Quantum matter without quasiparticles: random fermion models, black holes, graphene, and non-Fermi liquids

Workshop on Non-equilibrium Physics and Holography
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Quantum matter without quasiparticles:

1. Ground states disconnected from independent electron states: many-particle entanglement

2. No quasiparticles

- Quantum criticality near the superfluid-insulator transition of ultracold bosonic atoms in an optical lattice
- Graphene
- Solvable random fermion Sachdev-Ye-Kitaev (SYK) model
- Charged black hole horizons in anti-de Sitter space
- Non-Fermi liquids in two spatial dimensions

Note: Most states with long-range entanglement, like the fractional quantum Hall states, do have quasiparticles
Local thermal equilibration or phase coherence time, $\tau_\varphi$:

- As $T \rightarrow 0$, there is an *lower bound* on $\tau_\varphi$ in all many-body quantum systems of order $\hbar/(k_B T)$,

$$\tau_\varphi > C \frac{\hbar}{k_B T},$$

and the lower bound is realized by systems without quasiparticles.

- In systems *with* quasiparticles, $\tau_\varphi$ is parametrically larger at low $T$; e.g. in Fermi liquids $\tau_\varphi \sim 1/T^2$, and in gapped insulators $\tau_\varphi \sim e^{\Delta/(k_B T)}$ where $\Delta$ is the energy gap.

A bound on quantum chaos:

- The time over which a many-body quantum system becomes chaotic is given by $\tau_L = 1/\lambda_L$, where $\lambda_L$ is the “Lyapunov exponent” determining memory of initial conditions (the “butterfly effect”):

$$D(t) = \langle W(t)V(0)W(t)V(0) \rangle \sim c_0 - \epsilon c_1 e^{\lambda_L t},$$

where we make a (system-dependent) choice to arrange $\epsilon \ll 1$. As $T \to 0$, this Lyapunov time is argued to obey the lower bound

$$\tau_L \geq \frac{1}{2\pi} \frac{\hbar}{k_B T}.$$

- Theories holographically dual to Einstein gravity have the shortest possible $\tau_L = \hbar/(2\pi k_B T)$

A.I. Larkin and Y. N. Ovchinnikov, JETP 28, 6 (1969)
S. H. Shenker and D. Stanford, arXiv:1306.0622
A bound on quantum chaos:

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- Theories holographically dual to Einstein gravity have the shortest possible $\tau_L = \hbar/(2\pi k_B T)$

Quantum matter without quasiparticles

$\approx$ fastest possible many-body quantum chaos
Quantum matter without quasiparticles:

- Solvable random fermion Sachdev-Ye-Kitaev (SYK) model
- Charged black hole horizons in anti-de Sitter space
- Graphene
- Non-Fermi liquids - Ising-nematic criticality of a two-dimensional metal
Infinite-range model with quasiparticles

\[
H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \ldots
\]

\[
c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}
\]

\[
\frac{1}{N} \sum_i c_i^\dagger c_i = Q
\]

\[t_{ij}\] are independent random variables with \(\bar{t_{ij}} = 0\) and \(|t_{ij}|^2 = t^2\)

Fermions occupying the eigenstates of a \(N \times N\) random matrix
**Infinite-range model with quasiparticles**

Feynman graph expansion in $t_{ij}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)}$$

$$\Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = Q.$$

$G(\omega)$ can be determined by solving a quadratic equation.
Infinite-range model with quasiparticles

Now add weak interactions

\[ H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell \]

\( J_{ij;kl} \) are independent random variables with \( \overline{J_{ij;kl}} = 0 \) and \( |\overline{J_{ij;kl}|^2 = J^2} \). We compute the lifetime of a quasiparticle, \( \tau_\alpha \), in an exact eigenstate \( \psi_\alpha(i) \) of the free particle Hamiltonian with energy \( E_\alpha \). By Fermi’s Golden rule, for \( E_\alpha \) at the Fermi energy

\[
\frac{1}{\tau_\alpha} = \pi J^2 \rho_0^3 \int dE_\beta dE_\gamma dE_\delta f(E_\beta)(1 - f(E_\gamma))(1 - f(E_\delta))\delta(E_\alpha + E_\beta - E_\gamma - E_\delta) \\
= \frac{\pi^3 J^2 \rho_0^3}{4} T^2
\]

where \( \rho_0 \) is the density of states at the Fermi energy.

**Fermi liquid state:** Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as \( \sim T^{-2} \) at the Fermi level.
SY model without quasiparticles

\[ H = \frac{1}{(NM)^{1/2}} \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{M} J_{ij} \hat{S}_i,\alpha \beta \hat{S}_j,\beta \alpha \]

\[ = \frac{1}{(NM)^{1/2}} \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{M} J_{ij} c_{i\alpha} c_{i\beta} c_{j\beta} c_{j\alpha} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \quad , \quad c_{i\alpha} c_{j\beta}^\dagger + c_{j\beta}^\dagger c_{i\alpha} = \delta_{ij} \delta_{\alpha\beta} \]

\[ \frac{1}{M} \sum_{\alpha} c_{i\alpha} c_{i\alpha} = Q \]

Generalization of the classical Sherrington-Kirkpatrick model to quantum SU(M) spins.

\( J_{ij} \) are independent random variables with \( \overline{J_{ij}} = 0 \) and \( \overline{J_{ij}^2} = J^2 \)

\( N \to \infty \) at \( M = 2 \) yields spin-glass ground state.

\( N \to \infty \) and then \( M \to \infty \) yields critical strange metal

SYK model without quasiparticles

To obtain a non-Fermi liquid, we set $t_{ij} = 0$:

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{i,j;\ell,k} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$Q = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$H_{SYK}$ is similar, and has identical properties, to the SY model.

SYK model without quasiparticles

To obtain a non-Fermi liquid, we set $t_{ij} = 0$:

$$H_{\text{SYK}} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{i;j; k;\ell} c_{i}^{\dagger} c_{j}^{\dagger} c_{k} c_{\ell} - \mu \sum_{i} c_{i}^{\dagger} c_{i}$$

$$Q = \frac{1}{N} \sum_{i} c_{i}^{\dagger} c_{i}$$

$H_{\text{SYK}}$ is similar, and has identical properties, to the SY model.

A fermion can move only by entangling with another fermion: the Hamiltonian has “nothing but entanglement”.

SYK model without quasiparticles

Feynman graph expansion in $J_{ij...}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)}, \quad \Sigma(\tau) = -J^2G^2(\tau)G(-\tau)
\]

\[
G(\tau = 0^-) = Q.
\]

Low frequency analysis shows that the solutions must be gapless and obey

\[
\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots, \quad G(z) = \frac{A}{\sqrt{z}}
\]

for some complex $A$. The ground state is a non-Fermi liquid, with a continuously variable density $Q$. 

A better understanding of the above facts can be reached from the perspective of symmetry-protected topological (SPT) phases. As shown recently in Ref. 14, the complex SYK model can be thought of as the boundary of a 1D SPT system in the symmetry class AIII. The periodicity of 4 in $N_{\text{ar}}$ arises from the fact that we need to put 4 chains to gap out the boundary degeneracy without breaking the particle-hole symmetry. In the Majorana SYK case, the symmetric Hamiltonian can be constructed as a symmetric matrix in the Clifford algebra $\mathbb{C}P^{N_1}$, and the Bott periodicity in the real representation of the Clifford algebra gives rise to a $\mathbb{Z}_8$ classification[14]. Here, for the complex SYK case, we can similarly construct the Clifford algebra by dividing one complex fermion into two Majorana fermions, and then we will have a periodicity of 4.

### From the above definition of retarded Green’s function, we can relate them to the imaginary time Green’s function as defined in Eq. (16), $G_R(\!\tau\!) = G(\!\tau\! + i\!\tau\!)$.

In Fig. 3, we show a comparison between the imaginary part of the Green’s function from large $N$ and from the exact diagonalization computation. The spectral function from ED is particle-hole symmetric for all $N$, while $G_B(\!\tau\!)$ is not.

We identify the infinite time limit of $G_B$ as the Edward-Anderson order parameter $q_{\text{EA}}$, which can characterize long-time memory of spin-glass: $q_{\text{EA}} = \lim_{t \to \infty} G_B(t)$. This is quite different from the fermionic case, where we have $G_F(\!\tau\!)$ $\sim 1/\lambda^2$. This inverse square-root behavior also holds in the bosonic case without spin glass order[1]. Fig. 10 is our result from ED, with a comparison between $G_B$ with $G_F$. It is evident that the behavior of $G_B$ is qualitatively different from $G_F$, and so an inverse square-root behavior is ruled out. Instead, we can clearly see that, as system size gets larger, $G_B$’s peak value increases much faster than the $G_F$’s peak value. This supports the presence of spin glass order.

Large $N$ solution of equations for $G$ and $\Sigma$ agree well with exact diagonalization of the finite $N$ Hamiltonian.

However, exact diagonalization of the same model with hard-core bosons indicates the presence of spin-glass order in the ground state.

W. Fu and S. Sachdev, arXiv: 1603.05246
SYK model without quasiparticles

Local fermion density of states

\[ \rho(\omega) = -\text{Im} \ G(\omega) \sim \begin{cases} \omega^{-1/2}, & \omega > 0 \\ e^{-2\pi \mathcal{E}} |\omega|^{-1/2}, & \omega < 0. \end{cases} \]

\[ \mathcal{E} \text{ encodes the particle-hole asymmetry} \]

While \( \mathcal{E} \) determines the low energy spectrum, it is determined by the total fermion density \( Q \):

\[ Q = \frac{1}{4} (3 - \tanh(2\pi \mathcal{E})) - \frac{1}{\pi} \tan^{-1} (e^{2\pi \mathcal{E}}). \]

Analog of the relationship between \( Q \) and \( k_F \) in a Fermi liquid.

SYK model without quasiparticles

At non-zero temperature, $T$, the Green’s function also fully determined by $\mathcal{E}$.

$$G^R(\omega) = \frac{-iCe^{-i\theta}}{(2\pi T)^{1-2\Delta}} \frac{\Gamma\left(\Delta - \frac{i\hbar\omega}{2\pi T} + i\mathcal{E}\right)}{\Gamma\left(1 - \Delta - \frac{i\hbar\omega}{2\pi T} + i\mathcal{E}\right)}$$

where $\Delta = 1/4$ and $e^{2\pi\mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}$.

Note $G(\omega) \equiv f(\hbar\omega/k_B T)$.

A. Georges and O. Parcollet PRB 59, 5341 (1999)
SYK model without quasiparticles

The entropy per site, $S$, has a non-zero limit as $T \to 0$. This is not due to an extensive degeneracy, but due to an energy level spacing $\sim e^{-aN}$ in the entire many-body spectrum all the way down to the ground state. At low $T$ we write

$$S(T \to 0) = S_0 + \gamma T + \ldots$$

where the specific heat is $C = \gamma T$, and $S_0$ obeys

$$\frac{dS_0}{dQ} = 2\pi \mathcal{E},$$

with $\mathcal{E}$ the same spectral asymmetry parameter.

Note that $S_0$ and $\mathcal{E}$ involve low-lying states, while $Q$ depends upon all states, and details of the UV structure.

J. Maldacena and D. Stanford, arXiv:1604.07818
SYK model without quasiparticles

After integrating the fermions, the partition function can be written as a path integral with an action $S$ analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det [\delta(\tau_1 - \tau_2)(\partial_\tau_1 + \mu) - \Sigma(\tau_1, \tau_2)]$$

$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left[ G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2) \right]$$

At frequencies $\ll J$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = \left[ f'(\sigma_1)f'(\sigma_2) \right]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = \left[ f'(\sigma_1)f'(\sigma_2) \right]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
SYK model without quasiparticles

Let us write the large $N$ saddle point solutions of $S$ as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2}, \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$  

These are not invariant under the reparametrization symmetry but are invariant only under a SL(2,R) subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$  

So the (approximate) reparametrization symmetry is spontaneously broken.

Reparametrization zero mode

Expand about the saddle point by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))$$  

(and similarly for $\Sigma$) and obtain an effective action for $f(\tau)$. This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

J. Maldacena and D. Stanford, arXiv:1604.07818
See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768
SYK model without quasiparticles

However the effective action must vanish for $\text{SL}(2,\mathbb{R})$ transformations because $G_s, \Sigma_s$ are invariant under it. In this manner we obtain the effective action as a Schwarzian

$$NS_{\text{eff}} = -\frac{N\gamma}{4\pi^2} \int d\tau \{ f, \tau \} , \quad \{ f, \tau \} \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 ,$$

where the specific heat, $\mathcal{C} = \gamma T$.  

The Schwarzian effective action implies that the SYK model saturates the lower bound on the Lyapunov time

$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}$$
SYK model without quasiparticles

The Schwarzian describes fluctuations of the energy operator with scaling dimension $h = 2$.
Apart from the energy operator associated with the Schwarzian, there are an infinite number of other scalar operators with irrational scaling dimensions given by the roots of

$$\tan\left(\frac{\pi(2h - 1)}{4}\right) = \frac{1 - 2h}{3}$$

$$\Rightarrow \quad h = 3.77354\ldots, 5.67946\ldots, 7.63197\ldots, 9.60396\ldots, \ldots$$
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- Non-Fermi liquids - Ising-nematic criticality of a two-dimensional metal
SYK and AdS$_2$

Holographic Metals and the Fractionalized Fermi Liquid

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We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a “small” Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, AdS$_2 \times \mathbb{R}^2$ physics of Reissner-Nordström black holes.
SYK and AdS$_2$

AdS$_2$ boundary Green’s function of $\psi$ at $T = 0$

$\text{Im}G(\omega) \sim \begin{cases} \omega^{-(1-2\Delta)}, & \omega > 0 \\ e^{-2\pi \mathcal{E}} |\omega|^{-(1-2\Delta)}, & \omega < 0 \end{cases}$, where the fermion scaling dimension $\Delta$ is a function of $m$

$\mathcal{E}$ encodes the particle-hole asymmetry

T. Faulkner, Hong Liu, J. McGreevy, and D. Vegh, PRD 83, 125002 (2011)
Conformal mapping to $T > 0$

\[ ds^2 = \frac{d\zeta^2}{(1 - \zeta^2 / \zeta_0^2) - (1 - \zeta^2 / \zeta_0^2)dt^2} / \zeta^2 + d\bar{x}^2 \]

Gauge field: \( A = \mathcal{E}(1/\zeta - 1/\zeta_0)dt \) with \( \zeta_0 = 1/(2\pi T) \)

AdS$_2$ boundary Green's function of $\psi$ at $T > 0$

is fully determined by $\mathcal{E}$

\[ G^R(\omega) = \frac{-iCe^{-i\theta}}{(2\pi T)^{1-2\Delta}} \frac{\Gamma \left( \Delta - \frac{i\hbar \omega}{2\pi T} + i\mathcal{E} \right)}{\Gamma \left( 1 - \Delta - \frac{i\hbar \omega}{2\pi T} + i\mathcal{E} \right)} \]

where \( e^{2\pi \mathcal{E}} = \frac{\sin(\pi \Delta + \theta)}{\sin(\pi \Delta - \theta)} \).

T. Faulkner, Hong Liu, J. McGreevy, and D. Vegh, PRD 83, 125002 (2011)
SYK and AdS$_2$

Conformal mapping to $T > 0$

$\zeta = \zeta_0$

\[ ds^2 = \left[ \frac{d\zeta^2}{1 - \zeta^2/\zeta_0^2} - \frac{(1 - \zeta^2/\zeta_0^2) dt^2}{\zeta^2 + d\vec{x}^2} \right] \]

Gauge field: $A = \mathcal{E} \left( \frac{1}{\zeta} - \frac{1}{\zeta_0} \right) dt$ with $\zeta_0 = 1/(2\pi T)$

$\zeta = \infty$

- As $T \to 0$, there is a non-zero Bekenstein-Hawking entropy, $S_{BH}$.
- Using Gauss’s Law, it can be shown that $\mu(T) = -2\pi \mathcal{E} T + \text{constant as } T \to 0$.
- Using a thermodynamic Maxwell relation (also obeyed by gravity),

\[
\left( \frac{\partial S_{BH}}{\partial Q} \right)_T = - \left( \frac{\partial \mu}{\partial T} \right)_Q = 2\pi \mathcal{E}
\]

A. Sen
hep-th/0506177
S. Sachdev
PRX 5, 041025 (2015)

Also obeyed by Wald entropy in higher-derivative gravity.
SYK and AdS$_2$

The same Schwarzian effective action describes low energy fluctuations on the boundary theory of gravity theories with AdS$_2$ near-horizon geometries (including the AdS-Reissner-Nordstrom solution of Einstein-Maxwell theory in 4 space-time dimensions). And the co-efficient of the Schwarzian, $N\gamma/4\pi^2$, determines the specific heat $C = \gamma T$.

The Schwarzian effective action implies that both the SYK model and the AdS$_2$ theories *saturate* the lower bound on the Lyapunov time

$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}.$$
Quantum matter without quasiparticles:

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Graphene

Same “Hubbard” model as for ultracold atoms, but for electrons on the honeycomb lattice
Graphene

$T(K)$

Quantum critical Dirac liquid

Hole Fermi liquid

Electron Fermi liquid

$\mu < 0$

$\mu > 0$

Predicted “strange metal” without quasiparticles

M. Müller, L. Fritz, and S. Sachdev, PRB 78, 115406 (2008)

M. Müller and S. Sachdev, PRB 78, 115419 (2008)
Fermi liquids: quasiparticles moving ballistically between impurity (red circles) scattering events

Strange metals: electrons scatter frequently off each other, so there is no regime of ballistic quasiparticle motion. The electron “liquid” then “flows” around impurities
Graphene

Quantum critical

Electron Fermi liquid

Hole Fermi liquid

$T(K)$

$n \sim \sqrt{n}(1 + \lambda \ln \Lambda \sqrt{n})$

$T(K)$

$\mu < 0$

$\mu > 0$

$10^{12}/m^2$

M. Müller, L. Fritz, and S. Sachdev, PRB 78, 115406 (2008)

M. Müller and S. Sachdev, PRB 78, 115419 (2008)
Thermal and electrical conductivity with quasiparticles

- Wiedemann-Franz law in a Fermi liquid:

\[ L_0 = \frac{\kappa}{\sigma T} \approx \frac{\pi^2 k_B^2}{3e^2} \approx 2.45 \times 10^{-8} \frac{W \cdot \Omega}{K^2}. \]
Transport in Strange Metals

- hydrodynamics when \( l \gg l_{ee}, t \gg t_{ee} \)
- long time dynamics governed by conservation laws:

\[
\partial_\nu T^{\mu\nu} = J_\nu \left( F^{\text{ext}} \right)^{\mu\nu}, \quad \partial_\mu J^\mu = 0.
\]

- dynamics of relaxation to equilibrium

expand \( T^{\mu\nu}, J^\mu \) in perturbative parameter \( l_{ee} \partial_\mu \):

\[
T^{\mu\nu} = P \eta^{\mu\nu} + (\epsilon + P) u^\mu u^\nu
\]

\[
J^\mu = Q u^\mu - \sigma_Q \mathcal{P}^{\mu\rho} \left( \partial_\rho \mu - \frac{\mu}{T} \partial_\rho T - u^\nu F^{\text{ext}}_{\rho\nu} \right) + \cdots,
\]

\[
\mathcal{P}^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu,
\]

\[
Q^i = T^i - \mu J^i
\]

New (and only) transport co-efficient, \( \sigma_Q \):
- “quantum critical” conductivity at \( Q = 0 \).
more generally, measure thermoelectric transport:

\[
\begin{pmatrix}
\delta J_i \\
\delta Q_i
\end{pmatrix} = \begin{pmatrix}
\sigma_{ij} & \alpha_{ij} \\
T\bar{\alpha}_{ij} & \bar{\kappa}_{ij}
\end{pmatrix} \begin{pmatrix}
\delta E_j \\
-\partial_j \delta T \equiv T \delta \zeta_j
\end{pmatrix}.
\]

\[\sigma = \text{easy experiment; related to QFT correlators:}\]

\[\sigma_{ij}(\omega) = \frac{i}{\omega} \langle J_i(-\omega)J_j(\omega) \rangle, \quad \text{etc.}\]
Thermoelectric transport coefficients

Transport has two components: a “momentum drag” term, and a “quantum critical” term.

\[
\sigma = \frac{Q^2}{\mathcal{M}} \pi \delta(\omega) + \sigma_Q(\omega)
\]

\[
\alpha = \frac{SQ}{\mathcal{M}} \pi \delta(\omega) + \alpha_Q(\omega)
\]

\[
\bar{\kappa} = \frac{TS^2}{\mathcal{M}} \pi \delta(\omega) + \bar{\kappa}_Q(\omega)
\]

with entropy density \(S\), \(Q \equiv \chi_{J_x,P_x}\), and \(\mathcal{M} \equiv \chi_{P_x,P_x}\).

In theories which are relativistic at high energies (including graphene), \(T\alpha_Q(\omega) = -\mu\sigma_Q(\omega), T\bar{\kappa}_Q(\omega) = \mu^2\sigma_Q(\omega), \mathcal{M} = TS + \mu Q = \mathcal{H}\) the enthalpy density, and \(Q = n\) the electron density

Translational symmetry breaking

Momentum relaxation by an external source $h$ coupling to the operator $\mathcal{O}$

$$ H = H_0 - \int d^d x \, h(x) \, \mathcal{O}(x). $$

Leads to an additional term in equations of motion:

$$ \partial_\mu T^{\mu i} = \ldots - \frac{T^{it}}{\tau_{\text{imp}}} + \ldots $$

“Memory function” methods yield an explicit expression for $\tau_{\text{imp}}$:

$$ \frac{\mathcal{M}}{\tau_{\text{imp}}} = \lim_{\omega \to 0} \int d^d q \, |h(q)|^2 q_x^2 \frac{\text{Im} \left( G_{\mathcal{O}\mathcal{O}}^R (q, \omega) \right)}{\omega} H_0 + \ldots $$

Thermoelectric transport coefficients

Transport has two components: a “momentum drag” term, and a “quantum critical” term.

\[ \sigma = \frac{Q^2}{M} \pi \delta(\omega) + \sigma_Q(\omega) \]
\[ \alpha = \frac{S Q}{M} \pi \delta(\omega) + \alpha_Q(\omega) \]
\[ \kappa = \frac{TS^2}{M} \pi \delta(\omega) + \kappa_Q(\omega) \]

with entropy density \( S \), \( Q \equiv \chi_{J_x, P_x} \), and \( M \equiv \chi_{P_x, P_x} \).

 Obtained in hydrodynamics, and by memory functions

Thermoelectric transport coefficients

Transport has two components: a “momentum drag” term, and a “quantum critical” term.

\[
\sigma = \frac{Q^2}{\mathcal{M}} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} + \sigma_Q(\omega)
\]

\[
\alpha = \frac{SQ}{\mathcal{M}} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} + \alpha_Q(\omega)
\]

\[
\bar{\kappa} = \frac{TS^2}{\mathcal{M}} \frac{1}{(-i\omega + 1/\tau_{\text{imp}})} + \bar{\kappa}_Q(\omega)
\]

with entropy density \( S \), \( Q \equiv \chi_{J_x,P_x} \), and \( \mathcal{M} \equiv \chi_{P_x,P_x} \).

Obtained in hydrodynamics, and by memory functions

Prediction for transport in the graphene strange metal

For a strange metal with a “relativistic” Hamiltonian, hydrodynamic, holographic, and memory function methods yield for the Lorentz ratio \( L = \kappa / (T \sigma) \)

\[
\sigma = \sigma_Q \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right), \quad \kappa = \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T} \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right)^{-1}
\]

\[
L = \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T^2 \sigma_Q} \left( 1 + \frac{e^2 v_F^2 Q^2 \tau_{\text{imp}}}{\mathcal{H} \sigma_Q} \right)^{-2},
\]

where \( \mathcal{H} \) is the enthalpy density, \( \tau_{\text{imp}} \) is the momentum relaxation time (from impurities), while \( \sigma = \sigma_Q \), an intrinsic, finite, “quantum critical” conductivity.

- For \( Q = 0 \), as \( \tau_{\text{imp}} \to \infty \), \( \sigma \) remains finite, while \( \kappa \to \infty \), and so \( L \to \infty \).
- For \( Q \neq 0 \), as \( \tau_{\text{imp}} \to \infty \), \( \sigma \to \infty \), while \( \kappa \) remains finite, and so \( L \to 0 \).

**Prediction:** \( L \) diverges as \( 1/Q^4 \) near \( Q = 0 \) in clean graphene.

M. Müller and S. Sachdev, PRB 78, 115419 (2008)
Predicted strange metal

M. Müller, L. Fritz, and S. Sachdev, PRB 78, 115406 (2008)
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To minimize disorder, the monolayer graphene samples are locally doped to form electron-hole puddles with finite potential, and even when the sample is globally neutral, it represents the charge puddle energy, consistent with previous notes that the extraction of temperatures we find to the minimum conductivity \([\kappa_{\text{min}}]\) as a function of log(\(\sigma\)). At the CNP, the residual carrier density \(n\) can be estimated by extrapolating a linear fit of log(\(\sigma\)) away from neutrality (dashed grey lines). Curves have been offset vertically such that the neutrality point \(\sigma=0\) is at 0 V, electron-phonon coupling becomes appreciable and begins to dominate thermal transport at all measured gate voltages. All data from this figure is taken from the thermal conductivity is enhanced and the WF law is violated. Above \(100 \, \text{K}\), electron-phonon scattering rate becomes comparable to the electron-electron scattering rate. These two temperature set the experimental window in which the DF and the breakdown of the WF law can be observed. Above \(250 \, \text{K}\), formation of the DF is further complicated by phonon scattering at high temperature which can relax momentum by creating additional inelastic scattering.

FIG. 1. Temperature and density dependent electrical and thermal conductivity. (A) Resistance versus gate voltage for a representative device (see SM for all samples). From this, the neutrality point can be isolated. At low temperature and/or high doping (blue lines). At low temperature and/or high doping, the minimum density (green) aligns with the temperature axis to the right. Solid black lines correspond to 4 times the thermal energy be larger than the local chemical potential, and even when the sample is globally neutral, it represents the charge puddle energy, consistent with previous note that the extraction of temperatures we find to the minimum conductivity \([\kappa_{\text{min}}]\) as a function of log(\(\sigma\)). At the CNP, the residual carrier density \(n\) can be estimated by extrapolating a linear fit of log(\(\sigma\)) away from neutrality (dashed grey lines). Curves have been offset vertically such that the neutrality point \(\sigma=0\) is at 0 V, electron-phonon coupling becomes appreciable and begins to dominate thermal transport at all measured gate voltages. All data from this figure is taken from the thermal conductivity is enhanced and the WF law is violated. Above \(100 \, \text{K}\), electron-phonon scattering rate becomes comparable to the electron-electron scattering rate. These two temperature set the experimental window in which the DF and the breakdown of the WF law can be observed. Above \(250 \, \text{K}\), formation of the DF is further complicated by phonon scattering at high temperature which can relax momentum by creating additional inelastic scattering.

Red dots: data
Blue line: value for \(L = L_0\)
To minimize disorder, the monolayer graphene samples used in this report are encapsulated in hexagonal boron nitride (hBN). Formation of the Dirac fluid in graphene requires that the neutrality point, defined at position \( r \approx 0 \), is locally doped to form electron-hole puddles with finite potential, and even when the sample is globally neutral, it can be locally charged. At the neutrality point, the residual charge carriers contribute to the charge puddle energy, consistent with previous estimates the charge puddle energy, consistent with previous work.

Temperature and density dependent electrical and thermal conductivity. (A) Electrical conductivity (blue) as a function of the charge density set by the back gate for different temperatures. We see a clear crossover in the conductivity at high temperatures, which is consistent with the Wiedemann-Franz law (WF law) to hold. This is a non-trivial check on the quality of our measurement. In the non-degenerate regime, the conductivity is enhanced and the WF law is violated. Above the neutrality point, thermal excitations begin to dominate and the sample enters the non-degenerate regime near the minimum density (green) aligns with the temperature axis to the right. Solid black lines correspond to a linear fit of log(\( \sigma \)) as a function of log(\( n \)).

At low temperature and/or high doping, the electron-electron scattering rate becomes comparable to the electron-phonon scattering rate at high temperature which can reduce the conductivity.

FIG. 1. Resistance versus gate voltage at various temperatures. To minimize disorder, the monolayer graphene samples are used in this report are encapsulated in hexagonal boron nitride (hBN). Realization of the Dirac fluid in graphene requires that the neutrality point, defined at position \( r \approx 0 \), is locally doped to form electron-hole puddles with finite potential, and even when the sample is globally neutral, it can be locally charged. At the neutrality point, the residual charge carriers contribute to the charge puddle energy, consistent with previous work.

Red dots: data
Blue line: value for \( L = L_0 \)

J. Crossno et al., Science 351, 1058 (2016)
Two-terminal to keep a well-defined temperature profile. Nitride (hBN) used in this report are encapsulated in hexagonal boron nitride. The breakdown of the WF law can be observed.

A low momentum limit is locally doped to form electron-hole puddles with finite potential, and even when the sample is globally neutral, it may still have a finite residual carrier density due to disorder. At the neutrality point, the residual carrier density at the neutrality point (green) is estimated by the intersection of the minimum conductivity with a linear fit to log(κ_bath) versus log(T). All measurements are performed in a cryostat controlling bath temperatures.

Formation of the DF is further complicated by the thermal energy be larger than the local chemical potential, and even when the sample is globally neutral, it may still have a finite residual carrier density due to disorder. At the neutrality point, the residual carrier density at the neutrality point (green) is estimated by the intersection of the minimum conductivity with a linear fit to log(κ_bath) versus log(T). All measurements are performed in a cryostat controlling bath temperatures.

Temperature and density dependent electrical and thermal conductivity. (A) Electrical conductivity (blue) as a function of the charge density set by the back gate for different temperatures, the minimum density is limited by disorder (charge puddles). However, above the minimum density (green) aligns with the temperature axis to the right. Solid black lines correspond to 4 times the thermal conductivity with a linear fit to log(T) − |Δn| upon the temperature. The residual carrier density at the neutrality point (green) is estimated by the intersection of the minimum conductivity with a linear fit to log(κ_bath) versus log(T). All data from this figure is taken from the known sample dimensions. At the CNP, the residual carrier density at the neutrality point (green) is estimated by the intersection of the minimum conductivity with a linear fit to log(κ_bath) versus log(T). All measurements are performed in a cryostat controlling bath temperatures.

Red dots: data
Blue line: value for $L = L_0$
Graphene

\[ T(K) \]

Quantum Critical

Dirac Liquid

Electron

Fermi liquid

Hole

Fermi liquid

\[ \sim 1 / \sqrt{n} (1 + \lambda \ln \Lambda \sqrt{n}) \]

\[ n \rightarrow 10^{12} / m^2 \]

Impurity scattering dominates

M. Müller, L. Fritz, and S. Sachdev, PRB 78, 115406 (2008)
M. Müller and S. Sachdev, PRB 78, 115419 (2008)
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Strange metal in graphene

Wiedemann-Franz violated!

Wiedemann-Franz obeyed
Lorentz ratio \( L = \kappa / (T \sigma) \)

\[
= \frac{v_F^2 \mathcal{H} \tau_{\text{imp}}}{T^2 \sigma_Q} \frac{1}{(1 + e^2 v_F^2 Q^2 \tau_{\text{imp}} / (\mathcal{H} \sigma_Q))^2}
\]

\( Q \rightarrow \) electron density; \( \mathcal{H} \rightarrow \) enthalpy density

\( \sigma_Q \rightarrow \) quantum critical conductivity

\( \tau_{\text{imp}} \rightarrow \) momentum relaxation time from impurities


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Figure 1: A comparison of our hydrodynamic theory of transport with the experimental results of [33] in clean samples of graphene at $T = 75$ K. We study the electrical and thermal conductances at various charge densities $n$ near the charge neutrality point. Experimental data is shown as circular red data markers, and numerical results of our theory, averaged over 30 disorder realizations, are shown as the solid blue line. Our theory assumes the equations of state described in (27) with the parameters $C_0 = 11$, $C_2 = 9$, $C_4 = 200$, $\gamma_0 = 110$, $\gamma_0 = 0.7$, and (28) with $u_0 = 0.13$. The yellow shaded region shows where Fermi liquid behavior is observed and the Wiedemann-Franz law is restored, and our hydrodynamic theory is not valid in or near this regime. We also show the predictions of (2) as dashed purple lines, and have chosen the 3 parameter fit to be optimized for $\rho_n$.

Both of these predictions were observed in the recent experiment, and fits of the measured $\rho$ to (3) were quantitatively consistent, until large enough $n$ where Fermi liquid behavior was restored. However, the experiment also found that the conductivity did not grow rapidly away from $n = 0$ as predicted in (2), despite a large peak in $\rho_n$ near $n = 0$, as we show in Figure 1. Furthermore, the theory of [25] does not make clear predictions for the temperature dependence of $\tau$, which determines $\rho$. In this paper, we argue that there are two related reasons for the breakdown of (2). One is that the dominant source of disorder in graphene – fluctuations in the local charge density, commonly referred to as charge puddles [43, 44, 45, 46] – are not perturbatively weak, and therefore a non-perturbative treatment of their effect is necessary.

Figure 3: A cartoon of a nearly quantum critical fluid where our hydrodynamic description of transport is sensible. The local chemical potential $\mu(x)$ always obeys $|\mu| \ll k_B T$, and so the entropy density $s/k_B$ is much larger than the charge density $|n|$; both electrons and holes are everywhere excited, and the energy density $\epsilon$ does not fluctuate as much relative to the mean. Near charge neutrality the local charge density flips sign repeatedly. The correlation length of disorder $\xi$ is much larger than $l_{ee}$, the electron-electron interaction length.

Numerically solve the hydrodynamic equations in the presence of a $x$-dependent chemical potential. The thermoelectric transport properties will then depend upon the value of the shear viscosity, $\eta$. 

Solution of the hydrodynamic equations in the presence of a space-dependent chemical potential. Best fit of density dependence to thermal conductivity now gives a better fit to the density dependence of the electrical conductivity (for $\eta/s \approx 10$). The $T$ dependencies of other parameters also agree well with expectation.

Strange metal in graphene

Negative local resistance due to viscous electron backflow in graphene

Strange metal in graphene

Negative local resistance due to viscous electron backflow in graphene


Figure 1. Viscous backflow in doped graphene. (a,b) Steady-state distribution of current injected through a narrow slit for a classical conducting medium with zero $\nu$ (a) and a viscous Fermi liquid (b). (c) Optical micrograph of one of our SLG devices. The schematic explains the measurement geometry for vicinity resistance. (d,e) Longitudinal conductivity $\sigma_{xx}$ and $R_V$ for this device as a function of $n$ induced by applying gate voltage. $I = 0.3 \mu A; L = 1 \mu m$. For more detail, see Supplementary Information.
Signature of Navier-Stokes hydrodynamic flow in PdCoO$_2$

Experiment: Successively narrow the channel in factors of 2, measuring the resistance after every step.

Graphene: “a metal that behaves like water”
Quantum matter without quasiparticles:

- Solvable random fermion Sachdev-Ye-Kitaev (SYK) model
- Charged black hole horizons in anti-de Sitter space
- Graphene
- Non-Fermi liquids - Ising-nematic criticality of a two-dimensional metal
Quantum criticality of Ising-nematic ordering in a metal

A metal with a Fermi surface with full square lattice symmetry
Quantum criticality of Ising-nematic ordering in a metal

Pomeranchuk instability as a function of coupling $\lambda$

\[ \langle \phi \rangle \neq 0 \]

or

\[ \langle \phi \rangle = 0 \]
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering in a metal

- $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm \vec{k}_0$ and boson ($\phi$) kinetic energy about $\vec{q} = 0$. 
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L}[\psi_{\pm}, \phi] = \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \]

\[ -\phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Quantum criticality of Ising-nematic ordering in a metal

Thermodynamics and transport co-efficients in a non-Fermi liquid:

The entropy density, \( s \), obeys

\[
    s \sim T^{(d-\theta)/z}
\]

where \( z = 3/2 \) is the dynamic critical exponent for fermionic excitations dispersing normal to the Fermi surface, and \( d - \theta = 1 \) is the number of dimensions normal to the Fermi surface.

A RG analysis using a dimensionality expansion below \( d = 5/2 \) shows that the quantum critical conductivity obeys

\[
    \sigma_Q \sim T^{(d-\theta-2)/z}
\]

We also computed the shear viscosity and found

\[
    \eta \sim T^{(d-\theta-2)/z}
\]

Note that \( \eta/s \sim T^{-2/z} \) does not scale to a constant: this is a consequence of the anisotropic scaling between the directions normal and parallel to the Fermi surface.


Entangled quantum matter without quasiparticles

• No quasiparticle excitations

• Shortest possible “phase coherence” time, fastest possible local equilibration time, or fastest possible Lyapunov time towards quantum chaos, all of order $\frac{\hbar}{k_B T}$

• Realization in solvable SYK model, which saturates the lower bound on the Lyapunov time. Its properties have some similarities to non-rational, large central charge CFT2s.

• Remarkable match between SYK and quantum gravity of black holes with AdS$_2$ horizons, including a SL(2,R)-invariant Schwarzian effective action for thermal energy fluctuations.

• Experiments on graphene agree well with predictions of a theory of a nearly relativistic quantum liquid without quasiparticles.

• Non-Fermi liquids with a critical Fermi surface have a divergent $\eta/s$ as $T \to 0$. This is a consequence of the spatial anisotropy in the vicinity of a Fermi surface point, and unlike existing holographic models.