Quantum entanglement
and the phases of matter

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Quantum superposition and entanglement
Quantum Superposition

The double slit experiment

Let $|L\rangle$ represent the state with the electron in the left slit.

And $|R\rangle$ represents the state with the electron in the right slit.

Actual state of the electron is $|L\rangle + |R\rangle$. 
Quantum Entanglement: quantum superposition with more than one particle
Quantum Entanglement: quantum superposition with more than one particle

Hydrogen atom:
Quantum Entanglement: quantum superposition with more than one particle

Hydrogen atom: \[ |\uparrow\rangle \]

Hydrogen molecule:

\[
\begin{align*}
&= \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)/\sqrt{2} \\
&= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\end{align*}
\]

Superposition of two electron states leads to non-local correlations between spins
Quantum Entanglement: quantum superposition with more than one particle
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Quantum Entanglement: quantum superposition with more than one particle

Einstein-Podolsky-Rosen “paradox”: Non-local correlations between observations arbitrarily far apart
Quantum superposition and entanglement
Quantum superposition and entanglement

Quantum critical points of electrons in crystals

String theory and black holes
Quantum superposition and entanglement

Quantum critical points of electrons in crystals

String theory and black holes
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Examine ground state as a function of \( \lambda \)
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

\[ = \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow \rangle - | \downarrow \uparrow \rangle \right) \]

At large \( \lambda \) ground state is a “quantum paramagnet” with spins locked in valence bond singlets
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Nearest-neighbor spins are “entangled” with each other. Can be separated into an Einstein-Podolsky-Rosen (EPR) pair.

\[ \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle - |\downarrow\uparrow\rangle) \]
For $\lambda \approx 1$, the ground state has antiferromagnetic ("Néel") order, and the spins align in a checkerboard pattern.
$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$

For $\lambda \approx 1$, the ground state has antiferromagnetic ("Néel") order, and the spins align in a checkerboard pattern

No EPR pairs
\[ \lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
Pressure in TlCuCl$_3$ 

\[
\lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
\]

TlCuCl$_3$

An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
TlCuCl$_3$

Quantum paramagnet at ambient pressure
TlCuCl$_3$

Neel order under pressure

\[ \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
Excitation spectrum in the paramagnetic phase

Spin $S = 1$
“triplon”
Excitation spectrum in the paramagnetic phase

Spin $S = 1$  
"triplon"
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

“triplon”
Excitation spectrum in the Néel phase

Spin waves

\( \lambda_c \)
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

Spin waves
Excitations of TlCuCl$_3$ with varying pressure

Excitations of $\text{TlCuCl}_3$ with varying pressure

Broken valence bond ("triplon") excitations of the quantum paramagnet

Excitations of TlCuCl$_3$ with varying pressure

Spin waves above the Néel state

Excitations of TlCuCl$_3$ with varying pressure

![Graph showing excitations of TlCuCl$_3$ with varying pressure. The graph displays energy in meV against pressure in kbar. The graph is divided into two regions: Quantum Paramagnet and Néel. The graph highlights longitudinal excitations similar to the Higgs boson, with the first observation at the theoretically predicted energy.

S. Sachdev, arXiv:0901.4103

\[ \lambda_c = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Quantum critical point with non-local entanglement in spin wavefunction

$$\lambda = \frac{1}{\sqrt{2}} (|↑↓⟩ - |↓↑⟩)$$
Tensor network representation of entanglement at quantum critical point

• Long-range entanglement
• Long-range entanglement

• The low energy excitations are described by a theory which has the same structure as Einstein’s theory of special relativity, but with the spin-wave velocity playing the role of the velocity of light.
• Long-range entanglement

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• The theory of the critical point has an even larger symmetry corresponding to conformal transformations of spacetime: we refer to such a theory as a CFT3
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String theory and black holes

Quantum critical points of electrons in crystals
String theory

- Allows unification of the standard model of particle physics with gravity.
- Low-lying string modes correspond to gauge fields, gravitons, quarks ...
- A $D$-brane is a $d$-dimensional surface on which strings can end.
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• The low-energy theory on a $D$-brane has no gravity, similar to theories of entangled electrons of interest to us.
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In $d = 2$, we obtain strongly-interacting CFT3s. These are “dual” to string theory on anti-de Sitter space: AdS4.
• A $D$-brane is a $d$-dimensional surface on which strings can end.

• The low-energy theory on a $D$-brane has no gravity, similar to theories of entangled electrons of interest to us.

• In $d = 2$, we obtain strongly-interacting $\text{CFT3s}$. These are “dual” to string theory on anti-de Sitter space: $\text{AdS4}$. 
Tensor network representation of entanglement at quantum critical point.

$d$-dimensional space

Depth of entanglement

Sunday, February 19, 2012
String theory near a D-brane

Emergent direction of AdS4
Tensor network representation of entanglement at quantum critical point

Emergent direction of AdS4

Brian Swingle, arXiv:0905.1317
Measure strength of quantum entanglement of region A with region B.

\[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]

Entanglement entropy \( S_{EE} = -\text{Tr} (\rho_A \ln \rho_A) \)
Entanglement entropy

$d$-dimensional space

depth of entanglement
Entanglement entropy

Most links describe entanglement within A

Depth of entanglement

$d$-dimensional space
Entanglement entropy

$d$-dimensional space

Links overestimate entanglement between A and B

depth of entanglement

Sunday, February 19, 2012
Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

$d$-dimensional space

depth of entanglement
The entanglement entropy of a region $A$ on the boundary equals the minimal area of a surface in the higher-dimensional space whose boundary co-incides with that of $A$.

This can be seen both the string and tensor-network pictures

Brian Swingle, arXiv:0905.1317
Emergent holographic direction

\[ \text{AdS}_{d+2} \]

\[ \mathbb{R}^{d,1} \]

Minkowski

\[ \text{CFT}_{d+1} \]

Quantum matter with long-range entanglement
$\text{AdS}_{d+2}$

Emergent holographic direction

$R^{d,1}$

Minkowski

$CFT_{d+1}$

Quantum matter with long-range entanglement

$r$

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AdS$_{d+2}$

$\mathbb{R}^{d,1}$

Minkowski

CFT$_{d+1}$

Quantum matter with long-range entanglement

Area measures entanglement entropy

Emergent holographic direction

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\[ = \frac{1}{\sqrt{2}} (\left| \uparrow \downarrow \right> - \left| \downarrow \uparrow \right>) \]
Classical spin waves

Dilute trilpon gas

Quantum critical Neel order

$\lambda_c$
Classical spin waves

Quantum critical

Neel order

\( T \)

\( \lambda \)

\( \lambda_c \)

Thermally excited spin waves

Thermally excited triplon particles

Quantum critical
Classical spin waves

Dilute triplon gas

Quantum critical


Short-range entanglement

Thermally excited spin waves

Thermally excited triplon particles

Neel order

$$\lambda_c$$

Thermally excited spin waves

Thermally excited triplon particles

Quantum critical

Neel order

$\lambda_c$

Excitations of a ground state with long-range entanglement

Quantum critical

Thermally excited spin waves

Thermally excited triplon particles

Neel order

\( \lambda_c \)
Excitations of a ground state with long-range entanglement

Quantum critical

Needed: Accurate theory of quantum critical spin dynamics

Thermally excited spin waves

Thermally excited triplon particles

$\lambda_c$

Neel order

String theory at non-zero temperatures

A 2+1 dimensional system at its quantum critical point
String theory at non-zero temperatures

A 2+1 dimensional system at its quantum critical point

A “horizon”, similar to the surface of a black hole!
Objects so massive that light is gravitationally bound to them.
Horizon radius $R = \frac{2GM}{c^2}$

Objects so massive that light is gravitationally bound to them.

In Einstein’s theory, the region inside the black hole horizon is disconnected from the rest of the universe.
Around 1974, Bekenstein and Hawking showed that the application of the quantum theory across a black hole horizon led to many astonishing conclusions.
Quantum Entanglement across a black hole horizon
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Quantum Entanglement across a black hole horizon

Black hole horizon
Quantum Entanglement across a black hole horizon

Black hole horizon
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.

This entanglement leads to a black hole temperature (the Hawking temperature) and a black hole entropy (the Bekenstein entropy).
A "horizon", whose temperature and entropy equal those of the quantum critical point.
A “horizon”, whose temperature and entropy equal those of the quantum critical point

Friction of quantum criticality = waves falling into black brane

A 2+1 dimensional system at its quantum critical point

String theory at non-zero temperatures
A 2+1 dimensional system at its quantum critical point

An (extended) Einstein-Maxwell provides successful description of dynamics of quantum critical points at non-zero temperatures (where no other methods apply)

A “horizon”, whose temperature and entropy equal those of the quantum critical point

String theory at non-zero temperatures
Quantum superposition and entanglement

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String theory and black holes
Metals, “strange metals”, and high temperature superconductors

Insights from gravitational “duals”
High temperature superconductors

\[ \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \]
Iron pnictides:
a new class of high temperature superconductors
Resistivity \( \sim \rho_0 + AT^\alpha \)

BaFe\(_2\)(As\(_{1-x}\)P\(_x\))\(_2\)


Short-range entanglement in state with Neel (AF) order

Fe\textsubscript{2}(As\textsubscript{1-x}P\textsubscript{x})\textsubscript{2}

\sim \rho_0 + AT^\alpha

Physical Review B 81, 184519 (2010)
Superconductivity

Bose condensate of pairs of electrons

Short-range entanglement

Resistivity

\[ \rho \approx \rho_0 + A T^\alpha \]
Superconductivity

$\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$

Resistivity

$\sim \rho_0 + AT^\alpha$


Ordinary metal (Fermi liquid)
Sommerfeld-Bloch theory of ordinary metals

Momenta with electron states occupied

Momenta with electron states empty
Sommerfeld-Bloch theory of ordinary metals

Key feature of the theory: the Fermi surface

- Area enclosed by the Fermi surface $A = Q$, the electron density
- Excitations near the Fermi surface are responsible for the familiar properties of ordinary metals, such as resistivity $\sim T^2$. 
Superconductivity

Resistivity
\sim \rho_0 + AT^\alpha

Ordinary metal (Fermi liquid)

BaFe$_2$(As$_{1-x}$P$_x$)$_2$


*Physical Review B* 81, 184519 (2010)
Superconductivity

$\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$

Resistivity $\sim \rho_0 + AT^\alpha$


Superconductivity

\[ \text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2 \]

Resistivity

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*Physical Review B* 81, 184519 (2010)
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Dilute triplon gas

Quantum critical

Classical spin waves
Dilute trilop gas
Quantum critical
Neel order
Ordinary Metal

\[ T \]
\[ \lambda \]
\[ \lambda_c \]
Classical spin waves

Dilute triplon gas

Quantum critical

Strange Metal

Ordinary Metal

Neel order

$\lambda_c$
Strange Metal

$\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$

Resistivity $\sim \rho_0 + AT^{\alpha}$


Strange Metal

\[ \text{Resistivity } \sim \rho_0 + A T^\alpha \]


Excitations of a ground state with long-range entanglement

Resistivity $\sim \rho_0 + A T^\alpha$

Strange Metal

BaFe$_2$(As$_{1-x}$P$_x$)$_2$


Key (difficult) problem:

Describe quantum critical points and phases of systems with Fermi surfaces leading to metals with novel types of long-range entanglement.
Challenge to string theory:

Describe quantum critical points and phases of metals
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Describe quantum critical points and phases of metals

Can we obtain gravitational theories of superconductors and ordinary Sommerfeld-Bloch metals?
Challenge to string theory:

Describe quantum critical points and phases of metals

Can we obtain gravitational theories of superconductors and ordinary Sommerfeld-Bloch metals?

Yes

S. Sachdev, Physical Review D 84, 066009 (2011)
Challenge to string theory:

Describe quantum critical points and phases of metals

Do the “holographic” gravitational theories also yield metals distinct from ordinary Sommerfeld-Bloch metals?
Challenge to string theory:

Describe quantum critical points and phases of metals

Do the “holographic” gravitational theories also yield metals distinct from ordinary Sommerfeld-Bloch metals?

Yes, lots of them, with many “strange” properties!
Challenge to string theory:

Describe quantum critical points and phases of metals

How do we discard artifacts, and choose the holographic theories applicable to condensed matter physics?
**Challenge to string theory:**

Describe quantum critical points and phases of metals

How do we discard artifacts, and choose the holographic theories applicable to condensed matter physics?

Choose the theories with the proper entropy density

Checks: these theories also have the proper entanglement entropy and Fermi surface size!

The simplest example of a “strange metal” is realized by fermions with a Fermi surface coupled to an Abelian or non-Abelian gauge field.
Fermi surface of an ordinary metal
Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $A = Q$, the fermion density

Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $\mathcal{A} = Q$, the fermion density
- Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |k| - k_F$ is the distance from the Fermi surface and $z$ is the dynamic critical exponent.

Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $\mathcal{A} = Q$, the fermion density

- Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |\mathbf{k}| - k_F$ is the distance from the Fermi surface and $z$ is the dynamic critical exponent.

- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T^{d_{\text{eff}}/z}$ with $d_{\text{eff}} = 1$.

Holography of “strange metals”

J. McGreevy, arXiv0909.0518
Consider the following (most) general metric for the holographic theory

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]
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This metric transforms under rescaling as

\[ x_i \rightarrow \zeta x_i \]
\[ t \rightarrow \zeta^z t \]
\[ ds \rightarrow \zeta^{\theta/d} ds. \]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

L. Huijse, S. Sachdev, B. Swingle, arXiv:1112.0573
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This metric transforms under rescaling as

$$x_i \rightarrow \zeta x_i$$
$$t \rightarrow \zeta^z t$$
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This identifies $z$ as the dynamic critical exponent ($z = 1$ for “relativistic” quantum critical points).

What is $\theta$? ($\theta = 0$ for “relativistic” quantum critical points).
At $T > 0$, there is a “black-brane” at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$.
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The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$

Under rescaling $r \rightarrow \zeta^{(d-\theta)/d}r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$ measures “dimension deficit” in the phase space of low energy degrees of a freedom.
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$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$ measures “dimension deficit” in the phase space of low energy degrees of a freedom. For a strange metal should choose $\theta = d - 1$. 

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Holography of “strange metals”

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^2\theta/(d-\theta)}{d-\theta} dr^2 + dx_i^2 \right) \]

\[ \theta = d - 1 \]
The entanglement entropy exhibits logarithmic violation of the area law, expected for systems with Fermi surfaces, only for this value of $\theta$.

The coefficient of the logarithmic term is consistent with the Fermi surface size expected from $A = Q$.

Many other features of the holographic theory are consistent with a boundary theory which has "hidden" Fermi surfaces of gauge-charged fermions.

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^2\theta/(d-\theta)}{dr^2} + dx_i^2 \right) \]

$\theta = d - 1$

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Holography of “strange metals”

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^2\theta/(d-\theta)}{dx_i^2} \right) \]

\[ \theta = d - 1 \]

- The entanglement entropy exhibits logarithmic violation of the area law, expected for systems with Fermi surfaces, only for this value of \( \theta \)!

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- Many other features of the holographic theory are consistent with a boundary theory which has “hidden” Fermi surfaces of gauge-charged fermions.

Conclusions

Phases of matter with long-range quantum entanglement are prominent in numerous modern materials.
Conclusions

Simplest examples of long-range entanglement are at quantum-critical points of insulating antiferromagnets.
Conclusions

More complex examples in metallic states are experimentally ubiquitous, but pose difficult strong-coupling problems to conventional methods of field theory.
Conclusions

String theory and gravity in emergent dimensions offer a remarkable new approach to describing states with long-range quantum entanglement.
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Much recent progress offers hope of a holographic description of “strange metals”