1. Review of Fermi liquid theory

   Topological argument for the Luttinger theorem

2. Fractionalized Fermi liquid

   A Fermi liquid co-existing with topological order for the pseudogap metal

3. Quantum matter without quasiparticles

   (A) A mean-field model of a non-Fermi liquid, and charged black holes

   (B) Field theory of a non-Fermi liquid (Ising-nematic quantum critical point)

   (C) Theory of transport in strange metals: application to the (less) strange metal in graphene
Quantum criticality of Ising-nematic ordering in a metal

A metal with a Fermi surface with full square lattice symmetry
Quantum criticality of Ising-nematic ordering in a metal

Spontaneous elongation along $y$ direction:

Spontaneous elongation along $x$ direction:

Quantum criticality of Ising-nematic ordering in a metal
Ising-nematic order parameter

\[ \phi \sim \int d^2 k \left( \cos k_x - \cos k_y \right) c_{k \sigma}^\dagger c_{k \sigma} \]

Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian
Quantum criticality of Ising-nematic ordering in a metal

Spontaneous elongation along $x$ direction:
Ising order parameter $\phi > 0$. 
Quantum criticality of Ising-nematic ordering in a metal

Spontaneous elongation along $y$ direction:
Ising order parameter $\phi < 0$. 
Quantum criticality of Ising-nematic ordering in a metal

Pomeranchuk instability as a function of coupling $\lambda$

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

$\lambda_c$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$
Quantum criticality of Ising-nematic ordering in a metal

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Phase diagram as a function of $T$ and $\lambda$

- Strongly-coupled “non-Fermi liquid” metal with no quasiparticles
- $\langle \phi \rangle \neq 0$
- $\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

- Fermi liquid ($\langle \phi \rangle \neq 0$)
- Strongly-coupled “non-Fermi liquid” metal with no quasiparticles
- Quantum critical
- Fermi liquid ($\langle \phi \rangle = 0$)
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

Fermi liquid

\[ \langle \phi \rangle \neq 0 \]

Fermi liquid

\[ \langle \phi \rangle = 0 \]
The Fermi liquid

\[ \mathcal{L} = f_{\alpha}^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_{\alpha} \\
+ u f_{\alpha}^\dagger f_{\beta}^\dagger f_{\beta} f_{\alpha} \]

- Occupied states
- Empty states

\[ \kappa_F \]
The Fermi liquid: RG

\[
L = f^\dagger_\alpha \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\alpha \\
+ u f^\dagger_\alpha f^\dagger_\beta f_\beta f_\alpha
\]

- Expand fermion kinetic energy at wavevectors about \( \vec{k}_0 \), by writing \( f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q}) \)
The Fermi liquid: RG

\[ \mathcal{L} = f_\alpha^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\alpha \]

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- Expand fermion kinetic energy at wavevectors about \( \vec{k}_0 \), by writing \( f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q}) \)

\[ \mathcal{L}[\psi_\alpha] = \psi_\alpha^\dagger \left( \partial_\tau - i\partial_x - \partial_y^2 \right) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \]
The Fermi liquid: RG

\[S[\psi_\alpha] = \int d^{d-1}y\, dx\, d\tau \left[ \psi_\alpha^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \right]\]
The Fermi liquid: RG

\[ S[\psi_\alpha] = \int d^{d-1} y \, dx \, d\tau \left[ \psi_\alpha^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_\alpha + u \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha \right] \]

The kinetic energy is invariant under the rescaling \( x \to x/s, \ y \to y/s^{1/2}, \) and \( \tau \to \tau/s^z, \) provided \( z = 1 \) and

\[ \psi \to \psi \, s^{(d+1)/4}. \]
The kinetic energy is invariant under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, provided $z = 1$ and

$$\psi \rightarrow \psi s^{(d+1)/4}.$$ 

Then we find $u \to us^{(1-d)/2}$, and so we have the RG flow

$$\frac{du}{dl} = \frac{(1 - d)}{2} u$$

Interactions are *irrelevant* in $d = 2$!
The fermion Green’s function to order $u^2$ has the form (upto logs)

$$G(q, \omega) = \frac{A}{\omega - q_x - q_y^2 + ic\omega^2}$$

So the quasiparticle pole is sharp.
The Fermi liquid: RG

$$S[\psi_\alpha] = \int d^{d-1}y \, dx \, d\tau \left[ \psi^\dagger_\alpha (\partial_\tau - i \partial_x - \partial_y^2) \psi_\alpha + u \psi^\dagger_\alpha \psi^\dagger_\beta \psi_\beta \psi_\alpha \right]$$

The fermion Green’s function to order $u^2$ has the form (upto logs)

$$G(\vec{q}, \omega) = \frac{A}{\omega - q_x - q_y^2 + ic \omega^2}$$

So the quasiparticle pole is sharp. And fermion momentum distribution function $n(\vec{k}) = \left\langle f^\dagger_\alpha(\vec{k}) f_\alpha(\vec{k}) \right\rangle$ had the following form:
The Fermi liquid

\[ \mathcal{L} = f^{\dagger} \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]

+ 4 Fermi terms

- Fermi wavevector obeys the Luttinger relation \( k_F^d \sim Q \), the fermion density

![Diagram showing occupied and empty states with Fermi wavevector \( k_F \)].
The Fermi liquid

\[ \mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]
+ 4 Fermi terms

- Fermi wavevector obeys the Luttinger relation \( k_F^d \sim Q \), the fermion density

- Sharp particle and hole of excitations near the Fermi surface with energy \( \omega \sim |q|^z \), with dynamic exponent \( z = 1 \).
The Fermi liquid

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- Fermi wavevector obeys the Luttinger relation \( k_F^d \sim Q \), the fermion density

- Sharp particle and hole of excitations near the Fermi surface with energy \( \omega \sim |q|^z \), with dynamic exponent \( z = 1 \).

- The phase space density of fermions is effectively one-dimensional, so the entropy density \( S \sim T \). It is useful to write this as \( S \sim T^{(d-\theta)/z} \), with violation of hyperscaling exponent \( \theta = d - 1 \).
Quantum criticality of Ising-nematic ordering in a metal

Pomeranchuk instability as a function of coupling $\lambda$
Effective action for Ising order parameter

\[ S_\phi = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right] \]
Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

\[ S_\phi = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} \right] \]

\[ \equiv \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]
Quantum criticality of Ising-nematic ordering in a metal

Coupling between Ising order and electrons

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]

for spatially dependent \( \phi \)

\[ \langle \phi \rangle > 0 \quad \langle \phi \rangle < 0 \]
Quantum criticality of Ising-nematic ordering in a metal

\[ S_\phi = \int d^2rd\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_{k} \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k) c_{k\alpha} \]

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]
• \( \phi \) fluctuation at wavevector \( \vec{q} \) couples most efficiently to fermions near \( \pm \vec{k}_0 \).
Quantum criticality of Ising-nematic ordering in a metal

- $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm \vec{k}_0$ and boson ($\phi$) kinetic energy about $\vec{q} = 0$.  

$\mathcal{L}[\psi_\pm, \phi] =$

$$\begin{align*}
\psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_-
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2
\end{align*}$$

Quantum criticality of Ising-nematic ordering in a metal

\[
\mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2
\]

One loop \(\phi\) self-energy with \(N_f\) fermion flavors:

\[
\Sigma_\phi(\bar{q}, \omega) = N_f \int \frac{d^2k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{[-i(\Omega + \omega) + k_x + q_x + (k_y + q_y)^2] [-i\Omega - k_x + k_y^2]}
\]

\[
= \frac{N_f}{4\pi} \frac{|\omega|}{|q_y|}
\]

Landau-damping
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \]
\[ - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Electron self-energy at order $1/N_f$:

\[ \Sigma(\vec{k}, \Omega) = -\frac{1}{N_f} \int \frac{d^2q}{4\pi^2} \frac{d\omega}{2\pi} \frac{1}{[-i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2]} \left[ \frac{q_y^2}{g^2} + \frac{|\omega|}{|q_y|} \right] \]
\[ = -i \frac{2}{\sqrt{3}N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega)|\Omega|^{2/3} \]
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi^\dagger_+ (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi^\dagger_- (\partial_\tau + i \partial_x - \partial_y^2) \psi_- 
- \phi \left( \psi^\dagger_+ \psi_+ + \psi^\dagger_- \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Electron self-energy at order $1/N_f$:

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\[ = -i \frac{2}{\sqrt{3}N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega) |\Omega|^{2/3} \approx |\Omega|^{d/3} \text{ in dimension } d. \]
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Schematic form of \( \phi \) and fermion Green’s functions in \( d \) dimensions

\[ D(\vec{q}, \omega) = \frac{1/N_f}{q_\perp^2 + |\omega|/|q_\perp|} \quad , \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_\perp^2 - i\text{sgn}(\omega)|\omega|^{d/3}/N_f} \]

In the boson case, \( q_\perp^2 \sim \omega^{1/z_b} \) with \( z_b = 3/2 \).
In the fermion case, \( q_x \sim q_\perp^2 \sim \omega^{1/z_f} \) with \( z_f = 3/d \).

Note \( z_f < z_b \) for \( d > 2 \) \( \Rightarrow \) Fermions have higher energy than bosons, and perturbation theory in \( g \) is OK.
Strongly-coupled theory in \( d = 2 \).
Quantum criticality of Ising-nematic ordering in a metal

\[
\mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \\
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\]

Schematic form of $\phi$ and fermion Green’s functions in $d = 2$

\[
D(\vec{q}, \omega) = \frac{1/N_f}{q_y^2 + \frac{1}{q_y} \frac{\omega}{|\omega|}}, \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_y^2 - i\text{sgn}(\omega)|\omega|^{2/3}/N_f}
\]

In both cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in $G_f^{-1}$ is irrelevant.

Strongly-coupled theory without quasiparticles.
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \]
\[ - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Simple scaling argument for \( z = 3/2 \).
Quantum criticality of Ising-nematic ordering in a metal

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Simple scaling argument for \( z = 3/2 \).

Under the rescaling \( x \to x/s \), \( y \to y/s^{1/2} \), and \( \tau \to \tau/s^z \), we find invariance provided

\[
\begin{align*}
\phi & \to \phi s \\
\psi & \to \psi s^{(2z+1)/4} \\
g & \to g s^{(3-2z)/4}
\end{align*}
\]

So the action is invariant provided \( z = 3/2 \).
• $k_F^d \sim Q$, the fermion density

• Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

• Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$. 
- $k_F^d \sim Q$, the fermion density

- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

- NFL Nematic QCP

- Fermi surface with $k_F^d \sim Q$. 

\[ n(k) \]
• $k^d_F \sim Q$, the fermion density

• Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

• Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

• Fermi surface with $k^d_F \sim Q$.

• Diffuse fermionic excitations with $z = 3/2$ to three loops.

<table>
<thead>
<tr>
<th>FL</th>
<th>NFL</th>
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<tr>
<td>Fermi liquid</td>
<td>Nematic QCP</td>
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- $k_F^d \sim Q$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

- Fermi surface with $k_F^d \sim Q$.
- Diffuse fermionic excitations with $z = 3/2$ to three loops.
- $S \sim T^{(d-\theta)/z}$ with $\theta = d - 1$. 

**FL**

- Fermi liquid

**NFL**

- Nematic QCP
Quantum criticality of Ising-nematic ordering in a metal

\[ \langle \phi \rangle = 0 \]

\[ T_{l-n} \]

Fermi liquid

\[ \langle \phi \rangle \neq 0 \]

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

Strange Metal

Phase diagram as a function of \( T \) and \( \lambda \)