Conformal field theories in 3 dimensions, and holography

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Talk online at sachdev.physics.harvard.edu
“Complex entangled” states of quantum matter, not adiabatically connected to independent particle states

Gapped quantum matter

\( Z_2 \) Spin liquids, quantum Hall states

Conformal quantum matter

Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter

Strange metals, Bose metals

S. Sachdev, 100th anniversary Solvay conference (2011), arXiv:1203.4565
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Conformal quantum matter

A. Field theory:
  Honeycomb lattice
  Hubbard model

B. Gauge-gravity duality
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A. Field theory:
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B. Gauge-gravity duality
Honeycomb lattice
(describes graphene after adding long-range Coulomb interactions)

\[ H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) \]
Semi-metal with massless Dirac fermions at small $U/t$
The theory of free Dirac fermions is invariant under conformal transformations of spacetime. This is a realization of a simple conformal field theory in 2+1 dimensions: a CFT3.
The Hubbard Model at large $U$

$$H = -\sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

In the limit of large $U$, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

$$H_{AF} = \sum_{i<j} J_{ij} S_i^a S_j^a$$

where $a = x, y, z$,

$$S_i^a = \frac{1}{2} c_{i\alpha}^{a\dagger} \sigma_{\alpha\beta}^a c_{i\beta}$$

with $\sigma^a$ the Pauli matrices and

$$J_{ij} = \frac{4t_{ij}^2}{U}$$
Dirac semi-metal

Insulating antiferromagnet with Neel order

$U/t$
The lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field $\hat{a}$ where $J_A = \hat{a}$, $J_B = \hat{a}$.

The coupling between the field $\hat{a}$ and the fermions is given by

$$\hat{a} = \hat{a} \sigma^i \hat{c}^i \hat{c}^i \hat{a} \sigma^i = \hat{a} \sigma^z \hat{a} \sigma^z$$

Note that the matrix $\sigma^z$ commutes with all the $\hat{a}$; hence $\sigma^z$ is a matrix in "flavor" space. This is the Gross-Neveu model and it describes the quantum phase transition from the Dirac semi-metal to an insulating Neel state.

Low energy Gross-Neveu theory, with $\varphi^a$ the Néel order parameter.

$$\mathcal{L} = \bar{\Psi} \gamma_\mu \partial_\mu \Psi + \frac{1}{2} \left[ (\partial_\mu \varphi^a)^2 + s \varphi^a \varphi^a \right] + \frac{u}{24} (\varphi^a)^2 - \lambda \varphi^a \bar{\Psi} \rho^z \sigma^a \Psi$$
Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]
At the quantum critical point, the couplings $\lambda$ and $u$ reach non-zero fixed-point values under the RG flow (similar to the Wilson-Fisher fixed point). The critical theory is an interacting CFT3.
Interacting CFT3 with many-body entanglement and without quasiparticle excitations.

Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]

Free CFT3

Interacting CFT3 with many-body entanglement and without quasiparticle excitations
Electrical conductivity $\sigma(\omega)$

Dirac semi-metal

$\langle \varphi^a \rangle = 0$

Insulating antiferromagnet with Neel order

$\langle \varphi^a \rangle \neq 0$

$\sigma(\omega) = \frac{\pi e^2}{2h}$

$\sigma(\omega) = \frac{\kappa e^2}{\hbar}$
The electrical conductivity $\sigma(\omega)$ of a Dirac semi-metal is given by:

$$\sigma(\omega) = \frac{\pi e^2}{2h}$$

where $\omega$ is the frequency. The conductivity is independent of the temperature and can be computed in a $\epsilon$ or $1/N$ expansion, or by quantum Monte Carlo.

$\mathcal{K}$ is a universal number which can be computed (in principle) in a $\epsilon$ or $1/N$ expansion, or by quantum Monte Carlo.
Phase diagram at non-zero temperatures

- Insulator with thermally excited spin waves
- Quantum critical
- Semi-metal

Néel order
Phase diagram at non-zero temperatures

\[
\sigma(\omega \gg T) = \frac{\pi e^2}{2h}
\]

\[
\sigma(\omega \gg T) = \frac{Ke^2}{\hbar}
\]
Optical conductivity of graphene

Undoped graphene

Phase diagram at non-zero temperatures

- Quantum critical
- Insulator with thermally excited spin waves
- Semi-metal
- Néel order
Phase diagram at non-zero temperatures

- Insulator with thermally excited spin waves
- Semi-metal
- Quantum critical
- Néel order

Needed:
A theory for quantum-critical transport
Electrical transport in a free CFT3 for $T > 0$

$\sim T\delta(\omega)$
Electrical transport for a (weakly) interacting CFT3

\[ \sigma(\omega, T) = \frac{e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) \ ; \ \Sigma \rightarrow \text{a universal function} \]

Electrical transport for a (weakly) interacting CFT

\[ \sigma(\omega, T) = \frac{e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function} \]

\[ \mathcal{O}(u^*)^2, \]
where \( u^* \) is the fixed point interaction

Electrical transport for a (weakly) interacting CFT3

\[ \sigma(\omega, T) = \frac{e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) \sum \to \text{a universal function} \]

Re[\sigma(\omega)]

\[ \mathcal{O}(1/(u^*)^2) \]

\[ \mathcal{O}((u^*)^2), \text{ where } u^* \text{ is the fixed point interaction} \]

Electrical transport for a (weakly) interacting CFT3

\[ \sigma(\omega, T) = \frac{e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function} \]

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Electrical transport for a (weakly) interacting CFT3

\[ \sigma(\omega, T) = \frac{e^2}{\hbar} \sum \left( \frac{\hbar \omega}{k_B T} \right) \; ; \; \Sigma \rightarrow \text{a universal function} \]

Needed:

a method for computing the d.c. conductivity of interacting CFT3s (including that of pure graphene!)

Quantum critical transport

Quantum “nearly perfect fluid” with shortest possible equilibration time, $\tau_{eq}$

$$\tau_{eq} = \mathcal{C} \frac{\hbar}{k_B T}$$

where $\mathcal{C}$ is a universal constant

Quantum critical transport

Transport co-efficients not determined by collision rate, but by universal constants of nature

\[ \sigma = \frac{Q^2}{h} \times \text{[Universal constant } O(1) \text{]} \]

(Q is the “charge” of one particle)

Conformal quantum matter

A. Field theory:
   Honeycomb lattice
   Hubbard model

B. Gauge-gravity duality
Field theories in $d+1$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

$$u \frac{dg}{du} = \beta(g)$$

where $u$ is the energy scale. The RG equation is \textit{local} in energy scale, \textit{i.e.} the RHS does not depend upon $u$. 
Key idea: Implement $r$ as an extra dimension, and map to a local theory in $d + 2$ spacetime dimensions.
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation $(i = 1 \ldots d)$

$$x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds$$
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation $(i = 1 \ldots d)$

$$x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds$$

This gives the unique metric

$$ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2)$$

Reparametrization invariance in $r$ has been used to the prefactor of $dx_i^2$ equal to $1/r^2$. This fixes $r \rightarrow \zeta r$ under the scale transformation. This is the metric of the space $\text{AdS}_{d+2}$. 
AdS/CFT correspondence

AdS$_4$

$R^{2,1}$

Minkowski

$\mathcal{X}_i$

$\rho$
This emergent spacetime is a solution of Einstein gravity with a negative cosmological constant

$$S_E = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right]$$
For every primary operator $O(x)$ in the CFT, there is a corresponding field $\phi(x, r)$ in the bulk (gravitational) theory. For a scalar operator $O(x)$ of dimension $\Delta$, the correlators of the boundary and bulk theories are related by

$$\langle O(x_1) \ldots O(x_n) \rangle_{\text{CFT}} = Z^n \lim_{r \to 0} r_1^{-\Delta} \ldots r_n^{-\Delta} \langle \phi(x_1, r_1) \ldots \phi(x_n, r_n) \rangle_{\text{bulk}}$$

where the “wave function renormalization” factor $Z = (2\Delta - D)$. 
AdS/CFT correspondence

For a U(1) conserved current $J_\mu$ of the CFT, the corresponding bulk operator is a U(1) gauge field $A_\mu$. With a Maxwell action for the gauge field

$$S_M = \frac{1}{4g_M^2} \int d^{D+1}x \sqrt{g} F_{ab} F^{ab}$$

we have the bulk-boundary correspondence

$$\langle J_\mu(x_1) \ldots J_\nu(x_n) \rangle_{\text{CFT}} = (Z g_M^{-2})^n \lim_{r \to 0} r_1^{2-D} \ldots r_n^{2-D} \langle A_\mu(x_1, r_1) \ldots A_\nu(x_n, r_n) \rangle_{\text{bulk}}$$

with $Z = D - 2$. 

Sunday, December 16, 12
A similar analysis can be applied to the stress-energy tensor of the CFT, $T_{\mu\nu}$. Its conjugate field must be a spin-2 field which is invariant under gauge transformations: it is natural to identify this with the change in metric of the bulk theory. We write $\delta g_{\mu\nu} = (L^2/r^2)\chi_{\mu\nu}$, and then the bulk-boundary correspondence is now given by

$$
\langle T_{\mu\nu}(x_1) \ldots T_{\rho\sigma}(x_n) \rangle_{\text{CFT}} = 
\left( \frac{ZL^2}{\kappa^2} \right)^n \lim_{r \to 0} r_1^{-D} \ldots r_n^{-D} \langle \chi_{\mu\nu}(x_1, r_1) \ldots \chi_{\rho\sigma}(x_n, r_n) \rangle_{\text{bulk}},
$$

with $Z = D$. 

**AdS/CFT correspondence**
AdS/CFT correspondence

So the minimal bulk theory for a CFT with a conserved U(1) current is the *Einstein-Maxwell* theory with a cosmological constant

\[
S = \frac{1}{4g_M^2} \int d^4 x \sqrt{g} F_{ab} F^{ab} \\
+ \int d^4 x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right].
\]

This action is characterized by two dimensionless parameters: \( g_M \) and \( L^2/\kappa^2 \), which are related to the conductivity \( \sigma(\omega) = \mathcal{K} \) and the central charge of the CFT.
This minimal action also fixes multi-point correlators of the CFT: however these do not have the most general form allowed for a CFT. To fix these, we have to allow for higher-gradient terms in the bulk action. For the conductivity, it turns out that only a single 4 gradient term contributes

\[
S_{\text{bulk}} = \frac{1}{g_2^2 M} \int d^4 x \sqrt{g} \left[ \frac{1}{4} F_{ab} F^{ab} + \gamma L^2 C_{abcd} F^{ab} F^{cd} \right] \\
+ \int d^4 x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right],
\]

where \( C_{abcd} \) is the Weyl tensor. The parameter \( \gamma \) can be related to 3-point correlators of \( J_\mu \) and \( T_{\mu\nu} \). Both boundary and bulk methods show that \( |\gamma| \leq 1/12 \), and the bound is saturated by free fields.

D. Chowdhury, S. Raju, S. Sachdev, A. Singh, and P. Strack, arXiv:1210.5247
There is a family of solutions of Einstein gravity which describe non-zero temperatures.

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right] \]
AdS/CFT correspondence at non-zero temperatures

There is a family of solutions of Einstein gravity which describe non-zero temperatures

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r)dt^2 + dx^2 + dy^2 \right] \]

with \( f(r) = 1 - (r/R)^3 \)

A 2+1 dimensional system at its quantum critical point:

\[ k_B T = \frac{3\hbar}{4\pi R} \]
AdS/CFT correspondence at non-zero temperatures

AdS$_4$-Schwarzschild black-brane

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AdS\textsubscript{4}-Schwarzschild black-brane

A 2+1 dimensional system at its quantum critical point:
\[ k_B T = \frac{3\hbar}{4\pi R}. \]

Black-brane at temperature of 2+1 dimensional quantum critical system

Beckenstein-Hawking entropy of black brane = entropy of CFT3
AdS/CFT correspondence at non-zero temperatures

**AdS$_4$-Schwarzschild black-brane**

A 2+1 dimensional system at its quantum critical point:

\[ k_B T = \frac{3\hbar}{4\pi R}. \]

**Friction of quantum criticality = waves falling into black brane**

Black-brane at temperature of 2+1 dimensional quantum critical system
AdS4 theory of electrical transport in a strongly interacting CFT3 for $T > 0$

Conductivity is independent of $\omega/T$ for $\gamma = 0$. 

\[
\sigma = \frac{1}{g_M^2}
\]
AdS4 theory of electrical transport in a strongly interacting CFT3 for $T > 0$

Consequence of self-duality of Maxwell theory in 3+1 dimensions

Conductivity is independent of $\omega/T$ for $\gamma = 0$.

Electrical transport in a free CFT3 for $T > 0$

$\sigma \sim T \delta(\omega)$

Complementary $\omega$-dependent conductivity in the free theory
AdS4 theory of electrical transport in a strongly interacting CFT3 for $T > 0$

Conductivity is independent of $\omega/T$ for $\gamma = 0$. 

Conductivity \( \sigma \)

\[
\frac{1}{g_M^2}
\]

\( \omega/T \)
AdS$_4$ theory of “nearly perfect fluids”

The $\gamma > 0$ result has similarities to the quantum-Boltzmann result for transport of particle-like excitations.

The $\gamma < 0$ result can be interpreted as the transport of vortex-like excitations.

AdS$_4$ theory of “nearly perfect fluids”

- The $\gamma = 0$ case is the exact result for the large $N$ limit of SU($N$) gauge theory with $\mathcal{N} = 8$ supersymmetry (the ABJM model). The $\omega$-independence is a consequence of self-duality under particle-vortex duality ($S$-duality).

AdS$_4$ theory of “nearly perfect fluids”

- Stability constraints on the effective theory ($|\gamma| < 1/12$) allow only a limited $\omega$-dependence in the conductivity

AdS$_4$ theory of “nearly perfect fluids”

The holographic solutions for the conductivity satisfy two sum rules, valid for all CFT3s. (W. Witzack-Krempa and S. Sachdev, Phys. Rev. B 86, 235115 (2012))

$$\int_0^\infty d\omega \text{Re} \left[ \sigma(\omega) - \sigma(\infty) \right] = 0$$

$$\int_0^\infty d\omega \text{Re} \left[ \frac{1}{\sigma(\omega)} - \frac{1}{\sigma(\infty)} \right] = 0$$

The second rule follows from the existence of a EM-dual CFT3.

Boltzmann theory chooses a “particle” basis: this satisfies only one sum rule but not the other.

Holographic theory satisfies both sum rules.
Traditional CMT

- Identify quasiparticles and their dispersions
Traditional CMT

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- Compute scattering matrix elements of quasiparticles (or of collective modes)
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Holography and black-branes

- Start with strongly interacting CFT without particle- or wave-like excitations
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- Compute scattering matrix elements of quasiparticles (or of collective modes)
- These parameters are input into a quantum Boltzmann equation
- Deduce dissipative and dynamic properties at non-zero temperatures

Holography and black-branes

- Start with strongly interacting CFT without particle- or wave-like excitations
- Compute OPE co-efficients of operators of the CFT
- Relate OPE co-efficients to couplings of an effective gravitational theory on AdS
- Solve Einstein-Maxwell-... equations, allowing for a horizon at non-zero temperatures.
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Conclusions

Conformal quantum matter

New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points
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Conformal quantum matter

New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.
New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points.

The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.

Good prospects for experimental tests of frequency-dependent, non-linear, and non-equilibrium transport.