The AdS/CFT description of the quantum phases of matter

Penn State University, September 17, 2010

Talk online: sachdev.physics.harvard.edu
Outline

1. “Zero”-density quantum critical points
   Diffusion and transport in strongly interacting “perfect fluids”

2. Quantum matter at non-zero density
   Holographic superconductors and strange metals
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1. “Zero”-density quantum critical points
   Diffusion and transport in strongly interacting “perfect fluids”

2. Quantum matter at non-zero density
   Holographic superconductors and strange metals
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state has long-range Néel order

Order parameter is a single vector field \( \vec{\varphi} = \eta_i \vec{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\( \langle \vec{\varphi} \rangle \neq 0 \) in Néel state.
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Weaken some bonds to induce spin entanglement in a new quantum phase
$S = \int d^2 r d\tau \left[ (\partial_{\tau} \bar{\phi})^2 + c^2 (\nabla_r \bar{\phi})^2 + (\lambda - \lambda_c) \bar{\phi}^2 + u \left( \bar{\phi}^2 \right)^2 \right]$
Classical spin waves

Quantum critical

Dilute triplon gas

CFT3 at $T > 0$

Pressure in $\text{TI CuCl}_3$

Classical spin waves

Quantum critical

CFT3 at $T > 0$

Neel order

Dilute triplon gas

Pressure in TlCuCl$_3$

CFT3 at $T>0$

Strong coupling problem: dynamics and transport at times $> \frac{\hbar}{(k_B T)}$
where transport and damping constants are universally determined fundamental constants of nature.

Field theories in $D$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

$$u \frac{dg}{du} = \beta(g)$$

where $u$ is the energy scale. The RG equation is \textit{local} in energy scale, \textit{i.e.} the RHS does not depend upon $u$. 
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**Key idea:** $\Rightarrow$ Implement $u$ as an extra dimension, and map to a local theory in $D+1$ dimensions.
At the RG fixed point, $\beta(g) = 0$, the $D$ dimensional field theory is invariant under the scale transformation

$$x^\mu \rightarrow x^\mu / b \quad , \quad u \rightarrow b u$$
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This is an invariance of the metric of the theory in $D + 1$ dimensions. The unique solution is

$$ds^2 = \left( \frac{u}{L} \right)^2 dx^\mu dx_\mu + L^2 \frac{du^2}{u^2}.$$ 

Or, using the length scale $z = L^2 / u$

$$ds^2 = L^2 \frac{dx^\mu dx_\mu + dz^2}{z^2}.$$ 

This is the space $\text{AdS}_{D+1}$, and $L$ is the AdS radius.
Figure 1: The extra (‘radial’) dimension of the bulk is the resolution scale of the field theory. The left figure indicates a series of block spin transformations labelled by a parameter $z$. The right figure is a cartoon of AdS space, which organizes the field theory information in the same way. In this sense, the bulk picture is a hologram: excitations with different wavelengths get put in different places in the bulk image.

J. McGreevy, arXiv0909.0518
AdS/CFT correspondence
The quantum theory of a black hole in a 3+1-dimensional negatively curved AdS universe is holographically represented by a CFT (the theory of a quantum critical point) in 2+1 dimensions.
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Maldacena, Gubser, Klebanov, Polyakov, Witten
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3+1 dimensional AdS space

Quantum criticality in 2+1 dimensions

Black hole temperature = temperature of quantum criticality

Maldacena, Gubser, Klebanov, Polyakov, Witten

Wednesday, September 15, 2010
**AdS/CFT correspondence**

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3+1 dimensional AdS space

Black hole entropy = entropy of quantum criticality

Quantum criticality in 2+1 dimensions

Strominger, Vafa

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Friction of quantum criticality = waves falling into black hole

Quantum criticality in 2+1 dimensions

3+1 dimensional AdS space

Kovtun, Policastro, Son

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Friction of quantum criticality in 2+1 dimensions

Strong coupling problem: General solution of spin and magneto-thermo-electric transport in quantum critical region.


Kovtun, Policastro, Son
Quantum critical transport

Quantum “perfect fluid” with shortest possible relaxation time, $\tau_R$

$$\tau_R \gtrsim \frac{\hbar}{k_B T}$$

Quantum critical transport

Transport co-efficients not determined by collision rate, but by universal constants of nature

Spin/charge conductivity

\[ \sigma = \frac{Q^2}{\hbar} \times \text{[Universal constant } \mathcal{O}(1) \text{]} \]

(Q is the quantum of spin/charge)

Quantum critical transport

Transport co-efficients not determined by collision rate, but by universal constants of nature

Momentum transport

\[ \eta \equiv \frac{\text{viscosity}}{s} = \frac{\hbar}{k_B} \times \left[ \text{Universal constant } \mathcal{O}(1) \right] \]
Boltzmann theory of quantum critical transport

\[ \sigma = \frac{4e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function} \]

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Collisionless “critical” dissipation

Boltzmann theory of quantum critical transport

\[ \sigma = \frac{4e^2}{h} \sum \left( \frac{\hbar \omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function} \]

Collision-dominated hydrodynamics

AdS theory of strongly interacting “perfect fluids”

An infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$ with the following common properties:
AdS theory of strongly interacting “perfect fluids”

An infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS4 with the following common properties:

- Exact solutions for correlators of conserved densities as a function of k and ω exhibit the collisionless-hydrodynamic crossover.

AdS theory of strongly interacting “perfect fluids”

An infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$ with the following common properties:

- The viscosity/entropy-density ratio is

\[ \eta \propto \frac{\hbar}{s} = \frac{\hbar}{4\pi k_B} \]

AdS theory of strongly interacting “perfect fluids”

An infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$ with the following common properties:

- All continuous global symmetries are self dual.
- The conductivity $\sigma$ is $\omega$-independent and equal to the self-dual value. Frequency-dependent corrections are obtained from higher-derivative gravity theories.

AdS theory of strongly interacting “perfect fluids”

Examine quantum gravity effects by computing the consequences of higher-derivative corrections to the Einstein-Maxwell action.

- Stability constraints on the effective theory allow only a limited $\omega$-dependence in the conductivity

R. C. Myers, S. Sachdev, and A. Singh, arXiv:1009.xxxx
Resistivity of Bi films

Conductivity $\sigma$

$$\sigma_{\text{Superconductor}}(T \to 0) = \infty$$
$$\sigma_{\text{Insulator}}(T \to 0) = 0$$
$$\sigma_{\text{Quantum critical point}}(T \to 0) \approx \frac{4e^2}{h}$$

- Self-dual value $= \frac{4e^2}{h}$


Frequency dependency of integer quantum Hall effect

Little frequency dependence, and conductivity is close to self-dual value

FIG. 3. \( \text{Re}(\sigma_{xx}) \) vs \( B \) at three frequencies and two temperatures. Peaks are marked with Landau level index \( N \) and spin.

L. W. Engel, D. Shahar, C. Kurdak, and D. C. Tsui,
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Graphene

Conical Dirac dispersion

\[ \varepsilon_{\mathbf{k}} = \hbar v_F |\mathbf{k}| \]
Quantum phase transition in graphene tuned by a gate voltage

Electron Fermi surface

\[ \mu > 0 \]
Quantum phase transition in graphene tuned by a gate voltage

Hole Fermi surface

$\mu < 0$

Electron Fermi surface

$\mu > 0$
Quantum phase transition in graphene tuned by a gate voltage

There must be an intermediate quantum critical point where the Fermi surfaces reduce to a Dirac point
Quantum phase transition in graphene

\[ T(K) \]

\[ \frac{n}{10^{12}/m^2} \]

- Dirac liquid
- Hole Fermi liquid
- Quantum critical
- Electron Fermi liquid

\[ \sim \sqrt{n} \]

\[ (1 + \lambda \ln \Lambda \sqrt{n}) \]

\[ \mu < 0 \]

\[ \mu > 0 \]
Quantum phase transition in graphene

General solution of spin and magneto-thermo-electric transport in quantum critical region using AdS/CFT methods


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Quantum phase transition in graphene

What are the possible low $T$ phases at non-zero densities near strongly-interacting quantum critical points?

$T(K)$

$\mu < 0$

$\mu > 0$

Hole Fermi liquid

Electron Fermi liquid

$n \times 10^{12} / m^2$
AdS/CFT correspondence

3+1 dimensional AdS space

Quantum criticality in 2+1 dimensions
AdS/CFT correspondence

Move away from the quantum critical point to a system of matter at non-zero density: equivalent to adding an electrical charge to the black hole.

3+1 dimensional AdS space

Extremal Reissner-Nordstrom black hole

Finite density matter in 2+1 dimensions
One of the most common phases of finite density quantum matter at zero temperature is a superfluid or a superconductor.
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This is obtained in the dual gravity theory by condensing a scalar field in the background of a charged black hole.

The resulting theory has response functions indicating that the superfluid condensate is made up of pairs of fermions, just as in BCS theory.

This yields the first mean-field-theory of a superconductor which has a value of $\Delta/T_c$, (where $\Delta$ is the fermion energy gap) different from the BCS value.

The other common phase of finite density quantum matter at zero temperature is a metal.
AdS/CFT correspondence

Examine the free energy and Green’s function of a probe particle


3+1 dimensional AdS space

Finite density matter in 2+1 dimensions

Extremal Reissner-Nordstrom black hole
Green’s function of a fermion

\[ G(k, \omega) \approx \frac{1}{\omega - v_F(k - k_F) - i\omega \theta(k)} \]

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A Fermi surface: Green’s function is singular on a spherical surface at a non-zero \( k = k_F \). But the nature of the singularity differs from that in Landau’s Fermi liquid theory.

Green’s function of a fermion

\[ G(k, \omega) \approx \frac{\omega - v_F (k - k_F) - i\omega \theta(k)}{1 + \ldots} \]


See also S.-S. Lee, Phys. Rev. D 79, 086006 (2009);
M. Cubrovic, J. Zaanen, and K. Schalm, Science 325, 439 (2009);
The cuprate superconductors

Na-CCOC
- Cu
- Ca/Na
- O
- Cl

Antiferromagnet

PG

d-wave superconductor

Temperature

~2-3 %
~5-10 %
~15 %

hole concentration

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The cuprate superconductors

Na-CCOC
- Cu
- Ca/Na
- O
- Cl

Antiferromagnet
Strange Metal

~2-3 % ~5-10 % ~15 %
hole concentration

PG
ECG
d-wave superconductor
AdS theory of finite density quantum matter

$AdS_4$ Einstein-Maxwell theory of non-zero density quantum matter has a number of serious shortcomings:

- Non-zero ground state entropy density.
- Microscopic particle content is not clear.
- Single particle self energies are momentum independent, and their singular behavior is the same on and off the Fermi surface.
- Low energy singularities are described by "conformal quantum mechanics": a $0+1$ dimensional defect in a $2+1$ dimensional CFT. This is linked to the factorization of the near-horizon metric to $AdS_2 \times R^2$. 
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A Kondo lattice model for the AdS$_2 \times R^d$ region of an extremal Reissner-Nordstrom black hole

$$\sum_{i<j} J_H(i, j) \vec{S}_i \cdot \vec{S}_j$$

$J_H(i, j)$ Gaussian random variables.
A quantum Sherrington-Kirkpatrick model of SU($N$) spins.

S. Sachdev, arXiv:1006.3794
A Kondo lattice model for the $\text{AdS}_2 \times \mathbb{R}^d$ region of an extremal Reissner-Nordstrom black hole

Described by the conformal quantum mechanics of a quantum spin fluctuating in a self-consistent time-dependent magnetic field: a realization the finite entropy density $\text{AdS}_2 \times \mathbb{R}^d$ state

$$\sum_{i<j} J_H(i,j) \vec{S}_i \cdot \vec{S}_j$$

$J_H(i,j)$ Gaussian random variables.

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AdS$_2$ realization in the quantum SK model

Focus on a single $\vec{S}$ spin, and represent its imaginary time fluctuations by a unit vector $\vec{S} = \vec{n}(\tau)/2$ which is controlled by the partition function

$$Z = \int D\vec{n}(\tau) \delta(\vec{n}^2(\tau) - 1) \exp(-S)$$

$$S = \frac{i}{2} \int_0^1 du \int_0^{1/T} d\tau \vec{n} \cdot \left( \frac{\partial \vec{n}}{\partial u} \times \frac{\partial \vec{n}}{\partial \tau} \right) - \int_0^{1/T} d\tau \vec{h}(\tau) \cdot \vec{n}(\tau)$$

The first term is a Wess-Zumino term, with the “extra dimension” $u$ defined so that $\vec{n}(\tau, u = 1) \equiv \vec{n}(\tau)$ and $\vec{n}(\tau, u = 0) = (0, 0, 1)$.

The field $\vec{h}(\tau)$ represents the “environment”, which is determined as a Gaussian field obeying the self-consistency condition

$$\left< \vec{h}(\tau) \cdot \vec{h}(0) \right> \propto \left< \vec{n}(\tau) \cdot \vec{n}(0) \right>$$

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A small Fermi surface of probe fermions

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Low energy properties of the Sherrington-Kirkpatrick-Kondo model map onto the near-horizon physics of an extremal Reissner-Nordstrom black hole

$\sum_{i<j} J_{H}(i,j)$ Gaussian random variables.
A quantum Sherrington-Kirkpatrick model of SU($N$) spins.

S. Sachdev, arXiv:1006.3794
A Kondo lattice model for the $\text{AdS}_2 \times R^d$ region of an extremal Reissner-Nordstrom black hole

Much work remains in extending these solvable models (the AdS theories or the large-connectivity limits of Kondo lattice models) to a realistic theory of the cuprate superconductors.

$\sum_{i<j} J_{H}(i,j)$ Gaussian random variables.

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S. Sachdev, arXiv:1006.3794
Conclusions

New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle picture.
Conclusions

The AdS/CFT correspondence offers promise in providing a new understanding of strongly interacting quantum matter at non-zero density.