Outline

1. Introduction to the Hubbard model
   *Superexchange and antiferromagnetism*

2. Coupled dimer antiferromagnet
   *CFT3: the Wilson-Fisher fixed point*

3. Honeycomb lattice: semi-metal and antiferromagnetism
   *CFT3: Dirac fermions and the Gross-Neveu model*

4. Quantum critical dynamics
   *AdS/CFT and the collisionless-hydrodynamic crossover*

5. Hubbard model as a SU(2) gauge theory
   *Spin liquids, valence bond solids: analogies with SQED and SYM*
Outline

6. Square lattice: Fermi surfaces and spin density waves
   *Fermi pockets and Quantum oscillations*

7. Instabilities near the SDW critical point
   *d-wave superconductivity and other orders*

8. Global phase diagram of the cuprates
   *Competition for the Fermi surface*
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   Competition for the Fermi surface
The Hubbard Model

\[ H = - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_{i} \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_{i} c_{i\alpha}^\dagger c_{i\alpha} \]

\( t_{ij} \rightarrow \) “hopping”. \( U \rightarrow \) local repulsion, \( \mu \rightarrow \) chemical potential

Spin index \( \alpha = \uparrow, \downarrow \)

\[ n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha} \]

\[ c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta}^\dagger c_{i\alpha} = \delta_{ij} \delta_{\alpha\beta} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \]

Will study on the honeycomb and square lattices
The Hubbard Model

\[ H = - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha} \]

\( t_{ij} \rightarrow \text{“hopping”}. \) \( U \rightarrow \text{local repulsion}, \mu \rightarrow \text{chemical potential} \)

Spin index \( \alpha = \uparrow, \downarrow \)

\[ n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha} \]

\[ c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta} c_{i\alpha}^\dagger = \delta_{ij} \delta_{\alpha\beta} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \]

Will study on the honeycomb and \textit{square} lattices
The electron spin polarization obeys

$$\langle \vec{S}(r, \tau) \rangle = \varphi(r, \tau)e^{iK \cdot r}$$

where $K$ is the ordering wavevector.
Effective Hamiltonian for quasiparticles:

\[ H_0 = - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \equiv \sum_k \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} \]

with \( t_{ij} \) non-zero for first, second and third neighbor, leads to satisfactory agreement with experiments. The area of the occupied electron states, \( A_e \), from Luttinger’s theory is

\[ A_e = \begin{cases} 
2\pi^2(1 - p) & \text{for hole-doping } p \\
2\pi^2(1 + x) & \text{for electron-doping } x 
\end{cases} \]

The area of the occupied hole states, \( A_h \), which form a closed Fermi surface and so appear in quantum oscillation experiments is \( A_h = 4\pi^2 - A_e \).
Spin density wave theory

In the presence of spin density wave order, \( \varphi \) at wavevector \( \mathbf{K} = (\pi, \pi) \), we have an additional term which mixes electron states with momentum separated by \( \mathbf{K} \)

\[
H_{\text{sdw}} = \varphi \cdot \sum_{\mathbf{k},\alpha,\beta} c_{\mathbf{k},\alpha}^\dagger \sigma_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}
\]

where \( \sigma \) are the Pauli matrices. The electron dispersions obtained by diagonalizing \( H_0 + H_{\text{sdw}} \) for \( \varphi \propto (0, 0, 1) \) are

\[
E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right)^2 + \varphi^2}
\]

This leads to the Fermi surfaces shown in the following slides for electron and hole doping.
Hole-doped cuprates

Increasing SDW order

Hole-doped cuprates

Increasing SDW order

Hole-doped cuprates

Increasing SDW order

Hot spots

Hole-doped cuprates

Fermi surface breaks up at hot spots into electron and hole “pockets”

Hole-doped cuprates

Fermi surface breaks up at hot spots into electron and hole “pockets”

Evidence for small Fermi pockets

Fermi liquid behaviour in an underdoped high Tc superconductor

Suchitra E. Sebastian, N. Harrison, M. M. Altarawneh, Ruixing Liang, D. A. Bonn, W. N. Hardy, and G. G. Lonzarich

arXiv:0912.3022

FIG. 2: Magnetic quantum oscillations measured in YBa$_2$Cu$_3$O$_{6+x}$ with $x \approx 0.56$ (after background polynomial subtraction). This restricted interval in $B = |B|$ furnishes a dynamic range of $\sim 50$ dB between $T = 1$ and 18 K. The actual $T$ values are provided in Fig. 3.
Hole-doped cuprates

Increasing SDW order

Large Fermi surface breaks up into electron and hole pockets

Hole-doped cuprates

Increasing SDW order

$\phi$ fluctuations act on the large Fermi surface

Start from the “spin-fermion” model

\[
Z = \int Dc_\alpha D\phi \exp (-S)
\]

\[
S = \int d\tau \sum_k c_{k\alpha}^\dagger \left( \frac{\partial}{\partial \tau} - \varepsilon_k \right) c_{k\alpha}
\]

\[
- \lambda \int d\tau \sum_i c_{i\alpha}^\dagger \vec{\phi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{iK \cdot r_i}
\]

\[
+ \int d\tau d^2 r \left[ \frac{1}{2} \left( \nabla_r \phi \right)^2 + \frac{\zeta}{2} \left( \partial_\tau \phi \right)^2 + \frac{s}{2} \phi^2 + \frac{u}{4} \phi^4 \right]
\]
Low energy fermions

\[ \psi_{1\alpha}, \psi_{2\alpha} \]
\[ \ell = 1, \ldots, 4 \]

\[ L_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

\[ \mathbf{v}_{1}^{\ell=1} = (v_x, v_y), \quad \mathbf{v}_{2}^{\ell=1} = (-v_x, v_y) \]
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger \left( \zeta \partial_\tau - iv_1 \cdot \nabla_r \right) \psi_{1\alpha} + \psi_{2\alpha}^\dagger \left( \zeta \partial_\tau - iv_2 \cdot \nabla_r \right) \psi_{2\alpha} \]
\[ \mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\bar{\zeta}}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:
\[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling:
\[ \mathcal{L}_c = -\lambda \varphi \cdot \left( \psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]
\[ \mathcal{L}_f = \psi_{1\alpha}^{l\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^l \cdot \nabla_r) \psi_{1\alpha}^l + \psi_{2\alpha}^{l\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^l \cdot \nabla_r) \psi_{2\alpha}^l \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi^2) + \frac{\zeta}{2} (\partial_\tau \varphi^2) + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi \cdot \left( \psi_{1\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^l + \psi_{2\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^l \right) \]

**Hertz theory**

Integrate out fermions and obtain non-local corrections to \( \mathcal{L}_\varphi \)

\[ \mathcal{L}_\varphi = \frac{1}{2} \varphi^2 \left[ \mathbf{q}^2 + \gamma |\omega| \right] / 2 \]

\[ ; \quad \gamma = \frac{2}{\pi v_x v_y} \]

Exponent \( z = 2 \) and mean-field criticality (upto logarithms)
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\phi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \sigma_{\alpha\beta} \psi_{2\beta}^\dagger + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

Integrate out fermions and obtain non-local corrections to \( \mathcal{L}_\phi \)

\[ \mathcal{L}_\phi = \frac{1}{2} \bar{\varphi}^2 \left[ q^2 + \gamma |\omega| \right] / 2 \quad ; \quad \gamma = \frac{2}{\pi v_x v_y} \]

Exponent \( z = 2 \) and mean-field criticality (upto logarithms)

**OK in** \( d = 3 \), but higher order terms contain an infinite number of marginal couplings in \( d = 2 \)

Perform RG on both fermions and $\bar{\phi}$, using a *local* field theory.

$$\mathcal{L}_f = \psi_{1\alpha}^{l\dagger} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha}^l + \psi_{2\alpha}^{l\dagger} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha}^l$$

Order parameter:

$$\mathcal{L}_\phi = \frac{1}{2} (\nabla_r \bar{\phi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \bar{\phi})^2 + \frac{s}{2} \bar{\phi}^2 + \frac{u}{4} \bar{\phi}^4$$

“Yukawa” coupling:

$$\mathcal{L}_c = -\lambda \bar{\phi} \cdot \left( \psi_{1\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^l + \psi_{2\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^l \right)$$
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   *Competition for the Fermi surface*
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   *Competition for the Fermi surface*
Superconductivity by SDW fluctuation exchange
$d$-wave pairing near a spin-density-wave instability

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We investigate the three-dimensional Hubbard model and show that paramagnon exchange near a spin-density-wave instability gives rise to a strong singlet $d$-wave pairing interaction. For a cubic band the singlet ($d_{x^2-y^2}$ and $d_{3z^2-r^2}$) channels are enhanced while the singlet ($d_{xy},d_{xz},d_{yz}$) and triplet $p$-wave channels are suppressed. A unique feature of this pairing mechanism is its sensitivity to band structure and band filling.

Physical Review B 34, 8190 (1986)
Spin density wave theory in hole-doped cuprates
Spin-fluctuation exchange theory of d-wave superconductivity in the cuprates

Fermions at the large Fermi surface exchange fluctuations of the SDW order parameter $\tilde{\phi}$.

David Pines, Douglas Scalapino
Pairing by SDW fluctuation exchange

We now allow the SDW field $\vec{\varphi}$ to be dynamical, coupling to electrons as

$$H_{sdw} = - \sum_{k,q,\alpha,\beta} \vec{\varphi}_q \cdot c_{k,\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{k+K+q,\beta}.$$ 

Exchange of a $\varphi$ quantum leads to the effective interaction

$$H_{ee} = -\frac{1}{2} \sum_q \sum_{\gamma,\delta} \sum_{k,\alpha,\beta} V_{\alpha\beta,\gamma\delta}(q) c_{k,\alpha}^\dagger c_{k+q,\beta} c_{p,\gamma}^\dagger c_{p-q,\delta},$$

where the pairing interaction is

$$V_{\alpha\beta,\gamma\delta}(q) = \vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta} \frac{\chi_0}{\xi^{-2} + (q-K)^2},$$

with $\chi_0\xi^2$ the SDW susceptibility and $\xi$ the SDW correlation length.
In BCS theory, this interaction leads to the ‘gap equation’ for the pairing gap $\Delta_k \propto \langle c_{k\uparrow} c_{-k\downarrow} \rangle$.

$$\Delta_k = - \sum_p \left( \frac{3 \chi_0}{\xi^{-2} + (p - k - K)^2} \right) \frac{\Delta_p}{2 \sqrt{\varepsilon_p^2 + \Delta_p^2}}$$

Non-zero solutions of this equation require that $\Delta_k$ and $\Delta_p$ have opposite signs when $p - k \approx K$. 
Increasing SDW order

\[ \Gamma \]

\[ +\]

\[ \vec{\phi} \]

\[ K \]

\[ \langle c_{k\uparrow} c_{-k\downarrow} \rangle \propto \Delta_k = \Delta_0 (\cos(k_x) - \cos(k_y)) \]
\textbf{d-wave pairing in the theory of hotspots}

Low energy fermions

\[ \psi_{1\alpha}^\ell, \psi_{2\alpha}^\ell \]

\[ \ell = 1, \ldots, 4 \]
Hot spots have strong instability to $d$-wave pairing near SDW critical point. This instability is stronger than the BCS instability of a Fermi liquid.

Pairing order parameter: $\varepsilon^\alpha_\beta \left( \psi^3_1 \psi^1_\beta - \psi^3_2 \psi^1_\beta \right)$
$d$-wave Cooper pairing instability in particle-particle channel
Similar theory applies to the pnictides, and leads to $s_{\pm}$ pairing.
Emergent Pseudospin symmetry

Continuum theory of hotspots in invariant under:

\[
\begin{pmatrix}
\psi_{\ell}^\uparrow \\
\psi_{\ell\dagger}^\downarrow
\end{pmatrix}
\rightarrow
U^{\ell}
\begin{pmatrix}
\psi_{\ell}^\uparrow \\
\psi_{\ell\dagger}^\downarrow
\end{pmatrix}
\]

where \(U^{\ell}\) are arbitrary SU(2) matrices which can be different on different hotspots \(\ell\).
$d$-wave Cooper pairing instability in particle-particle channel
Bond density wave (with local Ising-nematic order) instability in particle-hole channel
$d$-wave pairing has a partner instability in the particle-hole channel

Density-wave order parameter:

$$\left( \psi_{1\alpha}^{3\dagger} \psi_{1\alpha}^{1} - \psi_{2\alpha}^{3\dagger} \psi_{2\alpha}^{1} \right)$$
Single ordering wavevector $Q$:

$$\langle c_{\mathbf{k}-Q/2,\alpha}^\dagger c_{\mathbf{k}+Q/2,\alpha}\rangle = \Phi(\cos k_x - \cos k_y)$$
No modulations on sites. Modulated bond-density wave with local Ising-nematic ordering:

\[
\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = \Phi(\cos k_x - \cos k_y)
\]
No modulations on sites. Modulated bond-density wave with local Ising-nematic ordering:

\[
\left\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \right\rangle = \Phi(\cos k_x - \cos k_y)
\]

“Bond density” measures amplitude for electrons to be in spin-singlet valence bond: VBS order
STM measurements of $Z(r)$, the energy asymmetry in density of states in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$.

STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$.

$O_N = Z_A + Z_B - Z_C - Z_D$

STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$.

Strong anisotropy of electronic states between $x$ and $y$ directions:

Electronic "Ising-nematic" order

$$O_N = Z_A + Z_B - Z_C - Z_D$$

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Antiferromagnetism

Fermi surface

d-wave superconductivity
Antiferromagnetism

Fermi surface

d-wave superconductivity
Theory of quantum criticality in the cuprates

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Spin density wave

Fermi surface

d-wave superconductivity
Theory of quantum criticality in the cuprates

Spin density wave (SDW)

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Theory of quantum criticality in the cuprates

Onset of $d$-wave superconductivity hides the critical point $x = x_m$
Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Spin density wave (SDW)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.


Theory of quantum criticality in the cuprates

\[ T* \]

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Spin density wave (SDW)

Competition between SDW order and superconductivity moves the actual quantum critical point to \( x = x_s < x_m \).


Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Magnetic quantum criticality

Spin gap

Thermally fluctuating SDW

Spin density wave (SDW)


Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.  

Monday, June 14, 2010
Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Thermally fluctuating SDW

Magnetic quantum criticality

Spin gap

Spin density wave (SDW)

Physics of competition: $d$-wave SC and SDW “eat up” same pieces of the large Fermi surface.


Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Magnetic quantum criticality

Spin gap

Thermally fluctuating SDW

d-wave SC

Fluctuating, paired Fermi pockets

$T^*$

$\mathcal{T}_S$

$\mathcal{T}_M$
Quantum oscillations
Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

Small Fermi pockets with pairing fluctuations

d-wave SC

T^*

T_{sdw}

SDW

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)


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**Magnetic field**

**AF+SC**

**d-wave**

**SC**

**Hole doping**

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"Normal" (Large Fermi surface)

Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Fluctuating, paired Fermi pockets

T

T*

x

x_s

x_m

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

d-wave SC

SC+ SDW

SDW

T_{sdw}
Similar phase diagram for CeRhIn$_5$
Similar phase diagram for the pnictides