A simple model of many-particle entanglement: how it describes black holes and superconductors

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February 5, 2021

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Quantum entanglement
Quantum Entanglement: quantum superposition with more than one particle

Hydrogen atom:

Hydrogen molecule:

\[ \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Principles of Quantum Mechanics: II. Quantum Entanglement

Quantum Entanglement: quantum superposition with more than one particle
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Quantum Entanglement: quantum superposition with more than one particle

Einstein-Podolsky-Rosen “paradox” (1935): Measurement of one particle instantaneously determines the state of the other particle arbitrarily far away.
Quantum entanglement
Quantum entanglement

A simple many-particle (SYK) model
The Sachdev-Ye-Kitaev (SYK) model

Pick a set of random positions
Place electrons randomly on some sites
Place electrons randomly on some sites.
The SYK model

Place electrons randomly on some sites
The SYK model

Entangle electrons pairwise randomly
The SYK model

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(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large $N$ limit; T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. 53, 385 (1981))

\[ \mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;kl} c_i^{\dagger} c_j^{\dagger} c_k c_\ell - \mu \sum_i c_i^{\dagger} c_i \]

\[ c_i c_j + c_j c_i = 0 \quad , \quad c_i^{\dagger} c_j^{\dagger} + c_j^{\dagger} c_i^{\dagger} = \delta_{ij} \]

\[ Q = \frac{1}{N} \sum_i c_i^{\dagger} c_i \]

$U_{ij;kl}$ are independent random variables with $\overline{U_{ij;kl}} = 0$ and $|\overline{U_{ij;kl}}|^2 = U^2$

$N \to \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL 70, 3339 (1993)
Complex multi-particle entanglement in the SYK model leads to a state without ‘quasiparticle’ excitations; i.e. multiple excitations cannot be built by composing an elementary set of ‘quasiparticle’ excitations.
Many-body chaos and thermal equilibration in the shortest possible Planckian time $\sim \frac{\hbar}{k_B T}$.

Complex multi-particle entanglement in the SYK model leads to a state without ‘quasiparticle’ excitations; i.e. multiple excitations cannot be built by composing an elementary set of ‘quasiparticle’ excitations.
Main result I

For $k_B T \ll U$

$$Z = \text{Tr} \exp \left( - \frac{\mathcal{H}}{k_B T} \right)$$

$$= \exp \left( N \frac{S_0}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left( -\frac{1}{\hbar} S_{\text{2D-gravity}} [f(\tau)] \right)$$
Main result I

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\[ = \exp \left( N \frac{S_0}{k_B} \right) \int \mathcal{D} f(\tau) \exp \left( -\frac{1}{\hbar} S_{2D-\text{gravity}}[f(\tau)] \right) \]

$S_0$ is the $T \to 0$ entropy of the SYK model.

\[ \frac{\partial S_0}{\partial Q} = 2\pi \mathcal{E}, \text{ where } \mathcal{E} \text{ characterizes} \]
the particle-hole asymmetry of the spectrum.


$S(T) = S_0 + \ldots$ will map on to the Bekenstein-Hawking entropy of charged black holes

A. Kitaev (2015)
Main result I

For $k_B T \ll U$

\[ \mathcal{Z} = \text{Tr} \exp \left( -\frac{\mathcal{H}}{k_B T} \right) \]

\[ = \exp \left( N \frac{S_0}{k_B} \right) \int \mathcal{D} f(\tau) \exp \left( -\frac{1}{\hbar} S_{2D-\text{gravity}} [f(\tau)] \right) \]

• $f(\tau)$ is the reparameterization of the imaginary time of the SYK model: $\tau$ on a circle of circumference $\hbar/(k_B T)$. 

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\[ = \exp \left( N \frac{S_0}{k_B} \right) \int \mathcal{D} f(\tau) \exp \left( -\frac{1}{\hbar} S_{2\text{D-gravity}} [f(\tau)] \right) \]

- $f(\tau)$ is the reparameterization of the imaginary time of the SYK model: $\tau$ on a circle of circumference $\hbar/(k_B T)$.

- $f(\tau)$ is also the fluctuation of the boundary of a theory of 2D-gravity in 1+1 spacetime dimensions: a ‘boundary graviton’.
Quantum entanglement

A simple many-particle (SYK) model

Low temperatures

Quantum gravity in 1+1 dimensions
Quantum entanglement

A simple many-particle (SYK) model

Black holes
Black Holes

Objects so dense that light is gravitationally bound to them.

In Einstein’s theory, the region inside the black hole horizon is disconnected from the rest of the universe.

Horizon radius \( R = \frac{2GM}{c^2} \)

\( G \) Newton’s constant, \( c \) velocity of light, \( M \) mass of black hole
Quantum Entanglement across a black hole horizon
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Quantum Entanglement across a black hole horizon
Quantum Entanglement across a black hole horizon

There is quantum entanglement between the inside and outside of a black hole.
Quantum Entanglement across a black hole horizon

Hawking used this to show that black hole horizons have an entropy and a temperature (because to an outside observer, the state of the electron inside the black hole is an unknown).
Quantum Black holes

- Black holes have an entropy and a temperature, $T_H$.
- The entropy, $S_{BH}$ is proportional to their surface area.

J. D. Bekenstein, PRD 7, 2333 (1973)
The ring-down time \( \frac{8\pi GM}{c^3} \sim 8 \text{ milliseconds.} \) Curiously, for essentially all types of black holes, the ring-down time equals

\[
\frac{\hbar}{k_B T_H},
\]

\( \hbar \) Planck’s constant, \( k_B \) Boltzmann’s constant.
Quantum Black holes

- Black holes have an entropy and a temperature, $T_H$
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a Planckian time $\sim \hbar/(k_BT_H)$. 
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge.
Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge. Zooming into the near-horizon region of a charged black hole at low temperature, yields a gravitational theory in one space (ζ) and one time dimension.
This 2D-gravity theory is precisely that appearing in the low T limit of the Sachdev-Ye-Kitaev (SYK) models.
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge.

\[ I_{EM} = \int d^{d+2}x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( \mathcal{R}_{d+2} + \frac{d(d + 1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right] \]

Metric \( g_{\mu\nu} \)
Ricci scalar in \( d + 2 \) dimensions, \( \mathcal{R}_{d+2} \)
Cosmological constant \( \Lambda = -d(d + 1)/L^2 \)
U(1) gauge field \( A_\mu \)
Electromagnetic field \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

Boundary conditions at spatial infinity:
Metric \( \rightarrow \text{AdS}_{d+2} \)
Electric field \( \rightarrow Q/(4\pi r^2) \)
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge

\[ I_{EM} = \int d^{d+2}x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( \mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right] \]

Quantum gravity is ‘defined’ by the path integral

\[ Z_{\text{gravity}} = \int \mathcal{D}g\mathcal{D}A \exp(-I_{EM}/\hbar) \]

This integral is evaluated exactly in a certain low temperature limit for charged black holes.
SYK model and charged black holes

Solution of Euler-Lagrange equations of the action of Einstein gravity and Maxwell electromagnetism
SYK model and charged black holes

The entropy $S_{BH}$, the charge $Q$, and the dimensionless electric field $\mathcal{E}$ obey the same thermodynamic relation as the SYK model

\[ \frac{dS_{BH}}{dQ} = 2\pi \mathcal{E} \]
SYK model and charged black holes

Fluctuations about the path integral saddle

Horizon

$ds^2 = R_2^2 \frac{(d\zeta^2 - dt^2)}{\zeta^2} + R_h^2 d\Omega_2^2$

Gauge field: $A = \frac{\mathcal{E}}{\zeta} dt$

Boundary graviton

AdS$_2 \times S^2$

AdS$_4$

Boundary

total charge $Q$
Main result II

For $T \ll 1/R_h$

$Z_{\text{charged black hole in EM theory}} =$

$$\exp \left( \frac{S_{BH}}{k_B} \right) \int \mathcal{D}f(\tau) \exp \left( -\frac{1}{\hbar} S_{2D-\text{gravity}} [f(\tau)] \right)$$
For $T \ll 1/R_h$

\[ Z_{\text{charged black hole in EM theory}} = \exp \left( \frac{S_{BH}}{k_B} \right) \int Df(\tau) \exp \left( -\frac{1}{\hbar} S_{2D-\text{gravity}} [f(\tau)] \right) \]

\[ S_{2D-\text{gravity}} [f(\tau)] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \}, \]

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, and we have used the Schwarzian:

\[ \{g, \tau\} \equiv \frac{d^3 g/d\tau^3}{dg/d\tau} - \frac{3}{2} \left( \frac{d^2 g/d\tau^2}{dg/d\tau} \right)^2. \]

The defining property of the Schwarzian is its invariance under SL(2,R) transformations

\[ \{ \frac{ag(\tau) + b}{cg(\tau) + d}, \tau\} = \{g(\tau), \tau\} \]

Remarkably, this path integral can be evaluated exactly, using the Duistermaat–Heckman formula (Stanford, Witten, arXiv:1703.04612).
Main result

A. Kitaev (2015)


J. Maldacena, D. Stanford, and Zhenbin Yang, PTEP 12C104 (2016)


J. Engelsoy, T.G. Mertens, and H. Verlinde, JHEP 1607 (2016) 139


P. Chaturvedi, Yingfei Gu, Wei Song, Boyang Yu, arXiv:1808.08062
S. Sachdev, arXiv:1902.04078
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Charged black holes

Low temperatures

Quantum gravity in 1+1 dimensions

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Complex multi-particle entanglement leads to quantum systems without quasiparticle excitations.

Many-body chaos and thermal equilibration in the shortest possible Planckian time $\sim \frac{\hbar}{k_B T}$. 
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Copper-based superconductors
High temperature superconductors

$\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$
Insulating antiferromagnet
Antiferromagnet doped with hole density $p$
$p$ mobile holes in a background of fluctuating spins
Real-space view

$p$ mobile holes in a background of fluctuating spins
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Real-space view

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Fermi liquid
Strange Metal
Pseudogap Metal
Momentum-space view
1-p mobile electrons = 1+p mobile holes in a filled band
Momentum-space view at large $p$

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**Momentum-space view at large $p$**

(d)

(0,0)

(π,π)

Overdoped Tl$_2$Ba$_2$CuO$_{6+δ}$

$T_c = 30K$

$1+p$ holes

$1-p$ electrons

“Large Fermi surface”:

$1+p$ mobile holes in a filled band

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Momentum-space view at small $p$

$\text{Ca}_{2-x}\text{Na}_x\text{CuO}_2\text{Cl}_2$

at $x = 0.10$

“Fermi arcs”?

Hidden magnetism at the pseudogap critical point of a high temperature superconductor
Nature Physics 16, 1064 (2020)

Mehdi Frachet\textsuperscript{1}, Igor Vinograd\textsuperscript{1}, Rui Zhou\textsuperscript{1,2}, Siham Benhabib\textsuperscript{1}, Shangfei Wu\textsuperscript{1}, Hadrien Mayaffre\textsuperscript{1}, Steffen Krämer\textsuperscript{1}, Sanath K. Ramakrishna\textsuperscript{3}, Arneil P. Reyes\textsuperscript{3}, Jérôme Debray\textsuperscript{4}, Tohru Kurosawa\textsuperscript{5}, Naoki Momono\textsuperscript{6}, Migaku Oda\textsuperscript{5}, Seiki Komiya\textsuperscript{7}, Shimpei Ono\textsuperscript{7}, Masafumi Horio\textsuperscript{8}, Johan Chang\textsuperscript{8}, Cyril Proust\textsuperscript{1}, David LeBoeuf\textsuperscript{1,*}, Marc-Henri Julien\textsuperscript{1,*}
Angle-dependent magnetoresistance in Nd-LSCO near $p = p_c \approx 0.23$. 

\[ \frac{1}{\tau} = \frac{1}{\tau_{\text{aniso}}(\vec{k})} + \frac{\alpha}{\hbar} k_B T \]
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} \hat{c}_{i\alpha}^{\dagger} \hat{c}_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \hat{S}_i \cdot \hat{S}_j$$

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

\[ H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

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Random \textbf{t-J} model

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We consider the hole-doped case, with no double occupancy.
One particle energy distribution function

\[ N(\epsilon) = \frac{1}{N} \sum_\lambda \delta(\epsilon - \epsilon_\lambda) \sum_{ij\sigma} \langle i| \langle c_{i\sigma}^{\dagger} c_{j\sigma} \rangle \langle j| \lambda \rangle \]

where \( |\lambda\rangle \) are one-particle eigenstates of the \( t_{ij} \). In a Fermi liquid, the Luttinger identity implies that \( N(\epsilon) \) has a discontinuity at the free particle Fermi energy \( \epsilon_F \). \( (\mathcal{D}(\epsilon) \) is the Wigner semi-circle density of states.)

Evidence for a “Large Fermi surface” for \( p > p_c \approx 0.4 \)
Dynamic spin susceptibility

\[ \chi''(\omega) = \sum_n |\langle 0 | S_{+i} | n \rangle|^{2} \delta(\hbar\omega - E_n + E_0), \text{ (at } T = 0) \]
Dynamic spin susceptibility

$$\chi''(\omega) = \sum_n |\langle 0 | S_{+i} | n \rangle|^2 \delta(\hbar \omega - E_n + E_0), \text{ (at } T = 0)$$

Spin glass order $q$ non-zero for $p < p_c \approx 0.4$
Dynamic spin susceptibility

\[ \chi''(\omega) = \sum_n |\langle 0 | S_{+i} | n \rangle|^2 \delta(\hbar \omega - E_n + E_0), \text{ (at } T = 0) \]

Critical spin susceptibility matches the SYK model!

\[ \chi''(\omega) \sim \text{sgn}(\omega) [1 - C \gamma |\omega| + \ldots] \]
Consequences of 2D-gravity for the SYK model

\[ \chi''(\omega) = \sum_n |\langle 0 | S_{+i} | n \rangle|^2 \delta(\hbar \omega - E_n + E_0), \text{ (at } T = 0) \]

\[ \chi''(\omega) \sim \tanh \left( \frac{\hbar \omega}{2k_B T} \right) \left[ 1 - C \gamma \omega \tanh \left( \frac{\hbar \omega}{2k_B T} \right) - \ldots \right] \]
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Conformally (SL(2,R)) invariant result with characteristic dissipative time \( \sim \hbar/(k_B T) \)

A. Georges and O. Parcollet
PRB 59, 5341 (1999)
Consequences of 2D-gravity for the SYK model

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Correction from the boundary graviton
The random $t$-$J$ model has

- Spin glass order for $p < p_c$.
- Fermi liquid with "large Fermi surface" for $p > p_c$.
- Maxima in entropy, specific heat, and entanglement entropy near $p = p_c$.
- SYK-Planckian criticality near $p_c$.
- *Boundary graviton* correction in critical spin susceptibility!
The random $t$-$J$ model has

- Spin glass order for $p < p_c$.
- Fermi liquid with "large Fermi surface" for $p > p_c$
- Maxima in entropy, specific heat, and entanglement entropy near $p = p_c$
- SYK-Planckian criticality near $p_c$.
- Boundary graviton correction in critical spin susceptibility!
- SYK criticality can be understood in a model in which the electron fractionalizes into spinons and holons: then both the $t$ and $J$ terms map onto 4-particle SYK terms.

D. G. Joshi, Chenyuan Li, G. Tarnopolsky, A. Georges, and S. Sachdev, PRX 10, 021033 (2020)
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Complex multi-particle entanglement leads to quantum systems without quasiparticle excitations.

Many-body chaos and thermal equilibration in the shortest possible Planckian time \( \sim \frac{\hbar}{k_B T} \).