Gauge theories of ultra quantum metals

MPS Conference on Ultra Quantum Matter II
Simons Foundation, New York
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Subir Sachdev

Talk online: sachdev.physics.harvard.edu
diagram for the IL Sr encountered in their growth, a similar temperature-doping phase superconductivity are made difficult by the severe difficulties mechanism. Although similar studies of antiferromagnetism and tions provide some indications of its role for a possible pairing ing range from range antiferromagnetic order for the reduced samples in the dop-
on, inelastic neutron scattering demonstrate that there is no long-
Fig. 5.

Fig. 4.

318

Fig. 3.

Oxygen removal provides additional carriers (electrons). Adapted from Ref. [29].

reader is referred to the web version of this article.)

Ref. [2]

magnetic correlations without long-range order persist (Section 3.1. T

Single crystals of the T

behaving very similarly to the T

phase have mostly been grown success-

in thin films, the growth phase diagram involves also oxygen pressure.

Unfortunately, these competing phases can lead to many erroneous instances, they may appear during the post-annealing process.

 electron-doped Ln

cuprates and may be absent (and perhaps

charge-density waves, [30]

as a function of doping. (b) Temperature-

2

from a few rare data in the

literature. The phase diagram already presented in

and leading to superconductivity is most probably the same for

In the light blue region above the superconducting dome, strong

doping phase diagram for as-grown (solid circles) and reduced (open symbols)
samples obtained by muon spin resonance

crystals. The solid grey circles result from a shift assuming that

large magnetic fluctua-

13–0.18 but that the large magnetic fluctua-

x

Ce

2


doped cuprates. There are solid evidence of two-band-like beha-

rates (except at very large doping

viours in the transport properties of electron-doped cuprates

compete in the temperature-composition phase diagram.

The growth of T

and IL electron-doped cuprates is generally

heterostructures with surprising results like

We should underline that Sr in SrCuO

has been inserted successfully in (CaCuO

as the pseudogap line which will be described in Section

light blue) where magnetic correlations are strong but long-range

order does not set in as will be discussed in Section

sharp transition between the antiferromagnetic and the supercon-

Hole doping / Sr content (x) 0.20 0.10 0.10 0.20

Electron doping / Ce content (x)

0.20 0.10 0.10 0.20

La_{2-x}Sr_xCuO_4

Ln_{2-x}Ce_xCuO_4

T_c

T^*

T_N

~ 300K

~ 30K

AF

SC

T_N

T_c

T^*

T_N

SC
Figure: K. Fujita and J. C. Seamus Davis
Fermi liquid

Area enclosed by Fermi surface = 1 + ρ

Pseudogap metal at low $p$
Strange Metal
Anomalous Criticality in the Electrical Resistivity of La$_{2-x}$Sr$_x$CuO$_4$

R. A. Cooper,$^1$ Y. Wang,$^1$ B. Vignolle,$^2$ O. J. Lipscombe,$^1$ S. M. Hayden,$^1$ Y. Tanabe,$^3$ T. Adachi,$^3$
Y. Koike,$^3$ M. Nohara,$^4$ H. Takagi,$^4$ Cyril Proust,$^2$ N. E. Hussey$^{1+}$
From the resistivity, they determined the value of the number $\alpha$ defined by

$$\rho(T) = \rho_0 + \alpha \frac{\hbar}{2e^2} \left( \frac{T}{T_F} \right)$$

where $T_F = (\pi \hbar^2 / k_B)(n/m^*)$ and $m^*$ is determined from the specific heat. This expression is obtained from the Drude form $\rho = m^*/(ne^2\tau)$ and $\hbar/\tau = \alpha k_B T$. 

Universal $T$-linear resistivity and Planckian limit in overdoped cuprates

A. Legros$^{1,2}$, S. Benhabib$^3$, W. Tabis$^{3,4}$, F. Laliberté$^1$, M. Dion$^1$, M. Lizaire$^1$,

B. Vignolle$^3$, D. Vignolles$^3$, H. Raffy$^5$, Z. Z. Li$^5$, P. Auban-Senzier$^5$,

N. Doiron-Leyraud$^1$, P. Fournier$^{1,6}$, D. Colson$^2$, L. Taillefer$^{1,6}$, and C. Proust$^{3,6}$

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2 Service de Physique de l'État Condensé (CEA, CNRS), Université Paris-Saclay, CEA Saclay, Gif-sur-Yvette 91191, France

3 Laboratoire National des Champs Magnétiques Intenses (CNRS, EMFL, INSA, UJF, UPS), Toulouse 31400, France

4 AGH University of Science and Technology, Faculty of Physics and Applied Computer Science, Al. Mickiewicza 30, 30-059 Krakow, Poland

5 Laboratoire de Physique des Solides, Université Paris-Sud, Université Paris-Saclay, CNRS UMR 8502, Orsay 91405, France

6 Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada
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<table>
<thead>
<tr>
<th>Material</th>
<th>$p$ or $x$</th>
<th>$n$ $(10^{27} \text{ m}^{-3})$</th>
<th>$m^*$ $(m_0)$</th>
<th>$A_1 / d$ $(\Omega / \text{K})$</th>
<th>$h / (2e^2 T_F)$ $(\Omega / \text{K})$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bi2212</td>
<td>$p = 0.23$</td>
<td>6.8</td>
<td>$8.4 \pm 1.6$</td>
<td>$8.0 \pm 0.9$</td>
<td>$7.4 \pm 1.4$</td>
<td>$1.1 \pm 0.3$</td>
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<tr>
<td>Bi2201</td>
<td>$p \sim 0.4$</td>
<td>3.5</td>
<td>$7 \pm 1.5$</td>
<td>$8 \pm 2$</td>
<td>$8 \pm 2$</td>
<td>$1.0 \pm 0.4$</td>
</tr>
<tr>
<td>LSCO</td>
<td>$p = 0.26$</td>
<td>7.8</td>
<td>$9.8 \pm 1.7$</td>
<td>$8.2 \pm 1.0$</td>
<td>$8.9 \pm 1.8$</td>
<td>$0.9 \pm 0.3$</td>
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<tr>
<td>Nd-LSCO</td>
<td>$p = 0.24$</td>
<td>7.9</td>
<td>$12 \pm 4$</td>
<td>$7.4 \pm 0.8$</td>
<td>$10.6 \pm 3.7$</td>
<td>$0.7 \pm 0.4$</td>
</tr>
<tr>
<td>PCCO</td>
<td>$x = 0.17$</td>
<td>8.8</td>
<td>$2.4 \pm 0.1$</td>
<td>$1.7 \pm 0.3$</td>
<td>$2.1 \pm 0.1$</td>
<td>$0.8 \pm 0.2$</td>
</tr>
<tr>
<td>LCCO</td>
<td>$x = 0.15$</td>
<td>9.0</td>
<td>$3.0 \pm 0.3$</td>
<td>$3.0 \pm 0.45$</td>
<td>$2.6 \pm 0.3$</td>
<td>$1.2 \pm 0.3$</td>
</tr>
<tr>
<td>TMTSF</td>
<td>$P = 11 \text{ kbar}$</td>
<td>1.4</td>
<td>$1.15 \pm 0.2$</td>
<td>$2.8 \pm 0.3$</td>
<td>$2.8 \pm 0.4$</td>
<td>$1.0 \pm 0.3$</td>
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Slope of $T$-linear resistivity vs Planckian limit in seven materials.
1. Review of spin-density-wave theory
2. SU(2) gauge theory of fluctuating spin density waves
3. Quantum oscillations and photoemission on the electron-doped cuprates
4. Strange metals and SYK models
5. Review of SYK models: Schwarzian theory
6. Connections to black holes with AdS$_2$ horizons
1. Review of spin-density-wave theory

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6. Connections to black holes with AdS$_2$ horizons
The Hubbard Model

\[ H = -\sum_{i<j} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^{\dagger} c_{i\alpha} \]

\[ t_{ij} \rightarrow \text{“hopping”}. \quad U \rightarrow \text{local repulsion}, \quad \mu \rightarrow \text{chemical potential} \]

Spin index \( \alpha = \uparrow, \downarrow \)

\[ n_{i\alpha} = c_{i\alpha}^{\dagger} c_{i\alpha} \]

\[ c_{i\alpha}^{\dagger} c_{j\beta} + c_{j\beta} c_{i\alpha}^{\dagger} = \delta_{ij} \delta_{\alpha\beta} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \]

Will study on the square lattice
Fermi surface + antiferromagnetism

The electron spin polarization obeys

$$\left\langle \vec{S}( \mathbf{r}, \tau) \right\rangle = \vec{\Phi}( \mathbf{r}, \tau) e^{i \mathbf{K} \cdot \mathbf{r}}$$

where \( \mathbf{K} = (\pi, \pi) \) is the ordering wavevector.
Fermi surface + antiferromagnetism

We use the operator equation (valid on each site $i$):

$$U \left( n_\uparrow - \frac{1}{2} \right) \left( n_\downarrow - \frac{1}{2} \right) = -\frac{2U}{3} \bar{S}^2 + \frac{U}{4}$$  \hspace{1cm} (1)

Then we decouple the interaction via

$$\exp \left( \frac{2U}{3} \sum_i \int d\tau \bar{S}_i^2 \right) = \int \mathcal{D} \tilde{J}_i(\tau) \exp \left( -\sum_i \int d\tau \left[ \frac{3}{8U} \tilde{J}_i^2 - \tilde{J}_i \bar{S}_i \right] \right)$$  \hspace{1cm} (2)

We now integrate out the fermions, and look for the saddle point of the resulting effective action for $\tilde{J}_i$. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field $\bar{\Phi}_i$ where

$$\tilde{J}_i = \bar{\Phi}_i e^{iK \cdot r_i}$$  \hspace{1cm} (3)
In this manner, we obtain the “spin-fermion” model

\[ Z = \int \mathcal{D}c_\alpha \mathcal{D}\Phi \exp (-S) \]

\[ S = \int d\tau \sum_k c^\dagger_{k\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon_k \right) c_{k\alpha} \]

\[ -\lambda \int d\tau \sum_i c^\dagger_{i\alpha} \bar{\Phi}_i \cdot \bar{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K} \cdot \mathbf{r}_i} \]

\[ + \int d\tau d^2r \left[ \frac{1}{2} \left( \nabla_r \Phi \right)^2 + \frac{1}{2} \left( \partial_\tau \Phi \right)^2 + \frac{s}{2} \Phi^2 + \frac{u}{4} \Phi^4 \right] \]
Fermi surface + antiferromagnetism

In the Hamiltonian form (ignoring, for now, the time dependence of \( \Phi \)), the coupling between \( \Phi \) and the electrons takes the form

\[
H_{sdw} = \lambda \sum_{k,q,\alpha,\beta} \vec{\Phi}_q \cdot \hat{c}^\dagger_{k+q,\alpha} \vec{\sigma}_{\alpha \beta} c_{k+K,\beta}
\]

where \( \vec{\sigma} \) are the Pauli matrices, the boson momentum \( q \) is small, while the fermion momentun \( k \) extends over the entire Brillouin zone. In the antiferromagnetically ordered state, we may take \( \Phi \propto (0, 0, 1) \), and the electron dispersions obtained by diagonalizing \( H_0 + H_{sdw} \) are

\[
E_{k\pm} = \frac{\varepsilon_k + \varepsilon_{k+K}}{2} \pm \sqrt{\left(\frac{\varepsilon_k - \varepsilon_{k+K}}{2}\right)^2 + \lambda^2 |\Phi|^2}
\]

This leads to the Fermi surfaces shown in the following slides as a function of increasing \( |\Phi| \).
Metal with “large” Fermi surface
Fermi surface + antiferromagnetism

Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$. 
Fermi surface + antiferromagnetism

“Hot” spots
Electron and hole pockets in antiferromagnetic phase with $\langle \Phi \rangle \neq 0$
Square lattice Hubbard model with hole doping

Increasing SDW order


Square lattice Hubbard model with hole doping

Increasing SDW order

Square lattice Hubbard model with hole doping

Increasing SDW order

Hot spots

where $\varepsilon_k = \varepsilon_{k + \mathbf{K}}$

Square lattice Hubbard model with hole doping

Fermi surface breaks up at hot spots into electron and hole “pockets”

Increasing SDW order

Hole pockets

Electron pockets

Hot spots

where $\varepsilon_k = \varepsilon_{k+K}$


Fermi surface breaks up at hot spots into electron and hole “pockets”

Square lattice Hubbard model with hole doping

- \( \langle \Phi \rangle \neq 0 \) and large
- \( \langle \Phi \rangle \neq 0 \) and small
- \( \langle \Phi \rangle = 0 \)

Metal with hole pockets
Metal with electron and hole pockets
Metal with “large” Fermi surface
Square lattice Hubbard model with electron doping

\[ \langle \Phi \rangle \neq 0 \]
and large

Metal with electron pockets

\[ \langle \Phi \rangle \neq 0 \]
and small

Metal with electron and hole pockets

\[ \langle \Phi \rangle = 0 \]
Metal with “large” Fermi surface
Nd$_{2-x}$Ce$_x$CuO$_{4\pm\delta}$

Doping Dependence of an n-Type Cuprate Superconductor Investigated by Angle-Resolved Photoemission Spectroscopy


Antiferromagnetism in the Hubbard Model

\[ H = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha} \]

\( t_{ij} \rightarrow \) “hopping”. \( U \rightarrow \) local repulsion, \( \mu \rightarrow \) chemical potential

Mean-field theory with a spin density wave (SDW) order parameter \( \bar{\Phi}_i = (-1)^{i_x + i_y} \frac{\langle c_{i\alpha}^\dagger \bar{\sigma}_{\alpha\beta} c_{i\beta} \rangle}{2} \)

- **SDW LRO**
  - Reconstructed Fermi surface
  - \( \langle \bar{\Phi} \rangle \neq 0 \)

- **SDW SRO**
  - Large Fermi surface
  - \( \langle \bar{\Phi} \rangle = 0 \)

\( U/t \)
Symmetry breaking and topological phase transition

SDW SRO
Emergent gauge fields and “topological order”.
Reconstructed Fermi surface.
\( \langle \Phi \rangle = 0 \)

SDW LRO
Reconstructed Fermi surface
\( \langle \Phi \rangle \neq 0 \)

SDW SRO
Large Fermi surface.
\( \langle \Phi \rangle = 0 \)
1. Review of spin-density-wave theory

2. SU(2) gauge theory of fluctuating spin density waves

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6. Connections to black holes with AdS$_2$ horizons

We can (exactly) transform the Hubbard model to the “spin-fermion” model: electrons $c_{i\alpha}$ on the square lattice with dispersion

$$\mathcal{H}_c = -\sum_{i,\rho} t_\rho \left( c_{i,\alpha}^\dagger c_{i+\mathbf{v}_\rho,\alpha} + c_{i+\mathbf{v}_\rho,\alpha}^\dagger c_{i,\alpha} \right) - \mu \sum_i c_{i,\alpha}^\dagger c_{i,\alpha} + \mathcal{H}_{\text{int}}$$

are coupled to an antiferromagnetic SDW order parameter $\Phi^\ell(i)$, $\ell = x, y, z$

$$\mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i \Phi^\ell(i) c_{i,\alpha}^\dagger \sigma^\ell_{\alpha\beta} c_{i,\beta} + V_\Phi$$

where $\eta_i = \pm 1$ on the two sublattices. (For suitable $V_\Phi$, integrating out the $\Phi^\ell$ yields back the Hubbard model).
For (fluctuating) SDW SRO, we transform to a rotating reference frame using the SU(2) rotation $R_i$

\[
\begin{pmatrix}
    c_{i\uparrow} \\
    c_{i\downarrow}
\end{pmatrix}
= R_i \begin{pmatrix}
    \psi_{i,+} \\
    \psi_{i,-}
\end{pmatrix},
\]

in terms of fermionic “chargons” $\psi_s$ and a **Higgs field** $H^a(i)$

\[
\sigma^\ell \Phi^\ell(i) = R_i \sigma^a H^a(i) R_i^\dagger
\]

The Higgs field is the SDW order in the rotating reference frame.

S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, PRB **80**, 155129 (2009)
For (fluctuating) SDW SRO, we transform to a rotating reference frame using the SU(2) rotation $R_i$

$$
\left( \begin{array}{c}
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$$
\sigma^\ell \Phi^\ell(i) = R_i \sigma^a H^a(i) R_i^\dagger
$$

The Higgs field is the SDW order in the rotating reference frame. Note that this representation is ambiguous up to a SU(2) gauge transformation, $V_i$

$$
\left( \begin{array}{c}
\psi_{i,+} \\
\psi_{i,-}
\end{array} \right) \rightarrow V_i \left( \begin{array}{c}
\psi_{i,+} \\
\psi_{i,-}
\end{array} \right)
$$

$$
R_i \rightarrow R_i V_i^\dagger
$$

$$
\sigma^a H^a(i) \rightarrow V_i \sigma^b H^b(i) V_i^\dagger.
$$

S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, PRB 80, 155129 (2009)
The simplest effective Hamiltonian for the fermionic chargons is the same as that for the electrons, with the SDW order replaced by the Higgs field.

\[
\mathcal{H}_\psi = - \sum_{i,\rho} t_\rho \left( \psi_{i,s}^\dagger \psi_{i+\mathbf{v}_\rho,s} + \psi_{i+\mathbf{v}_\rho,s}^\dagger \psi_{i,s} \right) - \mu \sum_i \psi_{i,s}^\dagger \psi_{i,s} + \mathcal{H}_{\text{int}}
\]

\[
\mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i H^a(i) \psi_{i,s}^\dagger \sigma^a_{ss'} \psi_{i,s'} + V_H
\]

S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, PRB 80, 155129 (2009)
### Particle content of theories

<table>
<thead>
<tr>
<th>Field</th>
<th>Symbol</th>
<th>Statistics</th>
<th>$\text{SU}(2)_{\text{gauge}}$</th>
<th>$\text{SU}(2)_{\text{spin}}$</th>
<th>$\text{U}(1)_{\text{e.m.\ charge}}$</th>
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<tr>
<td>Electron</td>
<td>$c$</td>
<td>fermion</td>
<td>1</td>
<td>2</td>
<td>-1</td>
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<td>AF order</td>
<td>$\Phi$</td>
<td>boson</td>
<td>1</td>
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<td>0</td>
</tr>
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S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, Phys. Rev. B 80, 155129 (2009)
D. Chowdhury and S. Sachdev, PRB 91, 115123 (2015)
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**Spin density wave theory**

S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, Phys. Rev. B 80, 155129 (2009)
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Antiferromagnetism in the Hubbard Model

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Mean-field theory with a spin density wave (SDW) order parameter \( \Phi_i = (-1)^{i_x+i_y} \left\langle c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta} \right\rangle / 2 \)

SDW LRO
Reconstructed Fermi surface
\( \langle \Phi \rangle \neq 0 \)

Symmetry breaking phase transition

SDW SRO
Large Fermi surface.
\( \langle \Phi \rangle = 0 \)
### Table I. Quantum numbers of the matter fields in $L_1$ and $L_2$.

The transformations under the SU(2)’s are labelled by the dimension of the SU(2) representation, while those under the electromagnetic U(1) are labeled by the U(1) charge. The antiferromagnetic spin correlations are characterized by (5.3).

The Higgs field determines local spin correlations via (5.12).

The SU(2) gauge invariance associated with (5.6), when we express $L$ in terms of $\xi$, we naturally obtain a SU(2) gauge theory with an emergent gauge field $A_a^\mu = (A_a^\xi, A_a^z)$, with $a = 1, 2, 3$.

The Lagrangian of the resulting gauge theory is $L_g = L + L_Y + L_R + L_H$ (5.9).

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D. Chowdhury and S. Sachdev, PRB 91, 115123 (2015)
**SDW SRO**

**Higgs phase**

Emergent $\mathbb{Z}_2$ or $U(1)$ gauge fields. $\mathbb{Z}_2$ vortices or hedgehogs expelled.

\[ \langle \Phi^\ell \rangle = 0, \quad \langle H^a \rangle \neq 0, \quad \langle R \rangle = 0 \]

Symmetry breaking and topological phase transition

\[ \langle \Phi^\ell \rangle \neq 0, \quad \langle H^a \rangle \neq 0, \quad \langle R \rangle \neq 0 \]

Symmetry breaking phase transition

**Confinement**

No topological order.

\[ \langle \Phi^\ell \rangle = 0, \quad \langle H^a \rangle = 0, \quad \langle R \rangle \neq 0 \]

Topological phase transition
1. Review of spin-density-wave theory

2. SU(2) gauge theory of fluctuating spin density waves

3. Quantum oscillations and photoemission on the electron-doped cuprates

4. Strange metals and SYK models

5. Review of SYK models: Schwarzian theory

6. Connections to black holes with AdS$_2$ horizons
diagram for the IL Sr superconductivity are made difficult by the severe difficulties mechanism. Although similar studies of antiferromagnetism and pairs provide some indications of its role for a possible pairing range from antiferromagnetic order for the reduced samples in the dop-

Fig. 5. $T_0$ oxygen removal provides additional carriers (electrons). Adapted from Ref. $\mathbf{30}$

doping phase diagram for as-grown (solid circles) and reduced (open symbols) samples obtained by muon spin resonance showing the Néel temperature $T_N$ as the pseudogap line $T_\text{pseudogap}$ which will be described in Section $\mathbf{4.1}$.

Fig. 4. (a) Temperature-doping phase diagram of Nd$_{2-x}$Ce$_x$CuO$_4$. The diagram shows the doping range for supercon-ductivity (in red), antiferromagnetism (in blue) and $T_\text{pseudogap}$ light blue) where magnetic correlations are strong but long-range order persist (Section $\mathbf{3.1}$).

In the case of thin films, the growth phase diagram involves also oxygen pressure. The growth of T$_{0}$ single crystals of the T$_{0}$ phase have mostly been grown success-

Hole doping / Sr content ($x$) Electron doping / Ce content ($x$)
Antiferromagnetic order disappears beyond $x=0.14$
Nd$_{2-x}$Ce$_x$CuO$_{4\pm\delta}$

Doping Dependence of an n-Type Cuprate Superconductor Investigated by Angle-Resolved Photoemission Spectroscopy


Correlation between Fermi surface transformations and superconductivity in the electron-doped high-$T_c$ superconductor Nd$_{2-x}$Ce$_x$CuO$_4$

T. Helm,$^{1,*}$ M. V. Kartsovnik,$^{1,†}$ C. Proust,$^2$ B. Vignolle,$^2$ C. Putzke,$^{3,‡}$ E. Kampert,$^3$ I. Sheikin,$^4$ E.-S. Choi,$^5$ J. S. Brooks,$^5$ N. Bittner,$^{1,§}$ W. Biberacher,$^1$ A. Erb,$^{1,6}$ J. Wosnitza,$^3$ and R. Gross$^{1,6,‖}$

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$^5$National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32310, USA
$^6$Physik-Department, Technische Universität München, D-85748 Garching, Germany
Correlation between Fermi surface transformations and superconductivity in the electron-doped high-$T_c$ superconductor $\text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4$

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5National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32310, USA
6Physik-Department, Technische Universität München, D-85748 Garching, Germany

- First, the quantitative analysis of the Shubnikov-de Haas effect shows that the weak superlattice potential responsible for the Fermi surface reconstruction in the overdoped regime extrapolates to zero at the doping level $x_c = 0.175$ corresponding to the onset of superconductivity.
Correlation between Fermi surface transformations and superconductivity in the electron-doped high-$T_c$ superconductor Nd$_{2-x}$Ce$_x$CuO$_4$

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• First, the quantitative analysis of the Shubnikov-de Haas effect shows that the weak superlattice potential responsible for the Fermi surface reconstruction in the overdoped regime extrapolates to zero at the doping level $x_c = 0.175$ corresponding to the onset of superconductivity.

• Second, the high-field Hall coefficient exhibits a sharp drop right below optimal doping $x_{\text{opt}} = 0.145$ where the superconducting transition temperature is maximum. This drop is most likely caused by the onset of long-range antiferromagnetic ordering.
Quantum oscillation measurements show the presence of small hole pockets in the electron-doped superconductor \( \text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4 \) at doping levels into the overdoped regime until \( x = 0.17 \) (which corresponds to the end of the superconducting dome).
Fermi surface reconstruction in electron-doped cuprates without antiferromagnetic long-range order


- Quantum oscillation measurements show the presence of small hole pockets in the electron-doped superconductor $\text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4$ at doping levels into the overdoped regime until $x = 0.17$ (which corresponds to the end of the superconducting dome).

- Antiferromagnetic order is only present up to a doping $x = 0.14$. 
Fermi surface reconstruction in electron-doped cuprates without antiferromagnetic long-range order


- Quantum oscillation measurements show the presence of small hole pockets in the electron-doped superconductor \( \text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4 \) at doping levels into the overdoped regime until \( x = 0.17 \) (which corresponds to the end of the superconducting dome).

- Antiferromagnetic order is only present up to a doping \( x = 0.14 \).

- It has been assumed that this discrepancy could be explained by field-induced antiferromagnetic order.
Fermi surface reconstruction in electron-doped cuprates without antiferromagnetic long-range order


- New photoemission measurements at zero magnetic field show Fermi surfaces in quantitative agreement with quantum oscillation measurements.
Fermi surface reconstruction in electron-doped cuprates without antiferromagnetic long-range order


- New photoemission measurements at zero magnetic field show Fermi surfaces in quantitative agreement with quantum oscillation measurements.

- The energy gap between the electron and hole pockets collapses near $x = 0.17$ like an order parameter.
Fermi surface reconstruction in electron-doped cuprates without antiferromagnetic long-range order


• New photoemission measurements at zero magnetic field show Fermi surfaces in quantitative agreement with quantum oscillation measurements.

• The energy gap between the electron and hole pockets collapses near $x = 0.17$ like an order parameter.

• “The totality of the data points to a mysterious order between $x = 0.14$ and $x = 0.17$, whose appearance favors the FS reconstruction and disappearance defines the quantum critical doping. A recent topological proposal provides an ansatz for its origin.”
SDW SRO

**Higgs phase**
Emergent $Z_2$ or U(1) gauge fields.
$Z_2$ vortices or hedgehogs expelled.

\[
\langle \Phi^\ell \rangle = 0,
\langle H^a \rangle \neq 0,
\langle R \rangle \neq 0
\]

**Confinement**
No topological order.

\[
\langle \Phi^\ell \rangle = 0,
\langle H^a \rangle = 0,
\langle R \rangle \neq 0
\]
**Higgs phase**
Emergent $Z_2$ or U(1) gauge fields.
$Z_2$ vortices or hedgehogs expelled.

\[ \langle \Phi^\ell \rangle = 0 \]
\[ \langle H^a \rangle \neq 0 \]
\[ \langle R \rangle = 0 \]

\( x = 0.14 \)

\[ \langle \Phi^\ell \rangle \neq 0 \]
\[ \langle H^a \rangle \neq 0, \langle R \rangle \neq 0 \]

\( x = 0.17 \)

\[ \langle \Phi^\ell \rangle = 0 \]
\[ \langle H^a \rangle = 0, \langle R \rangle \neq 0 \]

**Symmetry breaking phase transition**
**Confinement**
No topological order.

**SDW SRO**
**SDW LRO**

\( U/t, g \)
Figure 1 | Fermi surface reconstruction in optimal-doped NCCO.

a, Schematic diagram of a reconstructed Fermi surface with electron-like pockets near antinode and hole-like pockets near node. The dashed lines indicate the antiferro magnetic Brillouin zone.

b, Schematic band dispersion along a momentum cut on the electron-like pocket (near hotspot), marked by the red arrow in (a).

The original dispersion is split into conduction and valence bands by an AFM energy gap. The reconstructed bands bend back at the AFMZB.

c, Schematic band dispersion along a momentum cut on the hole-like pocket (nodal cut), marked by the blue arrow in (a).

The AFM energy gap is slightly above $E_F$, but the folded band (back-bent hole band) disperses below $E_F$. The gray (dashed) line in (b,c) represents the original (folded) band.

d-f, the same as a-c, but for the original Fermi surface without reconstruction.

g-i, Photoemission intensity plot (g), second derivative image with respect to energy (h) and raw energy distribution curves (EDCs) (i) for optimal-doped NCCO, measured along a momentum cut on the electron-like pocket (near hotspot, labelled by the red arrow in the inset of h).

Conduction and valence bands extracted from the EDCs (blue triangles in i) are also presented in g (black circles) and h (white circles). The EDC at the AFMZB is shown in red (i).

j-l, the same as g-i, but for the nodal cut (labelled by the blue arrow in the inset of h).
S. Sachdev, Topological order and Fermi surface reconstruction, arXiv:1801.01125

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Coupled SYK Islands

SYK quantum islands of electrons with random or regular hopping between them.

\[ H = \sum_{x} \sum_{i<j, k<l} U_{ijkl,x} c_{ix}^\dagger c_{jx}^\dagger c_{kx} c_{lx} \]

\[ + \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij,xx'} c_{i,x}^\dagger c_{j,x'} \]

\[ |U_{ijkl}|^2 = \frac{2U^2}{N^3} \quad \quad |t_{ij,xx'}|^2 = t_0^2/N. \]


Pengfei Zhang, PRB 96, 205138 (2017)

Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, PRX 8, 021049 (2018)


See also A. Georges and O. Parcollet PRB 59, 5341 (1999)
Assessment of SYK lattice models

- Reasonable model of incoherent metal behavior for $t^2/U < T < t$.

- Strange metal (marginal Fermi-liquid) at smaller $T$ has linear in $B$ magnetoresistance, and $B/T$ scaling with mesoscopic disorder.
Assessment of SYK lattice models

- Reasonable model of incoherent metal behavior for $t^2/U < T < t$.
- Strange metal (marginal Fermi-liquid) at smaller $T$ has linear in $B$ magnetoresistance, and $B/T$ scaling with mesoscopic disorder.
- Strange metal (marginal Fermi-liquid) behavior as $T \rightarrow 0$ only for small ($p$) carrier density, rather than large ($1 + p$).
- No special role for quantum criticality at $p = p_c$.
- What is the origin of the “foot” at overdoping??
Anomalous Criticality in the Electrical Resistivity of La$_{2-x}$Sr$_x$CuO$_4$

Fermions (chargons) with random hopping coupled to a fluctuating U(1) gauge field

\[ H = -\frac{1}{(MN)^{1/2}} \sum_{ij=1}^{N} \sum_{\alpha\beta=1}^{M} \left[ t_{ij}^{\alpha\beta} e^{iA_{ij}} \psi_{i\alpha}^{\dagger} \psi_{j\beta} + (MN)^{1/2} \mu \delta_{ij}^{\alpha\beta} \psi_{i\alpha}^{\dagger} \psi_{i\alpha} \right], \]

\[ \ll t_{ij}^{\alpha\beta} t_{ji}^{\beta\alpha} \gg = \ll |t_{ij}^{\alpha\beta}|^2 \gg = t^2, \quad A_{ji} = -A_{ij}. \]

\[ arXiv:1807.04754 \]
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A simple model of a metal with quasiparticles

Pick a set of random positions
A simple model of a metal with quasiparticles

Place electrons randomly on some sites
A simple model of a metal with quasiparticles

Place electrons randomly on some sites
A simple model of a metal with quasiparticles

Electrons move one-by-one randomly
A simple model of a metal with quasiparticles

Electrons move one-by-one randomly
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Electrons move one-by-one randomly
A simple model of a metal with quasiparticles

\[ H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i \]

\[ c_i^\dagger c_j + c_j^\dagger c_i = 0 \quad , \quad c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = \delta_{ij} \]

\[ \frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q} \]

\( t_{ij} \) are independent random variables with \( \bar{t}_{ij} = 0 \) and \( |t_{ij}|^2 = t^2 \)

Fermions occupying the eigenstates of a

\( N \times N \) random matrix
A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = Q.$$ 

$G(\omega)$ can be determined by solving a quadratic equation.

$$\rho(\omega) = -\frac{1}{\pi} \text{Im} \ G(\omega)$$
A simple model of a metal with quasiparticles

Let $\varepsilon_\alpha$ be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest $NQ$ eigenvalues, up to the Fermi energy $E_F$. The single-particle density of states is

$$\rho(\omega) = (1/N) \sum_\alpha \delta(\omega - \varepsilon_\alpha), \text{ and } \rho_0 \equiv \rho(\omega = 0).$$
A simple model of a metal with quasiparticles

There are $2^N$ many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha,$$

where $n_\alpha = 0, 1$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $\varepsilon_\alpha$ have a level spacing $\sim 1/N$. 

Many-body level spacing $\sim 2^{-N}$

Quasiparticle excitations with spacing $\sim 1/N$
A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low $T$ is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N \left( -\frac{\pi^2}{6} \rho_0 T^2 + \mathcal{O}(T^4) \right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the single particle density of states at the Fermi level.

We can also define the many body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E)e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$D(E) \sim \exp \left( \pi \sqrt{\frac{2N \rho_0 (E - E_0)}{3}} \right), \quad E > E_0, \quad \frac{1}{N} \ll \rho_0 (E - E_0) \ll N$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi \sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.
A simple model of a metal with quasiparticles

Now add weak interactions

\[ H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell \]

\( U_{ij;kl} \) are independent random variables with \( \overline{U_{ij;kl}} = 0 \) and \( \overline{|U_{ij;kl}|^2} = U^2 \). We compute the lifetime of a quasiparticle, \( \tau_\alpha \), in an exact eigenstate \( \psi_\alpha(i) \) of the free particle Hamiltonian with energy \( \varepsilon_\alpha \). By Fermi’s Golden rule, for \( \varepsilon_\alpha \) at the Fermi energy

\[ \frac{1}{\tau_\alpha} = \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1-f(\varepsilon_\gamma))(1-f(\varepsilon_\delta)) \delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \]

\[ = \frac{\pi^3 U^2 \rho_0^2}{4} T^2 \]

where \( \rho_0 \) is the density of states at the Fermi energy, and \( f(\epsilon) = 1/(e^{\epsilon/T} + 1) \) is the Fermi function.

**Fermi liquid state**: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as \( \sim T^{-2} \) at the Fermi level.
The Sachdev-Ye-Kitaev (SYK) model

Pick a set of random positions
The SYK model

Place electrons randomly on some sites
The SYK model

Place electrons randomly on some sites
The SYK model

Entangle electrons pairwise randomly
Entangle electrons pairwise randomly

The SYK model
The SYK model

Entangle electrons pairwise randomly
Entangle electrons pairwise randomly
The SYK model

Entangle electrons pairwise randomly
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Entangle electrons pairwise randomly
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Entangle electrons pairwise randomly
Entangle electrons pairwise randomly

The SYK model
The SYK model

Entangle electrons pairwise randomly
The SYK model

This describes both a strange metal and a black hole!
The SYK model

(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large $N$ limit;

$$H = rac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{i j; k \ell} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$Q = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$U_{i j; k \ell}$ are independent random variables with $\overline{U_{i j; k \ell}} = 0$ and $|\overline{U_{i j; k \ell}}|^2 = U^2$

$N \rightarrow \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL 70, 3339 (1993)
The SYK model

Feynman graph expansion in $U_{ijkl}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)}, \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$G(\tau = 0^-) = Q.$$
The SYK model

Feynman graph expansion in $U_{ijkl}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{\omega + \mu - \Sigma(i\omega)}, \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)
\]

\[
G(\tau = 0^-) = Q.
\]

Low frequency analysis shows that the solutions must be gapless and obey

\[
\Sigma(z) = \mu - \frac{e^{i(\pi/4+\theta)}}{A} \sqrt{z} + \ldots, \quad G(z) = \frac{A e^{-i(\pi/4+\theta)}}{\sqrt{z}}
\]

where $A = (\pi/U^2 \cos(2\theta))^{1/4}$. The value of $\theta$ is universally related to $Q$ by a Luttinger-Ward functional analysis similar to that used to establish the Luttinger theorem of Fermi liquid theory:

\[
Q = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}
\]
The SYK model

Feynman graph expansion in $U_{ijkl}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \\
G(\tau = 0^-) = Q.
\]

At $T > 0$, we obtain a solution with a conformal structure

\[
G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1 + e^{-4\pi \mathcal{E}}}} \left( \frac{T}{\sin(\pi T \tau)} \right)^{1/2} , \quad 0 < \tau < 1/T ,
\]

where the ‘particle-hole asymmetry’ is determined by $\mathcal{E}$

\[
e^{2\pi \mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}.
\]

A. Georges and O. Parcollet PRB 59, 5341 (1999)
S. Sachdev, PRX 5, 041025 (2015)
The SYK model

There are $2^N$ many body levels with energy $E$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\frac{ds_0}{dQ} = 2\pi \mathcal{E}.$$ 

At $Q = 1/2$,

$$s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848 \ldots$$

where $G$ is Catalan’s constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB 63, 134406 (2001)

W. Fu and S. Sachdev, PRB 94, 035135 (2016)
The SYK model

\[ \Omega(T) - E_0 = N \left[ -s_0 T - \frac{1}{2} (\gamma + 4 \pi^2 \mathcal{E}^2 K) T^2 + \mathcal{O}(T^3) \right] + 2T \ln \left( \frac{U}{T} \right) \ldots \]

is the grand potential, where \( K = dQ/d\mu \sim 1/U \) is the compressibility \(/N\), \( \gamma \sim 1/U \) will appear later in the co-efficient of the Schwarzian, and the \( N^0 \) term arises from fluctuations about the large \( N \) theory described by the Schwarzian.

The inversion from \( \Omega(T) \) to the many-body density of states, \( D(E) \),

\[ Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T} \]

requires terms in \( \Omega(T) \) which are exponentially small in \( N \) (not shown above) from the Schwarzian action, yielding terms which are not small in \( D(E) \). We obtain

The SYK model

\[ D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} d\left( E - \frac{p^2}{2NK} \right) \]

where \( NQ + p \) is the integer fermion number, and \( d(E) \) is the density of states at fixed fermion number. We have \( d(E) = 0 \) for \( E < E_0 \), and

\[ d(E) \sim \exp(\sqrt{2N\gamma(E-E_0)}) \sinh\left( \sqrt{2N\gamma(E-E_0)} \right), \quad E > E_0, \quad e^{-cN} \ll \gamma(E-E_0) \ll N \]

There are exponentially more low energy states than for the quasiparticle case, and \( D(E) \) self-averages down to energies exponentially small in \( N \).

We can understand the dependence on the integer charge \( p \) by the relationship \( ds_0/dQ = 2\pi \mathcal{E} \), and hence \( Ns_0(Q + p/N) \approx Ns_0 + 2\pi p \mathcal{E} \).
A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low $T$ is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N \left(-\frac{\pi^2}{6} \rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the single particle density of states at the Fermi level.

We can also define the many body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$D(E) \sim \exp \left(\pi \sqrt{\frac{2N \rho_0 (E - E_0)}{3}}\right), \quad E > E_0, \quad \frac{1}{N} \ll \rho_0 (E - E_0) \ll N$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi \sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.
The SYK model

\[ D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} d \left( E - \frac{p^2}{2NK} \right) \]

where \( N\mathcal{Q} + p \) is the integer fermion number, and \( d(E) \) is the density of states at fixed fermion number. We have \( d(E) = 0 \) for \( E < E_0 \), and

\[ d(E) \sim \exp(Ns_0) \sinh \left( \sqrt{2N\gamma(E - E_0)} \right) , \quad E > E_0 , \quad e^{-cN} \ll \gamma(E - E_0) \ll N \]

There are exponentially more low energy states than for the quasiparticle case, and \( D(E) \) self-averages down to energies exponentially small in \( N \).

We can understand the dependence on the integer charge \( p \) by the relationship \( ds_0/d\mathcal{Q} = 2\pi \mathcal{E} \), and hence \( Ns_0(\mathcal{Q} + p/N) \approx Ns_0 + 2\pi p \mathcal{E} \).
The SYK model

\[ D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} d \left( E - \frac{p^2}{2NK} \right) \]

where \( NQ + p \) is the integer fermion number, and \( d(E) \) is the density of states at fixed fermion number. We have \( d(E) = 0 \) for \( E < E_0 \), and

\[ d(E) \sim \exp(Ns_0) \sinh \left( \sqrt{2N\gamma}(E - E_0) \right) , \quad E > E_0 , \quad e^{-cN} \ll \gamma(E - E_0) \ll N \]

There are exponentially more low energy states than for the quasiparticle case, and \( D(E) \) self-averages down to energies exponentially small in \( N \).

We can understand the dependence on the integer charge \( p \) by the relationship \( ds_0/dQ = 2\pi \mathcal{E} \), and hence \( Ns_0(Q + p/N) \approx Ns_0 + 2\pi p \mathcal{E} \).

The \( \exp(Ns_0) \) prefactor suggests that the low-lying eigenstates are separated into analogs of ‘superselection sectors’ which cannot be connected by ‘simple’ operators, in the Schwarzian approximation.

The SYK model

No quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a ‘Planckian’ time

\[ \tau_{eq} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0. \]

Established by solution of Schwinger-Keldysh equations for a quench.

- Presence of quasiparticles should slow down thermalization, so all quantum systems obey

\[ \tau_{eq} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0. \]

Absence of quasiparticles \( \Leftrightarrow \) Fastest possible thermalization

A. Eberlein, V. Kasper, S. Sachdev, and J. Steinberg, PRB 96, 205123 (2017)

A. Georges and O. Parcollet
PRB 59, 5341 (1999)

S. Sachdev, Quantum Phase Transitions, Cambridge (1999)
The SYK model

\[
G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)
\]

\[
\Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}
\]
The SYK model

\[ G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \]

\[ \Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}} \]

At frequencies \( \ll U \), the \( i\omega + \mu \) can be dropped, and without it equations are invariant under the reparametrization and gauge transformations.

The singular part of the self-energy and the Green’s function obey

\[ \int_0^\beta d\tau_2 \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3) \]

\[ \Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1) \]
The SYK model

\[ \int_0^\beta d\tau_2 \Sigma(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3) \]

\[ \Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1) \]

These equations are invariant under

\[ \tau = f(\sigma) \]

\[ G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{G}(\sigma_1, \sigma_2) \]

\[ \Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2) \]

where \( f(\sigma) \) and \( g(\sigma) \) are arbitrary functions.

By using \( f(\sigma) = \text{tan}(\pi T\sigma)/(\pi T) \) we can now obtain the \( T > 0 \) solution from the \( T = 0 \) solution.
Basics of conformal field theory

In a space with metric tensor \( g_{\mu\nu} \) and proper distance

\[
ds^2 = g_{\mu\nu} dx_\mu dx_\nu
\]
after the co-ordinate transformation \( x_\mu \rightarrow x'_\mu \), the new metric tensor is

\[
g'_{\mu\nu} = g_{\rho\lambda} \frac{\partial x_\rho}{\partial x'_\mu} \frac{\partial x_\lambda}{\partial x'_\nu}.
\]

A conformal transformation is one which preserves all angles and so

\[
g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x).
\]

In a conformal field theory, two-point correlators of scalar fields transform as

\[
\langle \phi(x_1) \phi(x_2) \rangle = \left| \det \left[ \frac{\partial x'_1}{\partial x_1} \right] \right|^{\Delta/d} \left| \det \left[ \frac{\partial x'_2}{\partial x_2} \right] \right|^{\Delta/d} \langle \phi(x'_1) \phi(x'_2) \rangle
\]
Let us write the large $N$ saddle point solutions of $S$ as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2}$$

$$\Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

The saddle point will be invariant under a reparamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\tilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for $\Sigma$). It turns out this is true only for the SL(2, $\mathbb{R}$) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$ 

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, $\mathbb{R}$) by the saddle point.
After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp \left[ - \sum_{ia} \int_{0}^{\beta} d\tau c_{ia}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) c_{ia} ight]$$

$$- \frac{U^2}{4N^3} \sum_{ab} \int_{0}^{\beta} d\tau d\tau' \left[ \sum_{ia} c_{ia}(\tau) c_{ib}(\tau') \right]^{4}.$$  

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[ -N \int_{0}^{\beta} d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left( G(\tau_2, \tau_1) + \frac{1}{N} \sum_{i} c_{i}(\tau_2) c_{i}^\dagger(\tau_1) \right) \right].$$
Infinite-range (SYK) model without quasiparticles

Then the partition function can be written as a path integral with an action $S$ analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det [\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)]$$

$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) [G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

A. Georges and O. Parcollet
PRB 59, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX 5, 041025 (2015)
The SYK model

Reparametrization and phase zero modes
We can write the path integral for the SYK model as

$$Z = \int \mathcal{D}G(\tau_1, \tau_1)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G, \Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, $G_s$, $\Sigma_s$, and only focus on the “Nambu-Goldstone” modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for $\Sigma$). Then the path integral is approximated by

$$Z = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-NS_{\text{eff}}[f, \phi]}.$$

The Schwarzian theory of the SYK model

Symmetry arguments, and explicit computations, show that the effective action is

\[
S_{\text{eff}}[f, \phi] = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{\tan(\pi T f(\tau)), \tau\},
\]

where \(f(\tau)\) is a monotonic map from \([0, 1/T]\) to \([0, 1/T]\), the couplings \(K, \gamma,\) and \(\mathcal{E}\) can be related to thermodynamic derivatives and we have used the Schwarzian:

\[
\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'}\right)^2.
\]

Specifically, an argument constraining the effective at \(T = 0\) is

\[
S_{\text{eff}} \left[ f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0 \right] = 0,
\]

and this is origin of the Schwarzian.

Yingfei Gu and S. Sachdev, unpublished
1. Review of spin-density-wave theory
2. SU(2) gauge theory of fluctuating spin density waves
3. Quantum oscillations and photoemission on the electron-doped cuprates
4. Strange metals and SYK models
5. Review of SYK models: Schwarzian theory
6. Connections to black holes with AdS$_2$ horizons
Black holes have an entropy and a temperature, $T_H = \frac{\hbar c^3}{(8\pi GMk_B)}$.

The entropy is proportional to their surface area.

J. D. Bekenstein, PRD 7, 2333 (1973)
The ring-down is predicted by General Relativity to happen in a time \( \frac{8\pi GM}{c^3} \sim 8 \) milliseconds. Curiously this happens to equal \( k_B T_H \): so the ring down can also be viewed as the approach of a quantum system to thermal equilibrium at the fastest possible rate.

The ring-down is predicted by General Relativity to happen in a time \( \frac{8\pi GM}{c^3} \sim 8 \text{ milliseconds} \). Curiously, this happens to equal \( \frac{\hbar}{k_B T_H} \), so the ring down can also be viewed as the approach of a quantum system to thermal equilibrium at the fastest possible rate.
- Black holes have an entropy and a temperature, \( T_H = \frac{\hbar c^3}{8\pi G M k_B} \).
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a time \( \sim \frac{\hbar}{k_B T_H} \).
Black holes have an entropy and a temperature, $T_H = \frac{\hbar c^3}{8\pi GMk_B}$.

The entropy is proportional to their surface area.

They relax to thermal equilibrium in a time $\sim \frac{\hbar}{k_B T_H}$.

**Holography:**
Quantum black holes “look like” quantum many-particle systems without quasiparticle excitations, residing “on” the surface of the black hole.

Consider a charged black hole with the smallest possible mass: the extremal limit. Zoom in to the near-horizon region at low energies. In this limit, the quantum theory lives in one space ($\vec{x}$) and one time dimension ($\zeta$).
The near-horizon region of an extremal charged black hole has the geometry of (1+1)-dimensional anti-de Sitter spacetime. By holography, this should map to a zero-dimensional quantum system: this turns out to be the SYK model.
SYK models and black holes

Black hole horizon

$\text{AdS}_2 \times S^2$

$$ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + d\vec{x}^2$$

Gauge field: $A = (E/\zeta)dt$

$\zeta = \infty$

$\vec{x}$

Bekenstein-Hawking entropy of $\text{AdS}_2$ horizon at $T = 0 \Leftrightarrow Ns_0$ entropy of SYK model

The correspondence between the complex SYK model and extremal black holes holds also for the low $T$ thermodynamics and low energy density of states. Both obey

$$\Omega(T) - E_0 = N \left[ -s_0 T - \frac{1}{2} (\gamma + 4\pi^2 \mathcal{E}^2 K)T^2 + \mathcal{O}(T^3) \right] + 2T \ln \left( \frac{U}{T} \right) \ldots$$

for the grand potential, and for the density of states at a fixed charge $Q$

$$d(E) \sim \exp (N s_0) \sinh \left( \sqrt{2N \gamma (E - E_0)} \right) , \quad E > E_0 , \quad e^{-cN} \ll \gamma (E - E_0) \ll N$$

with the relation $\frac{ds_0}{dQ} = 2\pi \mathcal{E}$ also obtained from Einstein’s equations

A Sen, JHEP 0509, 038 (2005)
SYK models and black holes

- Reparameterization invariance is a defining property of Einstein’s theory of gravity.

- In imaginary time, AdS\(_2\) is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)

\[
\frac{ds^2}{\zeta^2} = \frac{(d\tau^2 + d\zeta^2)}{\zeta^2} \text{ is invariant under}
\]

\[
\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.
\]
SYK models and black holes

- Reparameterization invariance is a defining property of Einstein’s theory of gravity

- In imaginary time, AdS$_2$ is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R).

$$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$  

Their identical symmetries lead to the same low energy quantum theory for the SYK model and extremal charged black holes!

A. Kitaev, 2015

Einstein-Maxwell-theory

\[ S_{4D} = \int d^4x \sqrt{-\hat{g}} \left( \hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right), \]

- Has Reissner-Nördstrom-AdS charged black hole solution, with charge density \( Q \), a near-horizon \( \text{AdS}_2 \times S^2 \) geometry, and surface electric field \( \mathcal{E} \).
  (This analysis also applies in asymptotically Minkowski spacetime \( L \rightarrow \infty \) provided the black hole mass is extremal.)

\[ ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + d\vec{x}^2 \]

Gauge field: \( A = (\mathcal{E}/\zeta) dt \)
Einstein-Maxwell-theory

\[ S_{4D} = \int d^4x \sqrt{-\hat{g}} \left( \hat{R} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right), \]

- Has Reissner-Nordstrom-AdS charged black hole solution, with charge density \( Q \), a near-horizon \( \text{AdS}_2 \times S^2 \) geometry, and surface electric field \( \mathcal{E} \). (This analysis also applies in asymptotically Minkowski spacetime \( L \to \infty \) provided the black hole mass is extremal.)

- From Einstein’s equations, the Bekenstein-Hawking black hole entropy \( S_{4D} \) is found to obey the same relation as the entropy of the SYK model

\[ \frac{\partial S_{4D}}{\partial Q} = 2\pi \mathcal{E}, \]

where \( \mathcal{E} \) is identified from the spectral asymmetry of probe particle Green’s functions in both cases. This establishes that the SYK entropy \( N s_0 \) maps onto \((\text{Area of horizon})/(4G)\)

\[ A \text{ Sen, JHEP 0509, 038 (2005)} \]

\[ S. \text{ Sachdev, PRX 5, 041025 (2015)} \]
In the small black hole size limit, $T \ll 1/R$, where $R$ is the radius of the black hole, the theory dimensionally reduces to an Einstein-Maxwell-dilaton theory in two dimensions (the Jackiw-Teitelbaum model), along with Maxwell term

$$S_{4D} = \int d^4 x \sqrt{-\hat{g}} \left( \hat{R} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

The dilaton $\Phi$ represents the radial oscillations of the small black hole.

$$S_{2D} = N s_0 + \int d^2 x \sqrt{-g} \left( \Phi (\hat{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right).$$
Einstein-Maxwell-theory

\[ S_{2D} = N s_0 + \int d^2 x \sqrt{-g} \left( \Phi (\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right). \]

There are no bulk quantum fluctuations of the metric in two-dimensional gravity, and there a further dimensional reduction to a 0 + 1 dimensional theory representing fluctuations of the AdS$_2$ boundary: this 0 + 1 dimensional turns out to be precisely the Schwarzian theory obtained for the SYK model.

\[ S_{\text{eff}} [f, \phi] = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T) \partial_\tau f)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \}, \]
Quantum matter without quasiparticles

- Planckian dynamics is realized in the ‘solvable’ SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where $T_H$ is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with SL(2,R) symmetry, describes the quantum dynamics of
  - the SYK models
  - black holes with near-extremal AdS$_2$ horizons