Where is the quantum critical point in the cuprate superconductors?

arXiv:0907.0008
Hole dynamics in an antiferromagnet across a deconfined quantum critical point,
R. K. Kaul, A. Kolezhuk, M. Levin, S. Sachdev, and T. Senthil,

**Algebraic charge liquids**
R. K. Kaul, Yong-Baek Kim, S. Sachdev, and T. Senthil,

**Destruction of Neel order in the cuprates by electron doping,**
R. K. Kaul, M. Metlitski, S. Sachdev, and Cenke Xu,

**Paired electron pockets in the underdoped cuprates,**
V. Galitski and S. Sachdev,

**Competition between spin density wave order and superconductivity in**
**the underdoped cuprates,**
Eun Gook Moon and S. Sachdev,

**Fluctuating spin density waves in metals**
S. Sachdev, M. Metlitski, Yang Qi, and Cenke Xu
arXiv:0907.3732
Crossovers in transport properties of hole-doped cuprates

\[ \rho(T) \]

S-shaped

upturns in \( \rho(T) \)

d-wave SC

\[ \rho \sim T \]

\[ \rho \sim T + T^2 \]

or

\[ \rho \sim T^n \]

(1 < n < 2)

\[ \rho \sim T^2 \]

AFM

FM

T*

\[ T_{coh} \]

\[ T_{FL} \]

Classical spin waves

Quantum critical

Dilute triplon gas

Neel order

Pressure in TlCuCl$_3$

Crossovers in transport properties of hole-doped cuprates

Strange metal: quantum criticality of optimal doping critical point at $x = x_m$?

Strange metal

Hole doping $x$

$T$ (K)

AFM

d-wave SC


Only candidate quantum critical point observed at low $T$.

Spin and charge density wave order present below a quantum critical point at $x = x_s$ with $x_s \approx 0.12$ in the La series of cuprates.
Theory of quantum criticality in the cuprates

Fluctuating Fermi pockets

Strange Metal

Large Fermi surface

Spin density wave (SDW)

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Spin density wave theory in hole-doped cuprates

Increasing SDW order

The amplitude of the SDW order parameter $\bar{\varphi}$ vanishes at the transition to the Fermi liquid.

Spin density wave theory in hole-doped cuprates

Incommensurate order in $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$

Theory of quantum criticality in the cuprates

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Onset of $d$-wave superconductivity hides the critical point $x = x_m$. 

**Theory of quantum criticality in the cuprates**

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

Spin density wave (SDW)
Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave superconductor

Spin density wave (SDW)

Competing between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$. 
Theory of quantum criticality in the cuprates

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Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$. 

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Spin density wave (SDW)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$. 
1. Phenomenological quantum theory of competition between superconductivity and SDW order
   *Survey of recent experiments*

2. Superconductivity in the overdoped regime
   *BCS pairing by spin fluctuation exchange*

3. Superconductivity in the underdoped regime
   *U(1) gauge theory of fluctuating SDW order*
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   U(1) gauge theory of fluctuating SDW order
Write down a Landau-Ginzburg action for the quantum fluctuations of the SDW order ($\vec{\phi}$) and superconductivity ($\psi$):

$$S = \int d^2r d\tau \left[ \frac{1}{2} (\partial_\tau \vec{\phi})^2 + \frac{c^2}{2} (\nabla_x \vec{\phi})^2 + \frac{r}{2} \vec{\phi}^2 + \frac{u}{4} (\vec{\phi}^2)^2 + \kappa \vec{\phi}^2 |\psi|^2 \right]$$

$$+ \int d^2r \left[ |(\nabla_x - i(2e/\hbar c)A)\psi|^2 - |\psi|^2 + \frac{|\psi|^4}{2} \right]$$

where $\kappa > 0$ is the repulsion between the two order parameters, and $\nabla \times A = H$ is the applied magnetic field.

Write down a Landau-Ginzburg action for the quantum fluctuations of the SDW order ($\vec{\varphi}$) and superconductivity ($\psi$):

\[ S = \int d^2r d\tau \left[ \frac{1}{2} (\partial_\tau \vec{\varphi})^2 + \frac{c^2}{2} (\nabla_x \vec{\varphi})^2 + \frac{r}{2} \vec{\varphi}^2 + \frac{u}{4} (\vec{\varphi}^2)^2 \right. \]

\[ \left. + \kappa \vec{\varphi}^2 |\psi|^2 \right] \]

\[ + \int d^2r \left[ |(\nabla_x - i(2e/\hbar c)A)\psi|^2 - |\psi|^2 + \frac{|\psi|^4}{2} \right] \]

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Write down a Landau-Ginzburg action for the quantum fluctuations of the SDW order (\( \phi \)) and superconductivity (\( \psi \)):

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\right. \\
+ \kappa \phi^2 |\psi|^2 \\
+ \int d^2r \left[ |(\nabla_x - i(2e/\hbar c)A)\psi|^2 - |\psi|^2 + \frac{|\psi|^4}{2} \right]
\]

where \( \kappa > 0 \) is the repulsion between the two order parameters, and \( \nabla \times A = H \) is the applied magnetic field.
Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

- SDW order is more stable in the metal than in the superconductor: \( x_m > x_s \).

Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

- SDW order is more stable in the metal than in the superconductor: $x_m > x_s$.

Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

- For doping with $x_s < x < x_m$, SDW order appears at a quantum phase transition at $H = H_{sdw} > 0$.

Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

Neutron scattering on La$_{1.855}$Sr$_{0.145}$CuO$_4$


Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

$H_{c2}$

$H_{sdw}$

$\chi_s$

$\chi_m$

Neutron scattering on $\text{YBa}_2\text{Cu}_3\text{O}_{6.45}$


Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

Quantum oscillations without Zeeman splitting


Quantum oscillations

Electron pockets in the Fermi surface of hole-doped high-$T_c$ superconductors

David LeBoeuf$^1$, Nicolas Doiron-Leyraud$^1$, Julien Levallois$^2$, R. Daou$^1$, J.-B. Bonnemaison$^1$, N. E. Hussey$^3$, L. Balicas$^4$, B. J. Ramshaw$^5$, Ruixing Liang$^{5,6}$, D. A. Bonn$^{5,6}$, W. N. Hardy$^{5,6}$, S. Adachi$^7$, Cyril Proust$^2$ & Louis Taillefer$^{1,6}$

Nature 450, 533 (2007)
Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

Change in frequency of quantum oscillations in electron-doped materials identifies $x_m = 0.165$
Increasing SDW order

Nd$_{2-x}$Ce$_x$CuO$_4$

Phenomenological quantum theory of competition between superconductivity (SC) and spin-density wave (SDW) order

Neutron scattering at $H=0$ in same material identifies $x_s = 0.14 < x_m$
Nd$_{2-x}$Ce$_x$CuO$_4$

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

$H_{c2}$

$H_{sdw}$

$x_s$

$x_m$
"Normal" (Large Fermi surface)

SDW (Small Fermi pockets)

SC+ SDW

$X_S$, $X_M$, M

H

x
Fluctuating, paired Fermi pockets
Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Fluctuating, paired Fermi pockets

Thermally fluctuating SDW

Spin gap

Magnetic quantum criticality

d-wave SC

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

SC

SC+ SDW

M

x

T

H

x_m

X_S
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Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Magnetic quantal criticality

Spin gap

Thermally fluctuating SDW

d-wave SC

Fluctuating, paired Fermi pockets

SDW (Small Fermi pockets)

“Normal” (Large Fermi surface)
Theory of the onset of $d$-wave superconductivity from a large Fermi surface
Spin-fluctuation exchange theory of d-wave superconductivity in the cuprates

Fermions at the \textit{large} Fermi surface exchange fluctuations of the SDW order parameter $\vec{\phi}$.

Increasing SDW order

\[ \langle c_{k \uparrow} c_{-k \downarrow} \rangle \propto \Delta_k = \Delta_0 (\cos(k_x) - \cos(k_y)) \]

Approaching the onset of antiferromagnetism in the spin-fluctuation theory
Approaching the onset of antiferromagnetism in the spin-fluctuation theory

- $T_c$ increases upon approaching the SDW transition. SDW and SC orders do not compete, but attract each other.

Approaching the onset of antiferromagnetism in the spin-fluctuation theory

- $T_c$ increases upon approaching the SDW transition. SDW and SC orders do not compete, but attract each other.
- No simple mechanism for nodal-anti-nodal dichotomy.

“Nodal/anti-nodal dichotomy” in STM measurements

J.C. Davis and collaborators
Outline

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Strange Metal

Magnetic quantum criticality

Thermally fluctuating SDW

Fluctuating, paired Fermi pockets

Large Fermi surface

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

Spin gap

d-wave SC

M

$x_m$

$x_s$
Theory of the onset of $d$-wave superconductivity from small Fermi pockets
Begin with SDW ordered state, and rotate to a frame polarized along the local orientation of the SDW order $\hat{\phi}$

$$\begin{pmatrix} c_{\uparrow} \\ c_{\downarrow} \end{pmatrix} = R \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} ; \quad R^\dagger \hat{\phi} \cdot \vec{\sigma} R = \sigma^z ; \quad R^\dagger R = 1$$
Theory of underdoped cuprates

With

\[ R = \begin{pmatrix} z_{\uparrow} & -z^*_{\downarrow} \\ z_{\downarrow} & z^*_{\uparrow} \end{pmatrix} \]

the theory is invariant under the U(1) gauge transformation

\[ z_\alpha \rightarrow e^{i\theta} z_\alpha \quad ; \quad \psi_+ \rightarrow e^{-i\theta} \psi_+ \quad ; \quad \psi_- \rightarrow e^{i\theta} \psi_- \]

and the SDW order is given by

\[ \hat{\varphi} = z^*_\alpha \bar{\sigma}_{\alpha\beta} z_\beta \]
Starting from the “SDW-fermion” model with Lagrangian

\[ \mathcal{L} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \\
- E_{sdw} \sum_{i} c^\dagger_{i\alpha} \hat{\varphi}_i \cdot \hat{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K} \cdot \mathbf{r}_i} \\
+ \frac{1}{2t} \left( \partial_\mu \hat{\varphi} \right)^2 \]
we obtain a U(1) gauge theory of

- fermions $\psi_p$ with U(1) charge $p = \pm 1$ and pocket Fermi surfaces,

$$
\mathcal{L} = \sum_{\mathbf{k}, p = \pm} \left[ \psi_{\mathbf{k}p}^\dagger \left( \frac{\partial}{\partial \tau} - ipA_\tau + \varepsilon_{\mathbf{k}-pA} \right) \psi_{\mathbf{k}p} 
- E_{sdw} \psi_{\mathbf{k}p}^\dagger p \psi_{\mathbf{k}+\mathbf{K}, p} \right]
$$
Theory of underdoped cuprates

we obtain a U(1) gauge theory of

- fermions $\psi_p$ with U(1) charge $p = \pm 1$ and pocket Fermi surfaces,

- relativistic complex scalars $z_\alpha$ with charge 1, representing the orientational fluctuations of the SDW order

\[
\mathcal{L} = \sum_{\mathbf{k}, p = \pm} \left[ \psi^\dagger_{\mathbf{k}p} \left( \frac{\partial}{\partial \tau} - ipA_\tau + \varepsilon_{\mathbf{k}-pA} \right) \psi_{\mathbf{k}p} \right.

\left. - E_{sdw} \psi^\dagger_{\mathbf{k}p} p \psi_{\mathbf{k}+\mathbf{K}, p} \right] + \frac{1}{t} \left[ \left| (\partial_\tau - iA_\tau) z_\alpha \right|^2 + v^2 \left| \nabla - iA \right| z_\alpha \right|^2 \right] + i\lambda (|z_\alpha|^2 - 1) \]
Features of theory
Features of theory

- $d$-wave superconductivity.

\[
\begin{array}{cc}
\Gamma & + \\
- & - \\
+ & +
\end{array}
\]
Features of theory

- $d$-wave superconductivity.
- Nodal-anti-nodal dichotomy: strong pairing near $(\pi, 0)$, $(0, \pi)$, and weak pairing near zone diagonals.

Increasing SDW order

Strong "$s$-wave" pairing

Weak "$p$-wave" pairing
“Nodal/anti-nodal dichotomy” in STM measurements

J.C. Davis and collaborators
Features of theory

- $d$-wave superconductivity.

- Nodal-anti-nodal dichotomy: strong pairing near $(\pi, 0)$, $(0, \pi)$, and weak pairing near zone diagonals.

Increasing SDW order

\[ \Gamma \]

Strong "$s$-wave" pairing

Weak "$p$-wave" pairing
Features of theory

- $d$-wave superconductivity.

- Nodal-anti-nodal dichotomy: strong pairing near $(\pi, 0)$, $(0, \pi)$, and weak pairing near zone diagonals.

- Shift in quantum critical point of SDW ordering: gauge fluctuations are stronger in the superconductor. Onset of confinement, large Fermi surface, and possible charge order.
- Normal (Large Fermi surface)
- Small Fermi pockets with pairing fluctuations
- Strange Metal
- Magnetic quantum criticality
- Spin gap
- Thermally fluctuating SDW
- d-wave SC
- SDW (Small Fermi pockets)
Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

d-wave SC

Thermally fluctuating SDW

Magnetic quantum criticality

Spin gap

Fluctuating, paired Fermi pockets

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)
"Normal"
(Large Fermi surface)

SDW
(Small Fermi pockets)

SC+
SDW

Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Fluctuating, paired Fermi pockets

Thermally fluctuating SDW

Magnetic quantum criticality

Spin gap

d-wave

d-wave SC

T

x

H

\( x_m \)

\( x_s \)

M

SDW

SDW

(SC (Small Fermi pockets))

"Normal"

(Large Fermi surface)
Fluctuating, paired Fermi pockets

Thermally fluctuating SDW

Magnetic quantum criticality

Spin gap

SDW (Small Fermi pockets)

SC+ (Large Fermi surface)

Strange Metal

d-wave SC

"Normal" (Large Fermi surface)
Andrey Chubukov

“Normal”

(Large Fermi surface)

SDW

(Small Fermi pockets)

SC+

SDW

Fluctuating, paired Fermi pockets

Strange Metal

Large Fermi surface

T_{sdw}

d-wave

SC

\( \bar{x}_s \)

\( \bar{x}_m \)

M

Andrey Chubukov

“Normal”

(Large Fermi surface)
Fluctuating, paired Fermi pockets

Large Fermi surface

Strange Metal

Large Fermi surface

d-wave SC

T_{sdw}

Fermi pockets

Large Fermi surface

SC

SDW

(SC + SDW)

SDW (Small Fermi pockets)

M

"Normal" (Large Fermi surface)

Andrey Chubukov
Fluctuating, paired Fermi pockets

Large Fermi surface

Strange Metal

d-wave SC

$T_{sdw}$

$\chi_s$

$\chi_m$

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)
Fluctuating Fermi pockets

Strange Metal

Large Fermi surface

Spin density wave (SDW)

Increasing SDW order
Strange Metal

Fluctuating, paired Fermi pockets

Large Fermi surface

Small Fermi pockets with pairing fluctuations

d-wave SC

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)
Elusive optimal doping quantum critical point has been “hiding in plain sight”.

It is shifted to lower doping by the onset of superconductivity.
An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Weaken some bonds to induce spin entanglement in a new quantum phase
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state is a “quantum paramagnet” with spins locked in valence bond singlets

\[ = \frac{1}{\sqrt{2}} \left( \left| \uparrow\downarrow \right> - \left| \downarrow\uparrow \right> \right) \]
Pressure in TlCuCl$_3$
Excitation spectrum in the paramagnetic phase

\[
V(\varphi) = (\lambda - \lambda_c)\varphi^2 + u(\varphi^2)^2
\]

\[
\lambda > \lambda_c
\]
Excitation spectrum in the paramagnetic phase

\[ V(\vec{\phi}) = (\lambda - \lambda_c)\vec{\phi}^2 + u(\vec{\phi}^2)^2 \]

\[ \lambda > \lambda_c \]

Spin \( S = 1 \)

“triplon”
Excitation spectrum in the paramagnetic phase

\[ V(\phi) = (\lambda - \lambda_c)\phi^2 + u(\phi^2)^2 \]

\[ \lambda > \lambda_c \]

Spin \( S = 1 \) "triplon"
Excitation spectrum in the paramagnetic phase

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Spin $S = 1$

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Excitation spectrum in the paramagnetic phase

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\( \lambda > \lambda_c \)

Spin \( S = 1 \) “triplon”
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase
Excitation spectrum in the Néel phase

\[ V(\vec{\phi}) = (\lambda - \lambda_c)\vec{\phi}^2 + u(\vec{\phi}^2)^2 \]

\( \lambda < \lambda_c \)
Excitation spectrum in the Néel phase

Field theory yields spin waves ("Goldstone" modes) but also an additional longitudinal "Higgs" particle.

\[ V(\vec{\varphi}) = (\lambda - \lambda_c)\vec{\varphi}^2 + u(\vec{\varphi}^2)^2 \]

\[ \lambda < \lambda_c \]
Observation of $3 \rightarrow 2$ low energy modes, emergence of new Higgs particle in the Néel phase, and vanishing of Néel temperature at the quantum critical point.

Prediction of quantum field theory

Energy of “Higgs” particle

\[ \frac{E(p < p_c)}{E(p > p_c)} = \sqrt{2} \]

\[ V(\vec{\varphi}) = (\lambda - \lambda_c)\varphi^2 + u(\varphi^2)^2 \]

\[
\begin{align*}
\text{TlCuCl}_3 \\
p_c = 1.07 \text{ kbar} \\
T = 1.85 \text{ K}
\end{align*}
\]

\[ V(\vec{\varphi}) = (\lambda - \lambda_c)\vec{\varphi}^2 + u(\varphi^2)^2 \]