Entanglement, holography, and the quantum phases of matter

Utrecht University, June 13, 2012

Subir Sachdev

Lecture at the 100th anniversary Solvay conference, Theory of the Quantum World
arXiv:1203.4565

sachdev.physics.harvard.edu
Modern phases of quantum matter
Not adiabatically connected
to independent electron states:

many-particle
quantum entanglement
“Complex entangled” states of quantum matter in $d$ spatial dimensions

Gapped quantum matter
Spin liquids, quantum Hall states

Conformal quantum matter
Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter
Strange metals in higher temperature superconductors, spin liquids
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topological field theory

conformal field theory

?
$|\Psi\rangle \Rightarrow \text{Ground state of entire system, }$

$\rho = |\Psi\rangle \langle \Psi|$

$\rho_A = \text{Tr}_B \rho = \text{density matrix of region } A$

**Entanglement entropy** $S_E = -\text{Tr} (\rho_A \ln \rho_A)$
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Band insulators

An even number of electrons per unit cell
$S_E = aP - b \exp(-cP)$

where $P$ is the surface area (perimeter) of the boundary between A and B.
Mott insulator

Emergent excitations

An odd number of electrons per unit cell but electrons are localized by Coulomb repulsion; state has long-range entanglement
Mott insulator: Kagome antiferromagnet

\[ H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \]
Mott insulator: Kagome antiferromagnet

\[ H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \]

\[ \bullet \bullet = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]

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\[
\mathbf{\hat{S}} = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
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P. Fazekas and P. W. Anderson, 
*Philos. Mag.* 
Mott insulator: Kagome antiferromagnet

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Mott insulator: Kagome antiferromagnet

Alternative view

Pick a reference configuration

Mott insulator: Kagome antiferromagnet

Alternative view

A nearby configuration

Mott insulator: Kagome antiferromagnet

Alternative view

Difference: a closed loop

D. Rokhsar and S. Kivelson,
Mott insulator: Kagome antiferromagnet

Alternative view

Ground state: sum over closed loops

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Quantum “disordered” state with exponentially decaying spin correlations.

non-collinear Néel state

Quantum “disordered” state with exponentially decaying spin correlations.
Mott insulator: Kagome antiferromagnet

Entangled quantum state: $\mathbb{Z}_2$ spin liquid.

non-collinear Néel state

Mott insulator: Kagome antiferromagnet

\[ \mathbb{Z}_2 \text{ spin liquid: parton construction} \]

Write spin operators in terms of \( S = 1/2 \) ‘partons’

\[ \vec{S}_i = \frac{1}{2} b_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} b_{i\beta}. \]

The ground state is

\[ |\Psi\rangle = \mathcal{P}_{n_b=1} \exp \left( f(i - j) \varepsilon^{\alpha\beta} b_{i\alpha}^\dagger b_{j\alpha}^\dagger \right) |0\rangle \]

Leads to a description of fractionalized ‘spinon’ and ‘vison’ excitations coupled to an emergent \( \mathbb{Z}_2 \) gauge field.


Entanglement in the $\mathbb{Z}_2$ spin liquid
Entanglement in the $\mathbb{Z}_2$ spin liquid

Sum over closed loops: only an even number of links cross the boundary between A and B
Entanglement in the $Z_2$ spin liquid

\[ S_E = aP - \ln(2) \]

where $P$ is the surface area (perimeter) of the boundary between A and B.

Entanglement in the $\mathbb{Z}_2$ spin liquid

\[ S_E = aP - \ln(4) \]

where $P$ is the surface area (perimeter) of the boundary between A and B.

Entanglement in the $Z_2$ spin liquid

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Mott insulator: Kagome antiferromagnet

Strong numerical evidence for a $\mathbb{Z}_2$ spin liquid


Hong-Chen Jiang, Z. Wang, and L. Balents, arXiv:1205.4289

Mott insulator: Kagome antiferromagnet

Evidence for spinons
Young Lee,
APS meeting, March 2012

ZnCu$_3$(OH)$_6$Cl$_2$ (also called Herbertsmithite)

![Graphical representation of ZnCu$_3$(OH)$_6$Cl$_2$]

![Intensity plot for ZnCu$_3$(OH)$_6$Cl$_2$ at different temperatures]

(a) 6meV 1.5K
(b) 2meV 1.5K

(H H O)

Intensity (arb. units)
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Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Examine ground state as a function of \( \lambda \)
Quantum critical point described by a CFT3 (O(3) Wilson-Fisher)

Entanglement entropy obeys $S_E = aP - \gamma$, where $\gamma$ is a shape-dependent universal number associated with the CFT3.

Tensor network representation of entanglement at quantum critical point

$d$-dimensional space

Depth of entanglement

Tensor network representation of entanglement at quantum critical point

$d$-dimensional space

depth of entanglement

Brian Swingle, arXiv:0905.1317
Tensor network representation of entanglement at quantum critical point

Most links describe entanglement within A

$d$-dimensional space

depth of entanglement

Brian Swingle, arXiv:0905.1317
Tensor network representation of entanglement at quantum critical point

Links overestimate entanglement between A and B

Brian Swingle, arXiv:0905.1317
Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

Tensor network representation of entanglement at quantum critical point

$d$-dimensional space

depth of entanglement

Brian Swingle, arXiv:0905.1317
Key idea: ⇒ Implement $r$ as an extra dimension, and map to a local theory in $d + 2$ spacetime dimensions.
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation (\(i = 1 \ldots d\))

\[ x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds \]
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\[ (i = 1 \ldots d) \]

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This gives the unique metric

\[ ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2) \]

Reparametrization invariance in $r$ has been used to the prefactor of $dx_i^2$ equal to $1/r^2$. This fixes $r \rightarrow \zeta r$ under the scale transformation. This is the metric of the space $\text{AdS}_{d+2}$. 
AdS/CFT correspondence

$AdS_4$

$R^{2,1}$

Minkowski

CFT$_3$
AdS/CFT correspondence

$\text{AdS}_4$  \hspace{2cm} $\mathbb{R}^{2,1}$

Minkowski

CFT3

$A$

$r$
Associate entanglement entropy with an observer in the enclosed spacetime region, who cannot observe “outside” : i.e. the region is surrounded by an imaginary horizon.

AdS/CFT correspondence

The entropy of this region is bounded by its surface area (Bekenstein-Hawking-’t Hooft-Susskind)

AdS/CFT correspondence

Minimal surface area measures entanglement entropy

Entanglement entropy

Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

Brian Swingle, arXiv:0905.1317
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Emergent direction of AdS_{d+2}
AdS/CFT correspondence

Computation of minimal surface area yields

\[ S_E = aP - \gamma, \]

where \( \gamma \) is a shape-dependent universal number.

Computation of minimal surface area, or direct computation in CFT2, yield \( S_E = (c/6) \ln P \), where \( c \) is the central charge.

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The Fermi liquid
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- Area enclosed by the Fermi surface $\mathcal{A} = \mathcal{Q}$, the fermion density
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- Area enclosed by the Fermi surface $A = Q$, the fermion density
- Particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|$. 
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- Particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|$.

- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T^{d_{\text{eff}}}$ with $d_{\text{eff}} = 1$. 
Logarithmic violation of “area law”:

\[ S_E = \frac{1}{12} (k_F P) \ln(k_F P) \]

for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape.

Strange metals

To obtain a compressible state which is not a Fermi liquid, take a Fermi surface in $d = 2$, and couple it to any gapless scalar field, $\phi$, which has low energy excitations near $\mathbf{q} = 0$. 
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To obtain a compressible state which is not a Fermi liquid, take a Fermi surface in $d = 2$, and couple it to any gapless scalar field, $\phi$, which has low energy excitations near $\mathbf{q} = 0$. The field $\phi$ could represent

- ferromagnetic order
- breaking of point-group symmetry (Ising-nematic order)
- breaking of time-reversal symmetry
- circulating currents
- transverse component of an Abelian or non-Abelian gauge field.
- ...
Strange metals

- Area enclosed by the Fermi surface $\mathcal{A} = Q$, the fermion density
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• Particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$; three-loop computation shows $z = 3/2$.

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Entanglement entropy of Fermi surfaces

Logarithmic violation of “area law”: \( S_E = \frac{1}{12} (k_F P) \ln (k_F P) \)

for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape.

Non-Fermi liquids have, at most, the “1/12” prefactor modified.

Holography

$\mathbf{r}$
Consider the metric which transforms under rescaling as

\[ x_i \rightarrow \zeta x_i \]
\[ t \rightarrow \zeta^z t \]
\[ ds \rightarrow \zeta^\theta/d ds. \]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

\( \theta \) is the violation of hyperscaling exponent.
Consider the metric which transforms under rescaling as

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\[ ds \rightarrow \zeta^{\theta/d} ds. \]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

\( \theta \) is the violation of hyperscaling exponent.

The most general choice of such a metric is

\[
\begin{align*}
 ds^2 &= \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + d\bar{x}_i^2 \right)
\end{align*}
\]

We have used reparametrization invariance in \( r \) to choose so that it scales as \( r \rightarrow \zeta^{(d-\theta)/d} r \).
\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2 d(z-1)/(d-\theta)} + r^2 \frac{\theta}{(d-\theta)} dr^2 + dx_i^2 \right) \]

- The thermal entropy density scales as
  \[ S \sim T^{(d-\theta)/z}. \]

  The third law of thermodynamics requires \( \theta < d \).

- The entanglement entropy, \( S_E \), of an entangling region with boundary surface ‘area’ \( P \) scales as
  \[ S_E \sim \begin{cases} 
  P & \text{, for } \theta < d - 1 \\
  P \ln P & \text{, for } \theta = d - 1 \\
  P^{\theta/(d-1)} & \text{, for } \theta > d - 1 
  \end{cases} \]

  All local quantum field theories obey the “area law” (upto log violations) and so \( \theta \leq d - 1 \).

- The null energy condition implies \( z \geq 1 + \frac{\theta}{d} \).
The value of $\theta$ is fixed by requiring that the thermal entropy density $S \sim T^{1/z}$ for general $d$.

Conjecture: this metric then describes a compressible state with a hidden Fermi surface of quarks coupled to gauge fields

The value of $\theta$ is fixed by requiring that the thermal entropy density $S \sim T^{1/z}$ for general $d$. Conjecture: this metric then describes a compressible state with a hidden Fermi surface of quarks coupled to gauge fields.

The null energy condition yields the inequality $z \geq 1 + \theta/d$. For $d = 2$ and $\theta = 1$ this yields $z \geq 3/2$. The field theory analysis gave $z = 3/2$ to three loops!

The entanglement entropy exhibits logarithmic violation of the area law only for this value of $\theta$!!

The logarithmic violation is of the form $P \ln P$, where $P$ is the perimeter of the entangling region. This form is independent of the shape of the entangling region, just as is expected for a (hidden) Fermi surface!!

$$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + r^{2\theta}/(d-\theta) dr^2 + dx_i^2 \right)$$

$\theta = d - 1$

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$$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + \frac{r^{2\theta/(d-\theta)} dr^2 + dx_i^2}{d^2 - \theta} \right)$$

$\theta = d - 1$

This metric can be realized in a Maxwell-Einstein-dilaton theory, which may be viewed as a "bosonization" of the non-Fermi liquid state. The entanglement entropy of this theory has log-violation of the area law with

\[ S_E = \Xi \frac{Q^{(d-1)/d}}{P} \ln P, \]

where \( P \) is surface area of the entangling region, and \( \Xi \) is a dimensionless constant which is independent of all UV details, of \( Q \), and of any property of the entangling region. Note \( Q^{(d-1)/d} \sim k_F^{d-1} \) via the Luttinger relation, and then \( S_E \) is just as expected for a Fermi surface !!!!
Gauss Law and the “attractor” mechanism ⇔ Luttinger theorem on the boundary
Conclusions

Gapped quantum matter

Numerical and experimental observation of a spin liquid on the kagome lattice. Likely a $\mathbb{Z}_2$ spin liquid.
Conclusions

Conformal quantum matter

Numerical and experimental observation in coupled-dimer antiferromagnets, and at the superfluid-insulator transition of bosons in optical lattices.
Conclusions

Compressible quantum matter

Field theory of a non-Fermi liquid obtained by coupling a Fermi surface to a gapless scalar field with low energy excitations near zero wavevector. Obtained promising holographic dual of this theory.
**Conclusions**

**Compressible quantum matter**

Evidence for *hidden Fermi surfaces* in compressible states obtained for a class of holographic Einstein-Maxwell-dilaton theories. These theories describe a *non-Fermi liquid* (NFL) state of gauge theories at non-zero density.

After fixing $\theta = d - 1$ to obtain thermal entropy density $S \sim T^{1/z}$, we found:

- Log violation of the area law in entanglement entropy, $S_E$.
- Leading-log $S_E$ independent of shape of entangling region.
- The $d = 2$ bound $z \geq 3/2$, compared to $z = 3/2$ in three-loop field theory.
- Evidence for Luttinger theorem in prefactor of $S_E$. 

Wednesday, June 13, 2012
Thank you !