

(B.85)

There is another way to obtain the effective action using an RG scheme.\(^6\) The basic idea is to replace the UV perturbation by a scaling form that can flow to IR and the effective action results from the IR soft modes couple to this UV source. We will put more details in another paper where we can pin down the coefficient ratio of the Schwarzian part and the phase mode part.

Using the effective action Eq. (B.84) we can also extract an order-one piece of the free energy which arises from the 1-loop calculation. In addition to the Schwarzian part that has

\(^3\)One can justify this statement by a renormalization theory argument in Ref. [117]
been discussed in Ref. [135], we have a new piece from phase fluctuations \( \tilde{\phi} \):

\[
Z_{\tilde{\phi}}(\beta) = \sqrt{\det B^{-1}}, \quad B_{n,m} = \delta_{n+m} \frac{N_{C\phi}}{2} n^2
\]  

(B.86)

We can evaluate the determinant using the zeta function regularization:

\[
\log Z_{\tilde{\phi}} = -\left( \sum_{n=1}^{\infty} \log \frac{N_{C\phi}}{2} n^2 \right) = \frac{1}{2} \log \frac{N_{C\phi}}{8\pi^2} \sim -\frac{1}{2} \log \beta J
\]

(B.87)

Together with the contribution from Schwarzian \( \sim -\frac{3}{2} \log \beta J \), we conclude that the partition function \( Z(\beta) \) is proportional to \( \beta^{-2} \) at large \( \beta \). From this, one can further extract the low energy density of state \( \rho(E) \) from inverse Laplace transformation of \( Z(\beta) \), and show that \( \rho(E) \) is proportional to \( E \) at small \( E \ll \frac{J}{N} \).

We have also numerically computed a variation of partition function \( |Z(\beta + it)| \) as in Ref. [39] using exact diagonalization, the result is shown in Fig. B.2. The slope is around \(-2.07\) in the "slope" regime which is naively outside the validity of the one-loop computation \((1 \ll |\beta + it| \ll N)\), this is an indication of the one-loop exactness[183] of the complex SYK model.

### B.6 Couplings in effective action of the SYK model

This appendix will present another derivation for the values of the couplings in the Schwarzian and phase fluctuation effective action in Eq. (3.41). Here, we will only obtain the leading quadratic terms in the gradient expansion, which have two temporal derivatives, although Eq. (3.41) contains many higher order terms. Just by matching these low order terms we will fix the couplings as in Eq. (3.50).
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Figure B.2: Blue line is $|Z(\beta + it)|$ computed from exact diagonalization data for $N = 15$ and 1000 realizations at $\beta J = 5$. The orange line is the linear fit for the "slope" regime and the slope is around $-2.07$.

First we examine phase fluctuations, under which by Eq. (3.35)

$$G(\tau, \tau') = e^{i\phi(\tau)}G_s(\tau - \tau')e^{-i\phi(\tau')}$$

$$\Sigma(\tau, \tau') = e^{i\phi(\tau)}\Sigma_s(\tau - \tau')e^{-i\phi(\tau')}$$

(\text{B.88})

We insert the ansatz (B.88) into the action (B.58), and perform a gradient expansion in derivatives of $\phi(\tau)$. It is evident that the entire contribution comes from the Tr ln term, as the other terms are independent of $\phi$. Furthermore, we can use the identity

$$\text{Tr} \ln \left[ \delta(\tau - \tau') \left( -\frac{\partial}{\partial \tau} + \mu \right) - e^{i\phi(\tau)}\Sigma_s(\tau - \tau')e^{-i\phi(\tau')} \right]$$

$$= \text{Tr} \ln \left[ \delta(\tau - \tau') \left( -\frac{\partial}{\partial \tau} + \mu + i\partial_\tau \phi(\tau) \right) - \Sigma_s(\tau - \tau') \right],$$

(\text{B.89})

which is easily derived by a gauge transformation of the fermion fields that were integrated to obtain the determinant. In a gradient expansion about a saddle point at a fixed $\mu$, after all other modes (other than the reparameterization mode mentioned below) have been
integrated out, we expect an effective action of the form

\[
\frac{S_{\phi}}{N} = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi)^2.
\] (B.90)

We can determine \( K \) by evaluating the effective action for the special case where \( \partial_\tau \phi \) a constant; under these conditions, we note from (B.89) that all we have to do in the effective action is to make a small change in \( \mu \) by \( i\partial_\tau \phi \). Therefore, we have established that

\[
K = - \left( \frac{\partial^2 \Omega}{\partial \mu^2} \right)_T
\] (B.91)

is indeed the compressibility, as in Eq. (3.50).

A similar argument can be made for energy fluctuations. Now we consider the temporal reparameterization

\[
\tau \rightarrow \tau + \epsilon(\tau)
\] (B.92)

After integrating out all other high energy modes at a fixed chemical potential (other than the phase mode above), we postulate an effective action for \( \epsilon(\tau) \), and assume that the lowest order gradient expansion leads to

\[
\frac{S_{\epsilon}}{N} = \frac{\tilde{K}}{2} \int_0^{1/T} d\tau (\partial_\tau \epsilon)^2.
\] (B.93)

We can now relate the coefficient to a thermodynamic derivative. As for (B.90), consider the case where \( \partial_\tau \epsilon \) is a constant. Then (B.92) implies a change in temperature

\[
-\frac{\delta T}{T^2} = \frac{\partial_\tau \epsilon}{T}
\] (B.94)

Inserting (B.94) into (B.93), we conclude that

\[
\frac{\tilde{K}}{T^2} = \left( \frac{\partial^2 \Omega}{\partial T^2} \right)_\mu
\] (B.95)
Finally, we can also fix the cross term by a similar argument, and so obtain the complete Gaussian effective action for $\phi$ and $\epsilon$ fluctuations, after all other modes have been integrated out

$$S_{\phi\epsilon} = \int_0^{1/T} dt \left[ -\frac{1}{2} \left( \frac{\partial^2 \Omega}{\partial \mu^2} \right)_T (\partial_\tau \phi)^2 - iT \frac{\partial^2 \Omega}{\partial T \partial \mu} (\partial_\tau \epsilon) (\partial_\tau \phi) + \frac{T^2}{2} \left( \frac{\partial^2 \Omega}{\partial T^2} \right)_\mu (\partial_\tau \epsilon)^2 \right].$$ (B.96)

After application of thermodynamic identities, this is found to agree with the second order temporal derivatives in Eq. (3.41), and the identifications in Eqs. (3.16) and (3.50).

### B.7 Diffusion Constants of the Higher-Dimensional SYK Model

The generalization of the zero-dimensional SYK results in Appendix B.5 to the higher dimensional models closely follows the lines of Ref. [83]. In high dimensional models, the quadratic form $Q$ acquires a spatial dependence, formally we have $Q \rightarrow Q_{xy}$ where $Q_{xy}$ contains a hopping matrix for the fluctuations, which can be easily diagonalized by going to $k$-space. For long wavelength limit, we can expand its eigenvalue around $k = 0$: $Q(k) = Q(0) + c k^2 + \ldots$ where $c$ is a constant depends on $J_0$ and $J_1$ that captures the band structure at long wavelength, and $Q(0)$ is the quadratic form at $k = 0$ which reproduces the quadratic form in $(0 + 1)$-dimension. In general, the hopping matrix acts differently on anti-symmetric fluctuation $g^\phi$ and symmetric fluctuation $g^\epsilon$, which will induce two different band structures $Q(k)^\phi = Q(0)^\phi + c_1 k^2 + \ldots$ and $Q(k)^\epsilon = Q(0)^\epsilon + c_2 k^2 + \ldots$ for charge and energy fluctuation respectively.

Inserting this back into the effective action derivation in Appendix B.5, we notice that for the the $\phi$ modes, we need to replace the UV correction for $Q$ from $Q^\phi(0) \sim \alpha_\phi \frac{|n| T}{J}$ to $Q^\phi(k) = \alpha_\phi \frac{|n| T}{J} + c_1 k^2$ Similarly, for $\epsilon$ modes, we need to replace $\alpha_\epsilon \frac{|n| T}{J}$ to $\alpha_\epsilon \frac{|n| T}{J} + c_2 k^2$, where $J = \sqrt{J_0^2 + J_1^2}$. This replacement leads to the effective action in Eq. (3.62) with

$$D_1 = \frac{2 \pi c_1 J}{\alpha_\phi}, \quad D_2 = \frac{2 \pi c_2 J}{\alpha_\epsilon}. \quad \text{(B.97)}$$
For the specific model we discussed in main text, the special form of the hopping term Eq. (3.60) leads to $c_1 = c_2$. Using Eq. (B.85), we then obtain the ratio of the diffusion constants

$$\frac{D_2}{D_1} = \frac{\alpha_\phi}{\alpha_c} = \frac{4\pi^2 \Delta^2 K}{3 \gamma}$$

which was presented in Eq. (3.63).

### B.8 Thermodynamics and Transport

This part follows Ref. [123]. In general, we work in the grand canonical ensemble, the grand partition function is defined to be

$$Q = \text{Tr} e^{-\beta (H - \mu N)}$$

We also define the grand potential

$$\Omega(\mu, T) = -\frac{1}{\beta} \log Q$$

Then we can define

$$\overline{N} = -\frac{\partial \Omega}{\partial \mu}, \quad S = -\frac{\partial \Omega}{\partial T}$$

Let us now consider the fluctuations in particle numbers in the grand canonical ensemble

$$\langle \delta N^2 \rangle = \langle (N - \overline{N})^2 \rangle = \frac{\text{Tr} \left[ N^2 e^{-\beta (H - \mu N)} \right]}{Q} - \left( \frac{\text{Tr} \left[ Ne^{-\beta (H - \mu N)} \right]}{Q} \right)^2 = -T \frac{\partial^2 \Omega}{\partial \mu^2}$$

Similar we can find the fluctuations in energy and cross fluctuations

$$\langle (\delta E - \mu N)^2 \rangle = -T^3 \frac{\partial^2 \Omega}{\partial T^2}$$

$$\langle (\delta E - \mu N) \delta N \rangle = -T^2 \frac{\partial^2 \Omega}{\partial T \partial \mu}$$
So we conclude that, the static susceptibility

\[
\chi_s \equiv \begin{pmatrix}
\langle \partial N / \partial T \rangle & \langle \partial N / \partial \mu \rangle \\
T \langle \partial N / \partial T \rangle & T \langle \partial S / \partial T \rangle
\end{pmatrix}
= \begin{pmatrix}
\langle (\delta N)^2 \rangle / T & \langle (\delta E - \mu \delta N) \delta N \rangle / T^2 \\
\langle (\delta E - \mu \delta N) \delta N \rangle / T & \langle (\delta E - \mu \delta N)^2 \rangle / T^2
\end{pmatrix}
\] (B.105)

This resembles Eq. (3.49)

The equations of motion for local energy and number densities are conservation laws

\[
\begin{align*}
\frac{\partial N}{\partial t} + \nabla \cdot J &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot J_E &= 0
\end{align*}
\] (B.106)

where the currents are specified by their response to the external sources. Here in the hydrodynamic limit, \(N, J, E, J_E\) will be considered as small fluctuations. We define \(J_Q = J_E - \mu J\)

\[
\begin{pmatrix}
J \\
J_Q
\end{pmatrix} = -\Sigma
\begin{pmatrix}
\nabla \mu \\
\nabla T
\end{pmatrix}
\] (B.107)

where \(\Sigma\) is the conductivity matrix:

\[
\Sigma \equiv \begin{pmatrix}
\sigma & \alpha \\
\alpha T & \kappa
\end{pmatrix}
\] (B.108)

The static function measures the response of particle number and energy

\[
\begin{pmatrix}
\nabla N \\
\nabla E - \mu \nabla N
\end{pmatrix} = \chi_s
\begin{pmatrix}
\nabla \mu \\
\nabla T
\end{pmatrix}
\] (B.109)

The second row can be verified by \(dE - \mu dN = T dS\) and the Maxwell relation \(\frac{\partial N}{\partial T} = \frac{\partial S}{\partial \mu} T\).
So the drift current can be expressed as

\[
\begin{pmatrix}
J \\
J_Q
\end{pmatrix} = -\Sigma \chi_s^{-1} \begin{pmatrix}
\nabla N \\
\nabla E - \mu \nabla N
\end{pmatrix}
\] (B.110)

From the continuation equation, we have

\[
\begin{pmatrix}
\partial N/\partial t \\
\partial E/\partial t - \mu \partial Q/\partial t
\end{pmatrix} = - \begin{pmatrix}

abla \cdot J \\
\nabla \cdot J_Q
\end{pmatrix} = D \begin{pmatrix}
\nabla^2 N \\
\nabla^2 E - \mu \nabla^2 N
\end{pmatrix}
\] (B.111)

where \( D = \Sigma \chi_s^{-1} \) and we have ignored higher order term \( \nabla \mu \cdot \nabla N \).

Now we will relate diffusion constant, static susceptibility with dynamical susceptibility.

We first consider simple diffusion of local density \( n(t, \mathbf{x}) \)

\[
\partial_t n - D \nabla^2 n = 0
\] (B.112)

If we consider the Fourier component in space,

\[
n(t, \mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i \mathbf{k} \cdot \mathbf{x}} n(t, \mathbf{k})
\] (B.113)

we obtain the solution

\[
n(t, \mathbf{k}) = e^{-Dk^2 t} n_0(\mathbf{k})
\] (B.114)

where \( n_0(\mathbf{k}) \equiv n(t = 0, \mathbf{k}) \). We will further work in frequency domain, and \( n(t, \mathbf{k}) \) is only defined for \( t > 0 \). So we can perform a Laplace transformation

\[
n(z, \mathbf{k}) = \int dt e^{izt} n(t, \mathbf{k})
\] (B.115)

which gives the solution

\[
n(z, \mathbf{k}) = \frac{n_0(\mathbf{k})}{-iz + Dk^2}
\] (B.116)
Here $n_0(k)$ should be considered as the fluctuation above an uniform background. Then we can further express $n_0(k)$ as $n_0(k) = \chi \mu_0(k)$, where $\chi$ is the static susceptibility. We consider the source for the density has the form $\mu(t, x) = e^{et} \mu(x) \theta(-t)$, from linear response theory, we obtain

$$\langle n(t, x) \rangle = -\int_{-\infty}^{0} dt' e^{t'} \int d^d x' \mu(x') G^R_{nn}(t - t', x - x')$$  \hspace{1cm} (B.117)$$

where $G_{nn}^R$ is the retarded Green’s function

$$G_{nn}^R(t - t', x - x') = -i \delta(t - t') \langle [n(t, x), n(t', x')] \rangle$$  \hspace{1cm} (B.118)$$

Our goal is to express $\langle n(t, k) \rangle$ in terms of the retarded Green’s function and compare with form obtained by the diffusion equation Eq. (B.116). Performing the Fourier transformation in space

$$\langle n(t, k) \rangle = -\int_{-\infty}^{0} dt' e^{t'} \mu_0(k) G_{nn}^R(t - t', k)$$  \hspace{1cm} (B.119)$$

Integrating $t'$, we obtain

$$\langle n(t, k) \rangle = -\mu_0(k) \int \frac{d\omega}{2\pi} G_{nn}^R(\omega, k) \frac{e^{-i\omega t}}{i\omega + \epsilon}$$  \hspace{1cm} (B.120)$$

Performing a Laplace transformation

$$\langle n(z, k) \rangle = -\mu_0(k) \int \frac{d\omega}{2\pi} \frac{G_{nn}^R(\omega, k)}{(i\omega + \epsilon)(i\omega - z) + \epsilon}$$  \hspace{1cm} (B.121)$$

Using the property that $G_{nn}^R(\omega, k)$ is analytic in the upper half plane,

$$\langle n(z, k) \rangle = -\mu_0(k) \frac{G_{nn}^R(z, k) - G_{nn}^R(z = 0, k)}{iz}$$  \hspace{1cm} (B.122)$$
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Comparing with Eq. (B.116)

\[ G_{nn}^R(z, k) - G_{nn}^R(z = 0, k) = \frac{-iz\chi}{-iz + Dk^2} \]  

(B.123)

On the other hand, Eq. (B.119) tells us that \( G_{nn}^R(z = 0, k) = -\chi \) for small momentum \( k \), thus

\[ G_{nn}^R(z, k) = \frac{\chi Dk^2}{iz - Dk^2} \]  

(B.124)

We can see that \( G_{nn}^R(\omega, k) \) is analytic in the upper half plane and the analytical continuation function has a pole at \( \omega = -iDk^2 \) corresponding to the diffusive mode.

From the above relation, we can obtain the standard Kubo formula for the diffusion constant

\[ D\chi = -\lim_{\omega \to 0} \lim_{k \to 0} \frac{\omega}{k^2} \text{Im}G_{nn}^R(\omega, k) \]  

(B.125)

The above analysis can be generalized to a matrix form where we have several fields at present. Then we have the matrix relation

\[ G^R(z, k) = -(1 + izK^{-1})\chi \]  

(B.126)

where \( K_{ab} = -iz\delta_{ab} + D_{ab}(k) \). Here the retarded Green's function is the same as the dynamical susceptibility \( -\chi(\omega, k) \). And we have resembled Eq. (3.53).
Appendix for chapter 4

C.1 The large $\hat{q}$ limit of the supersymmetric SYK model

It is interesting to take the large $\hat{q}$ limit of the model since then we can find an exact solution interpolating between the short and long distance behavior. The analysis is very similar to the one in [135]. We expand the functions as follows

$$G_{\psi\psi}(\tau) = \frac{1}{2}\epsilon(\tau) + \frac{1}{2\hat{q}}g_{\psi\psi}(\tau), \quad G_{bb}(\tau) = -\delta(\tau) + \frac{1}{2\hat{q}}g_{bb}$$

where we neglected higher order terms in the $1/\hat{q}$ expansion. We can then Fourier transform, compute $\Sigma_{\psi\psi}, \Sigma_{bb}$ to first order in the $1/\hat{q}$ expansion. This gives $\Sigma_{\psi\psi}(i\omega) = \frac{\omega^2}{2\hat{q}}[\text{sgn}g_{\psi\psi}](i\omega)$, and $\Sigma_{bb}(i\omega) = \frac{1}{2\hat{q}}g_{bb}(i\omega)$. Replacing this the equations (4.26) we find

$$\partial^2_{\tau}g_{\psi\psi} = J^2e^{2g_{\psi\psi}}, \quad g_{bb} = Je^{g_{\psi\psi}}, \quad J \equiv \frac{\hat{q}J}{2\hat{q}^2-2}$$

(C.2)
where we take the large $\hat{q}$ limit keeping $J$ fixed. The solution obeying the boundary conditions $g_{\psi\psi}(0) = g_{\psi\psi}(\beta) = 0$ is

$$e^{g_{\psi\psi}} = \frac{1}{\beta J} \frac{v}{\sin(v^2 + b)}, \quad \beta J = \frac{v}{\cos \frac{v}{2}}, \quad b = \frac{\pi - v}{2}$$ (C.3)

where $v, b$ are integration constants fixed by the boundary conditions. It is interesting to note that the UV supersymmetry condition $g_{b\psi} = -\partial_\tau g_{\psi\psi}$ is only approximately true at short distances, distances shorter than the temperature.

It is also interesting to compute the free energy. Again, this is conveniently done by taking a derivative with respect to $J$ and using the equations of motion.

$$J \partial_J \left( \frac{\log Z}{N} \right) = -\frac{\beta}{2(q - 1)} \partial_\tau G_{\psi\psi} \bigg|_{\tau=0^+} = -\frac{\beta}{2q^2} \partial_\tau g_{\psi\psi} \bigg|_{\tau=0^+}$$ (C.4)

where the first equality holds in general and the second only for large $\hat{q}$. Expressing it in term of the parameters in (C.3) we get

$$\frac{\log Z}{N} = \frac{1}{2} \log 2 + \frac{1}{4q^2} \left( -\frac{v^2}{4} + v \tan \frac{v}{2} \right), \quad \text{with} \quad \beta J = \frac{v}{\cos \frac{v}{2}}$$

$$\sim \frac{1}{2} \log 2 + \frac{1}{q^2} \left[ \frac{\beta J}{4} - \frac{\pi^2}{16} + \frac{\pi^2}{8\beta J} - \frac{\pi^2}{4(\beta J)^2} + \cdots \right], \quad \text{for} \quad \beta J \gg 1$$ (C.5)

we can also easily compute the small $(\beta J)$ expansion, which, as expected, goes in powers of $(\beta J)^2$. We have used the entropy of the free fermion system, at $\beta J \to 0$, as an integration constant in going from (C.4) to (C.5). The constant term in the large $(\beta J)$ expansion agrees with the large $\hat{q}$ expansion of the ground state entropy (4.38). The $1/(\beta J)$ term can also be obtained form the Schwarzian and this can serve as a way to fix the coefficient of the Schwarzian action at large $\hat{q}$. The linear term in $\beta J$ represents the ground state energy and it should be subtracted off.

All these results have the same form as the large $q$ limit of the usual SYK model [135]. This is not a coincidence. What happens is that the leading boson propagator is simply the
Appendix C. Appendix for chapter 4

delta function in (C.1) which collapses the diagrams to those of the large $q$ limit of the usual SYK model.
Appendix for chapter 5

D.1 The large $q$ limit of the $MN$ model

We follow the same analysis as in Chapter 4. We expand the Green’s function as

$$G_f(\tau) = \frac{1}{2} \text{sgn}(\tau)(1 + \frac{g_f(\tau)}{q}), \quad G_b(\tau) = -\delta(\tau) + \frac{1}{2q} g_b(\tau)$$  \hspace{1cm} (D.1)

We will take the large $q$ while keeping $J$ fixed. Now the SD equations reduce to

$$\Sigma_f(\tau) = \sqrt{\alpha} \frac{J}{2} \frac{g_b(\tau)}{q} e^{g_f(\tau)}, \quad \Sigma_b(\tau) = \frac{1}{\sqrt{\alpha}} \frac{J}{2} \frac{e^{g_f(\tau)}}{q}$$  \hspace{1cm} (D.3)
and also

$$\Sigma_f(i\omega) = -\frac{\omega^2}{2q} \text{sgn } g_f, \quad \Sigma_b(i\omega) = \frac{1}{2q} g_b(i\omega) \tag{D.4}$$

Combined together, we have

$$g_b(\tau) = \frac{J e^{g_f(\tau)}}{\sqrt{\alpha}}, \quad \partial_\tau^2 \left[ \text{sgn } g_f(\tau) \right] = \sqrt{\alpha} g_b(\tau) J e^{g_f(\tau)} = J^2 e^{2g_f(\tau)} \tag{D.5}$$

With the boundary condition $g_f(0) = g_f(\beta) = 0$, we find

$$e^{g_f(\tau)} = \frac{1}{\beta J} \frac{v}{\sin \left( \frac{v \pi}{2\beta} + b \right)}, \quad b = \frac{\pi - v}{2}, \quad \beta J = \frac{v}{\cos \frac{v}{2}}, \quad g_b(\tau) = \frac{1}{\sqrt{\alpha \beta \sin \left( \frac{v \pi}{2} + b \right)}} \tag{D.6}$$

Notice that this solution is quite similar as the supersymmetric large $q$ solution. To be more explicit, we have the same solution $g_f$, while $g_b$ just differs by a factor $\frac{1}{\alpha}$. But because of this, to the $\frac{1}{q}$ order, the supersymmetric constrain is not satisfied $g_b(\tau) \neq -\partial_\tau g_f(\tau)$.

Another thing we have noticed is that, the solution is close to the conformal solution with $\Delta_f = \frac{1}{2q}$. Here we are taking the large $q$ limit while keeping fixed $\alpha$, it is not the small $\alpha$ limit we considered before. In fact, $\alpha$ will appear in the $O(q^{-2})$ order in $\Delta_f$. 

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Appendix for chapter 6

E.1 Spectral functions

This appendix is similar to Appendix. B.1, here we also extend the analysis to bosonic case.

At the Matsubara frequencies, the fermion Green’s function is defined by

$$G(\tau) = -\langle T (f(\tau)f^\dagger(0)) \rangle$$  \hspace{1cm} (E.1)

$$G(i\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n \tau} G(\tau),$$  \hspace{1cm} (E.2)

and this is continued to all complex frequencies $z$ via the spectral representation

$$G(z) = \int_{-\infty}^{\infty} d\Omega \frac{\rho(\Omega)}{\pi} \frac{1}{z - \Omega}.$$  \hspace{1cm} (E.3)

The spectral density $\rho(\Omega) > 0$ for all real $\Omega$ and $T$. The retarded Green’s function is
Appendix E. Appendix for chapter 6

\[ G^R(\omega) = G(\omega + i\eta) \] with \( \eta \) a positive infinitesimal, while the advanced Green’s function is \( G^A(\omega) = G(\omega - i\eta) \). From these expressions we obtain

\[ G(\tau) = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \rho(\Omega) \frac{e^{(\beta-\tau)\Omega}}{e^{\beta\Omega} + 1}, \quad 0 < \tau < \beta. \] (E.4)

So in the limit \( T \to 0 \) we have

\[
G(\tau) = \begin{cases} 
-\int_0^\infty \frac{d\Omega}{\pi} \rho(\Omega) e^{-\Omega \tau}, & \tau > 0 \\
\int_0^\infty \frac{d\Omega}{\pi} \rho(-\Omega) e^{\Omega \tau}, & \tau < 0.
\end{cases} \] (E.5)

We will focus on the particle-hole symmetric case, in which case \( \rho(\Omega) = \rho(-\Omega) \).

For bosons, the Green’s function is defined by

\[ \chi(\tau) = \langle T (\phi(\tau)\phi(0)) \rangle \] (E.6)

\[ \chi(i\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n \tau} \chi(\tau), \] (E.7)

and we have the spectral representation

\[ \chi(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\zeta(\Omega)}{z - \Omega}. \] (E.8)

Now the positivity condition is \( \Omega \zeta(\Omega) < 0 \). For the real bosons \( \zeta(-\Omega) = -\zeta(\Omega) \), and we will assume this from now. The analog of Eq. (E.4) is

\[ \chi(\tau) = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \zeta(\Omega) \frac{e^{(\beta-\tau)\Omega}}{e^{\beta\Omega} - 1}, \quad 0 < \tau < \beta. \] (E.9)
So in the limit $T \to 0$ we have

$$
\chi(\tau) = \begin{cases} 
- \int_0^\infty \frac{d\Omega}{\pi} \zeta(\Omega) e^{-\Omega \tau}, & \tau > 0 \\
\int_0^\infty \frac{d\Omega}{\pi} \zeta(-\Omega) e^{\Omega \tau}, & \tau < 0.
\end{cases}
$$

(E.10)

### E.2 High temperature expansion

We consider the high-temperature regime where the solution is close to the free solution and we can expand the solution in terms of $J^2$ and $\tilde{J}^2$. The free solutions are

$$
G_0(\omega_n) = \frac{1}{i\omega_n}, \quad \chi_0(\omega_n) = \frac{1}{\omega_n^2 + \Delta_0^2}
$$

(E.11)

Here we denote $\Delta_0^2 = \chi_0^{-1}$. We systematically do the high temperature expansion by defining $G(\omega_n) = \sum_{i=0,j=0} J^{2i}\tilde{J}^{2j} G^{(i,j)}(\omega_n)$, $\chi(\omega_n) = \sum_{i=0,j=0} J^{2i}\tilde{J}^{2j} \chi^{(i,j)}(\omega_n)$, $\Sigma(\omega_n) = \sum_{i=0,j=0} J^{2i}\tilde{J}^{2j} \Sigma^{(i,j)}(\omega_n)$ and $P(\omega_n) = \sum_{i=0,j=0} J^{2i}\tilde{J}^{2j} P^{(i,j)}(\omega_n)$. Then we plug in the Schwinger-Dyson equations Eqs. (6.18)(6.19), and obtain the expansion order by order. The zeroth order is the free result:

$$
G^{(0,0)}(\omega_n) = \frac{1}{i\omega_n}, \quad \chi^{(0,0)}(\omega_n) = \frac{1}{\omega_n^2 + \Delta_0^2}
$$

(E.12)

Plugging $\chi^{(0,0)}(\omega_n)$ in the constrain equation Eq. (6.22), we obtain:

$$
\frac{1}{2\Delta_0} \coth \frac{\Delta_0}{2T} = \frac{1}{g}
$$

(E.13)
The next order we find that

\[
\Sigma^{(1,0)} = \frac{1}{4i\omega_n}, \quad \Sigma^{(0,1)} = k \left[ \frac{1}{2i\Delta_0^2\omega_n} + \frac{\cosh \frac{\Delta_n}{T}}{2i\Delta_0^2\omega_n(4\Delta_0^2 + \omega_n^2) \sinh \frac{\Delta_n^2}{2T}} \right], \quad P^{(0,1)} = \frac{1}{2(\omega_n^2 + \Delta_0^2)}
\]  
(E.14)

This gives:

\[
G^{(1,0)} = -\frac{1}{4i\omega_n^2}, \quad G^{(0,1)} = -\frac{k}{\omega_n^2} \left[ \frac{1}{2i\Delta_0^2\omega_n} + \frac{\cosh \frac{\Delta_n}{T}}{2i\Delta_0^2\omega_n(4\Delta_0^2 + \omega_n^2) \sinh \frac{\Delta_n^2}{2T}} \right], \quad \chi^{(0,1)} = \frac{1}{2(\omega_n^2 + \Delta_0^2)^3}
\]  
(E.15)

The summation of \(\chi^{(0,0)}\) and \(\chi^{(0,1)}\) gives

\[
\frac{1}{2\Delta_0} \coth \frac{\Delta_0}{2T} + \frac{T}{6} \coth \frac{\Delta_n}{T} \left( 3 + \frac{\Delta_n}{T} \coth \frac{\Delta_n}{T} / \sinh \frac{\Delta_n^2}{2T} \right) = \frac{1}{g}
\]  
(E.16)

Further we have the expansion

\[
P^{(1,1)} = \frac{2T\Delta(-3\omega_n^2 + \Delta^2) - (\omega_n^4 + \Delta^4) \coth \frac{\Delta}{2T}}{16T\Delta(\omega_n^2 + \Delta^2)^3}
\]  
(E.17)

This gives

\[
\chi^{(1,1)} = \frac{2T\Delta(-3\omega_n^2 + \Delta^2) - (\omega_n^4 + \Delta^4) \coth \frac{\Delta}{2T}}{16T\Delta(\omega_n^2 + \Delta^2)^5}
\]  
(E.18)

The summation of \(\chi^{(0,0)}\), \(\chi^{(0,1)}\) and \(\chi^{(1,1)}\) gives

\[
\frac{1}{2\Delta_0} \coth \frac{\Delta_0}{2T} + \frac{T}{6} \coth \frac{\Delta_n}{T} \left( 3 + \frac{\Delta_n}{T} \coth \frac{\Delta_n}{T} / \sinh \frac{\Delta_n^2}{2T} \right) + \frac{T}{6} \left( 5T \coth \left( \frac{\Delta}{2T} \right) - 4\Delta \right) - \Delta \left( 9T^3 + \Delta (\Delta^2 + 3T^2) \coth \left( \frac{\Delta}{2T} \right) + 2\Delta^2 T \right) \text{csch}^2 \left( \frac{\Delta}{2T} \right) = \frac{1}{g}
\]  
(E.19)
Figure E.1: $g_{HTE} - g_{num}$ as a function of temperature for fixed $k = 1$, $J = 5$, $\tilde{t} = 0.3$, $\Delta_0 = 2$. $g_{num}$ is obtained by numerically solving EOM (6.18)(6.19) and plugging in (6.22); $g_{HTE}$ to get blue line is obtained from (E.13), red line is obtained from (E.16), orange line is obtained from (E.19).

We plot the difference between the numerical result and the high temperature expansion (E.13)(E.16)(E.19) in Figure E.1. We see that they agree relatively well in the high temperature regime.
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