Entanglement, holography, and strange metals

Quantum Criticality and Novel Phases,
Dresden
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FIG. 1. (a) Schematic phase diagram for CeCu$_2$Si$_2$ at zero field, indicating existence ranges for phase A, superconductivity (S), and coexistence range (A + S). For samples labeled...
Resistivity \( \sim \rho_0 + AT^n \)

“Complex entangled” states of quantum matter, not adiabatically connected to independent particle states

Gapped quantum matter
Spin liquids, quantum Hall states

Conformal quantum matter
Quantum critical points in antiferromagnets, superconductors, and ultracold atoms; graphene

Compressible quantum matter
Non-Fermi liquids, strange metals, Bose metals
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Gapped quantum matter
\textit{Spin liquids, quantum Hall states}

Conformal quantum matter
\textit{Quantum critical points in antiferromagnets, superconductors, and dilute cold atoms and ultracold gases}

Compressible quantum matter
\textit{Non-Fermi liquids, strange metals, Bose metals}
• Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved U(1) charge \( Q \) (the “electron density”) in spatial dimension \( d > 1 \).

• Describe zero temperature phases where \( d\langle Q\rangle/d\mu \neq 0 \), where \( \mu \) (the “chemical potential”) which changes the Hamiltonian, \( H \), to \( H - \mu Q \).
The only compressible phase of traditional condensed matter physics which does not break the translational or $U(1)$ symmetries is the Landau Fermi liquid.
Our strategy here will be to start from conformal field theories, with a Hamiltonian $\mathcal{H}_\text{CFT}$ and a globally conserved U(1) charge $Q$, which are (reasonably) well understood examples of quantum states without quasiparticle excitations.

Then we will “dope” them by applying a chemical potential, $\mu$ and classify compressible states of $\mathcal{H}_\text{CFT} - \mu Q$. 

Conformal field theory (CFT) in 2+1 dimensions with a global U(1) symmetry

Apply a chemical potential

Superfluids, solids, and strange metals

AdS/CFT correspondence

Gravity and gauge theories on AdS\(_4\)

Apply a chemical potential

Gravity and gauge theories on 4-dim spacetimes with generalized scaling symmetries

Holographic theory of strange metals
\[ |\Psi\rangle \Rightarrow \text{Ground state of entire system}, \]
\[ \rho = |\Psi\rangle \langle \Psi| \]

\[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]

**Entanglement entropy** \( S_E = -\text{Tr} (\rho_A \ln \rho_A) \)

Entanglement entropy of a band insulator

\[ S_E = aP - b \exp(-cP) \]

where \( P \) is the surface area (perimeter) of the boundary between A and B.
**Mott insulator: the $\mathbb{Z}_2$ spin liquid**

\[ H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \]

\[ \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]


The $\mathbb{Z}_2$ spin liquid was introduced in
$S_E = aP - \ln(2)$

where $P$ is the surface area (perimeter) of the boundary between A and B.
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Coupled dimer XY antiferromagnet (hard core Bose gas)

\[ \mathcal{H} = \sum_{\langle ij \rangle} J_{ij} (S_{xi}S_{xj} + S_{yi}S_{yj}) = \sum_{\langle ij \rangle} J_{ij} (b_i^\dagger b_j + \text{H.c.}) \]

U(1) conserved charge :

\[ Q = \sum_i b_i^\dagger b_i \]

Examine ground state as a function of \( \lambda \)
Quantum critical point described by a CFT3 (O(2) Wilson-Fisher)

\[ \Phi = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]

Quantum critical point in a frustrated square lattice XY antiferromagnet

Valence bond solid (VBS) state with a nearly gapless, emergent "photon"

\[ \mathcal{H} = \sum_{\langle ij \rangle} J(S_{xi}S_{xj} + S_{yi}S_{yj}) + \ldots = \sum_{\langle ij \rangle} J(b_i^+b_j + \text{H.c.}) + \text{ring exchange terms.} \]

U(1) conserved charge : \[ Q = \sum_i b_i^+b_i \]


Quantum critical point in a frustrated square lattice XY antiferromagnet

Valence bond solid (VBS) state with a nearly gapless, emergent “photon”

CFT3 of spinons ($z_\uparrow$, $z_\downarrow$), with $b_i = (-1)^i z^*_i z_j$, coupled to an emergent photon ($A_\mu$)

$$ S_z = \int d^2r d\tau \left[ (\partial_\mu - i A_\mu) z_\alpha \right]^2 + s |z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{2e_0^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right] $$

• Entanglement entropy obeys $S_E = aP - \gamma$, where $\gamma$ is a shape-dependent universal number associated with the CFT3.

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Superfluids, solids, and strange metals
Key idea: Implement $r$ as an extra dimension, and map to a local theory in $d + 2$ spacetime dimensions.
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation $(i = 1 \ldots d)$

$$x_i \rightarrow \zeta x_i \quad , \quad t \rightarrow \zeta t \quad , \quad ds \rightarrow ds$$
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This gives the unique metric

$$ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2)$$

Reparametrization invariance in $r$ has been used to the prefactor of $dx_i^2$ equal to $1/r^2$. This fixes $r \rightarrow \zeta r$ under the scale transformation. This is the metric of the space $\text{AdS}_{d+2}$. 
AdS/CFT correspondence

AdS$_4$ \quad \quad R^{2,1} \quad \text{Minkowski}

CFT$_3$
Associate entanglement entropy with an observer in the enclosed spacetime region, who cannot observe “outside” : i.e. the region is surrounded by an imaginary horizon.

AdS/CFT correspondence

The entropy of this region is bounded by its surface area (Bekenstein-Hawking-’t Hooft-Susskind)

AdS/CFT correspondence

Minimal surface area measures entanglement entropy

Tensor network representation of entanglement at quantum critical point

\[ \text{depth of entanglement} \rightarrow \text{d-dimensional space} \]

Tensor network representation of entanglement at quantum critical point

Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

Brian Swingle, arXiv:0905.1317
Tensor network representation of entanglement at quantum critical point

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Emergent direction of AdS_{d+2}

Brian Swingle, arXiv:0905.1317
The AdS/CFT correspondence

AdS$_4$ \hspace{2cm} $\mathbb{R}^{2,1}$

Minkowski \hspace{2cm} CFT$_3$

- Computation of minimal surface area yields

$$S_E = aP - \gamma,$$

where $\gamma$ is a shape-dependent universal number.

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Quantum critical point in a frustrated square lattice XY antiferromagnet

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+ ring exchange terms. \(- \mu Q \)

U(1) conserved charge : \( Q = \sum_i b_i^\dagger b_i \)

Valence bond solid (VBS) state with a nearly gapless, emergent "photon"
Compressible quantum phases of a doped CFT3

- Superfluid (Néel order): breaks $U(1)$ symmetry
- Solid (Wigner crystal): breaks translational symmetry
- Strange metal: a *Bose metal*, a compressible phase which breaks no symmetries.
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NFL, the non-Fermi liquid *Bose metal*. The boson, \( b \), fractionalizes into (say) 2 fermions, \( f_1 \) and \( f_2 \) ("quarks"), each of which forms a Fermi surface. Both fermions necessarily couple to an emergent gauge field, and so the Fermi surfaces are "hidden".

\[ Q = b^\dagger b \]
\[ A_f = \langle Q \rangle \]

S. Sachdev, to appear
- **NFL**, the non-Fermi liquid *Bose metal*. The boson, $b$, fractionalizes into (say) 2 fermions, $f_1$ and $f_2$ ("quarks"), each of which forms a Fermi surface. Both fermions necessarily couple to an emergent gauge field, and so the Fermi surfaces are "hidden".

\[ b \rightarrow f_1 f_2 \]

Gauge invariance:
\[ f_1(x) \rightarrow f_1(x)e^{i\theta(x)}, \]
\[ f_2(x) \rightarrow f_2(x)e^{-i\theta(x)} \]

S. Sachdev, to appear
- $k_F^d \sim Q$, the fermion density
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$Q$, the fermion density

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Fermi liquid

- $k_F^d \sim Q$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

NFL Bose metal

- Hidden Fermi surface with $k_F^d \sim Q$.
- Diffuse fermionic excitations with $z = 3/2$ to three loops.
• $k_F^d \sim Q$, the fermion density

• Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

• Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

• $S \sim T^{(d-\theta)/z}$ with $\theta = d - 1$.

• Hidden Fermi surface with $k_F^d \sim Q$.

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Holographic theory of strange metals
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Monday, August 27, 12
Consider the metric which transforms under rescaling as

\[ x_i \rightarrow \zeta x_i \]
\[ t \rightarrow \zeta^z t \]
\[ ds \rightarrow \zeta^{\theta/d} ds. \]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

\( \theta \) is the violation of hyperscaling exponent.
Consider the metric which transforms under rescaling as

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This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for "relativistic" quantum critical points).

\( \theta \) is the violation of hyperscaling exponent.
The most general choice of such a metric is

\[
ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)
\]

We have used reparametrization invariance in \( r \) to choose so that it scales as \( r \rightarrow \zeta^{(d-\theta)/d} r \).

At $T > 0$, there is a horizon, and computation of its Bekenstein-Hawking entropy shows

$$S \sim T^{(d-\theta)/z}.$$ 

So $\theta$ is indeed the violation of hyperscaling exponent as claimed. For a compressible quantum state we should therefore choose $\theta = d - 1$. No additional choices will be made, and all subsequent results are consequences of the assumption of the existence of a holographic dual.

Holography of strange metals

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

\[ \theta = d - 1 \]

The null energy condition (stability condition for gravity) yields a new inequality

\[ z \geq 1 + \frac{\theta}{d} \]

In \( d = 2 \), this implies \( z \geq 3/2 \). So the lower bound is precisely the value obtained from the field theory.

Application of the Ryu-Takayanagi minimal area formula to a dual Einstein-Maxwell-dilaton theory yields

\[ S_E \sim Q^{(d-1)/d} P \ln P \]

with a co-efficient independent of UV details and of the shape of the entangling region. These properties are just as expected for a circular Fermi surface with \( Q \sim k_F^d \).

Holographic theory of a non-Fermi liquid (NFL)

Hidden Fermi surfaces of “quarks”

Electric flux

\[ \mathcal{E}_r = Q \]

Fully fractionalized state has all the electric flux exiting to the horizon at \( r = \infty \)
Holographic theory of a fractionalized-Fermi liquid (FL*)

A state with partial fractionalization, and partial electric flux exiting horizon:
has a small Fermi surface of electrons

Visible Fermi surfaces of “electrons”
Hidden Fermi surfaces of “quarks”

\[ \mathcal{E}_r = Q - Q_{\text{mesino}} \]
\[ \mathcal{E}_r = Q \]

S. Sachdev,
*Phys. Rev. Lett.* 105, 151602 (2010);
*Phys. Rev. D* 84, 066009 (2011)
Holographic theory of a Fermi liquid (FL)

Visible Fermi surfaces of “mesinos”

- Confining geometry leads to a state which has all the properties of a Landau Fermi liquid.

S. Sachdev, Physical Review D 84, 066009 (2011)
Holography, fractionalization, and hidden Fermi surfaces

- Electric flux exiting the horizon corresponds to fractionalized component of the conserved density $Q$, which is proposed to be associated with “hidden” Fermi surfaces of gauge-charged particles.

- Gauss Law and the “attractor” mechanism in the bulk $\iff$ Luttinger theorem on the boundary theory.
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Holographic method offers new approaches to quantum critical transport, and non-equilibrium dynamics. Related to dynamics of black hole horizons.
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Obtained holographic representation as a doped conformal field theory. Yields models of non-Fermi liquids (NFL), fractionalized Fermi liquids (FL*), and Fermi liquids (FL), in close correspondence with the phases expected from field theory.