Theory of Quantum Matter: from Quantum Fields to Strings

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Subir Sachdev

Talk online: sachdev.physics.harvard.edu
Outline

1. The simplest models without quasiparticles
   A. Superfluid-insulator transition of ultracold bosons in an optical lattice
   B. Conformal field theories in 2+1 dimensions and the AdS/CFT correspondence

2. Metals without quasiparticles
   A. Review of Fermi liquid theory
   B. A “non-Fermi” liquid: the Ising-nematic quantum critical point
   C. Holography, entanglement, and strange metals
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   A. Review of Fermi liquid theory
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   C. Holography, entanglement, and strange metals
(a) Entanglement
(b) Holography, entanglement, and CFTs
(c) Generalized holography beyond CFTs
(d) Holography of strange metals
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\[
|\Psi\rangle \Rightarrow \text{Ground state of entire system,} \\
\rho = |\Psi\rangle\langle\Psi| \\
\rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \\
\text{Entanglement entropy } S_E = -\text{Tr} (\rho_A \ln \rho_A)
\]
\[ |\Psi\rangle \Rightarrow \text{Ground state of entire system,} \]
\[ \rho = |\Psi\rangle\langle\Psi| \]

Take \[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B) \]

Then \[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]
\[ = \frac{1}{2} (|\uparrow\rangle_A \langle \uparrow|_A + |\downarrow\rangle_A \langle \downarrow|_A) \]

**Entanglement entropy** \[ S_E = -\text{Tr} (\rho_A \ln \rho_A) \]
\[ = \ln 2 \]
Entanglement entropy of a band insulator

Band insulators

An even number of electrons per unit cell
Entanglement entropy of a band insulator

\[ S_E = aP - b \exp(-cP) \]

where \( P \) is the surface area (perimeter) of the boundary between A and B.
Logarithmic violation of “area law”: \[ S_E = C_E k_F P \ln(k_F P) \]

for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape. The prefactor \( C_E \) is independent of the shape of the entangling region, and dependent only on IR features of the theory.

Entanglement entropy of Fermi surfaces

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Quantum state with complex, many-body, “long-range” quantum entanglement

\[
S = \int d^2 r dt \left[ |\partial_t \Psi|^2 - c^2 |\nabla_r \Psi|^2 - V(\Psi) \right]
\]

\[
V(\Psi) = (\lambda - \lambda_c) |\Psi|^2 + u (|\Psi|^2)^2
\]

\[
\langle \Psi \rangle \neq 0 \Rightarrow \text{Superfluid}
\]

\[
\langle \Psi \rangle = 0 \Rightarrow \text{Insulator}
\]

\[
\lambda \sim \frac{U}{t}
\]
Entanglement at the quantum critical point

- Entanglement entropy obeys $S_E = aP - \gamma$, where $\gamma$ is a shape-dependent universal number associated with the CFT3.

Tensor network representation of entanglement at quantum critical point

Tensor network representation of entanglement at quantum critical point

d-dimensional space

depth of entanglement
Tensor network representation of entanglement at quantum critical point

Most links describe entanglement within $A$
Tensor network representation of entanglement at quantum critical point

$A$ in $d$-dimensional space

Links overestimate entanglement between $A$ and $B$
Tensor network representation of entanglement at quantum critical point

Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

Brian Swingle, arXiv:0905.1317
Tensor network representation of entanglement at quantum critical point
String theory near a D-brane

Emergent direction of AdS4
Tensor network representation of entanglement at quantum critical point

Emergent direction of AdS4

Brian Swingle, arXiv:0905.1317
(a) Entanglement
(b) Holography, entanglement, and CFTs
(c) Generalized holography beyond CFTs
(d) Holography of strange metals
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Field theories in $d + 1$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

$$u \frac{dg}{du} = \beta(g)$$

where $u$ is the energy scale. The RG equation is local in energy scale, i.e. the RHS does not depend upon $u$. 
Key idea:  ⇒ Implement $u$ as an extra dimension, and map to a local theory in $d+2$ spacetime dimensions.
Key idea: Implement $u$ as an extra dimension, and map to a local theory in $d+2$ spacetime dimensions. We identify the extra-dimensional co-ordinate $r = 1/u$. 
For a relativistic CFT in $d$ spatial dimensions, the proper length, $ds$, in the holographic space is fixed by demanding the scale transformation ($i = 1 \ldots d$)

$$x_i \rightarrow \zeta x_i \ , \quad t \rightarrow \zeta t \ , \quad ds \rightarrow ds$$
This gives the unique metric

\[ ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2) \]

This is the metric of anti-de Sitter space AdS\(_{d+2}\).
AdS/CFT correspondence

$\text{AdS}_4$

$\mathbb{R}^{2,1}$

Minkowski

CFT$_3$

$r$

$x^i$
Holography and Entanglement

AdS$_4$  

$R^{2,1}$  

Minkowski  

CFT$_3$
Associate entanglement entropy with an observer in the enclosed spacetime region, who cannot observe "outside": i.e. the region is surrounded by an imaginary horizon.

Holography and Entanglement

AdS$_4$

$R^{2,1}$ Minkowski

Minimal surface area measures entanglement entropy

CFT3

$r$

$\mathcal{X}_i$

Computation of minimal surface area yields

\[ S_E = aP - \gamma, \]

where \( \gamma \) is a shape-dependent universal number.

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Generalized holography
Consider a metric which transforms under rescaling as

\[ x_i \rightarrow \zeta x_i, \quad t \rightarrow \zeta^z t, \quad ds \rightarrow \zeta^{\theta/d} ds. \]

Recall: conformal matter has \( \theta = 0, \ z = 1, \) and the metric is anti-de Sitter
The most general such metric is

\[
    ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)
\]
Generalized holography

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2 d(z-1)/(d-\theta)} + r^{2\theta/(d-\theta)} \, dr^2 + dx_i^2 \right) \]

This is the most general metric which is invariant under the scale transformation

\[
\begin{align*}
    x_i & \rightarrow \zeta x_i \\
    t & \rightarrow \zeta^z t \\
    ds & \rightarrow \zeta^{\theta/d} \, ds.
\end{align*}
\]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for "relativistic" quantum critical points). We will see shortly that \( \theta \) is the violation of hyperscaling exponent.

We have used reparametrization invariance in \( r \) to define it so that it scales as

\[ r \rightarrow \zeta^{(d-\theta)/d} \, r. \]

At $T > 0$, there is a “black-brane” at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$


\[
ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + \frac{r^{2\theta/(d-\theta)} dr^2 + dx_i^2}{d} \right)
\]
Under rescaling $r \rightarrow \zeta^{(d-\theta)/d}r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$, the "dimension deficit", is now identified as the violation of hyperscaling exponent.
The null energy condition (stability condition for gravity) yields a new inequality

\[ z \geq 1 + \frac{\theta}{d} \]

\[
\begin{multline}
 ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d}(z-1)/(d-\theta)} + \frac{r^{2d}/(d-\theta) dr^2}{dz^2} + dx_i^2 \right) \\
\end{multline}

Generalized holography

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + r^2\theta/(d-\theta)\,dr^2 + dx_i^2 \right) \]

The null energy condition (stability condition for gravity) yields a new inequality

\[ z \geq 1 + \frac{\theta}{d} \]

The non-Fermi liquid in \( d = 2 \) has \( \theta = d - 1 \), and this implies \( z \geq 3/2 \). So the lower bound is precisely the value obtained for the non-Fermi liquid!
Application of the Ryu-Takayanagi minimal area formula to a dual Einstein-Maxwell-dilaton theory yields

\[ S_E \sim \begin{cases} 
  P & \text{, for } \theta < d - 1 \\
  P \ln P & \text{, for } \theta = d - 1 \\
  P^{\theta/(d-1)} & \text{, for } \theta > d - 1 
\end{cases} \]
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The non-Fermi liquid has log-violation of “area law”, and this appears precisely at the correct value \( \theta = d - 1 \)!
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\end{align*}\]

The non-Fermi liquid has log-violation of “area law”, and this appears precisely at the correct value \(\theta = d - 1\)!

Moreover, the co-efficient of \(P \ln P\) computed holographically is independent of the shape of the entangling region just as expected for a circular Fermi surface!!

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Begin with a CFT
Holographic representation: AdS$_4$

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right] \]

A 2+1 dimensional CFT at $T=0$
Holographic representation: AdS$_4$

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r) dt^2 + dx^2 + dy^2 \right] \]

with \( f(r) = 1 \)
Apply a chemical potential
AdS$_4$ theory of “nearly perfect fluids”

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

\[
S_{EM} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4g_4^2} F_{ab} F^{ab} \right].
\]

This is to be solved subject to the constraint

\[
A_\mu (r \to 0, x, y, t) = A_\mu (x, y, t)
\]

where $A_\mu$ is a source coupling to a conserved U(1) current $J_\mu$ of the CFT3

\[
S = S_{CFT} + i \int dxdydt A_\mu J_\mu
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$$S = S_{CFT} + i \int dxdydt A_\mu J_\mu$$

At non-zero chemical potential we simply require $A_\tau = \mu$. 
The Maxwell-Einstein theory of the applied chemical potential yields a AdS\(_4\)-Reissner-Nordström black-brane

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) - \frac{1}{4g_4^2} F_{ab} F^{ab} \right] \]

The Maxwell-Einstein theory of the applied chemical potential yields a AdS$_4$-Reissner-Nordström black-brane

\[ \mathcal{E}_r = \langle Q \rangle \]

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r) dt^2 + dx^2 + dy^2 \right] \]

with \( f(r) = \left( 1 - \frac{r}{R} \right)^2 \left( 1 + \frac{2r}{R} + \frac{3r^2}{R^2} \right) \) and \( R = \frac{\sqrt{6}Lg_4}{\kappa \mu} \), and \( A_r = \mu \left( 1 - \frac{r}{R} \right) \)
The Maxwell-Einstein theory of the applied chemical potential yields a AdS$_4$-Reissner-Nordström black-brane

At $T = 0$, we obtain an extremal black-brane, with a near-horizon (IR) metric of AdS$_2 \times R^2$

\[
ds^2 = \frac{L^2}{6} \left( -\frac{dt^2 + dr^2}{r^2} \right) + dx^2 + dy^2
\]

Holography of a non-Fermi liquid

Einstein-Maxwell-dilaton theory

\[ S = \int d^{d+2}x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R - 2(\nabla \Phi)^2 - \frac{V(\Phi)}{L^2} \right) - \frac{Z(\Phi)}{4e^2} F_{ab} F^{ab} \right] \]

with \( Z(\Phi) = Z_0 e^{\alpha \Phi} \), \( V(\Phi) = -V_0 e^{-\beta \Phi} \), as \( \Phi \to \infty \).

Holography of a non-Fermi liquid

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

The \( r \to \infty \) limit of the metric of the Einstein-Maxwell-dilaton (EMD) theory has the most general form with

\[ \theta = \frac{d^2 \beta}{\alpha + (d-1)\beta} \]

\[ z = 1 + \frac{\theta}{d} + \frac{8(d(d - \theta) + \theta)^2}{d^2(d - \theta)\alpha^2} \].
Computation of the entanglement entropy in the EMD theory via the Ryu-Takayanagi formula for $\theta = d - 1$ yields

$$S_E = C_E Q^{(d-1)/d} P \ln P$$

where $C_E$ is independent of UV details.
Holography of a non-Fermi liquid

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + \frac{r^{2\theta/(d-\theta)}}{d} dr^2 + dx_i^2 \right) \]

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\[ S_E = C_E Q^{(d-1)/d} P \ln P \]

where \( C_E \) is independent of UV details. This is precisely as expected for a Fermi surface, when combined with the Luttinger theorem!
Open questions:

- Is there any selection principle for the values of $\theta$ and $z$?
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- Are there $N^2$ Fermi surfaces of ‘quarks’ with $k_F \sim 1$, or 1 Fermi surface of a ‘baryon’ with $k_F \sim N^2$?
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- Why is $k_F$ not observable as Friedel oscillations in correlators of the density, and other gauge-neutral operators?

  Answer: need non-perturbative effects of monopole operators
  T. Faulkner and N. Iqbal, arXiv:1207.4208;
Strongly-coupled quantum criticality leads to a novel regime of quantum dynamics without quasiparticles.

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Holographic theories provide an excellent quantitative description of quantum Monte studies of quantum-critical boson models.

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