Progress on the physics of the underdoped cuprates

International Institute of Physics
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Subir Sachdev
Quantum oscillations and the Fermi surface in an underdoped high-$T_c$ superconductor

Nicolas Doiron-Leyraud$^1$, Cyril Proust$^2$, David LeBoeuf$^1$, Julien Levallois$^2$, Jean-Baptiste Bonnemaison$^1$, Ruixing Liang$^{3,4}$, D. A. Bonn$^{3,4}$, W. N. Hardy$^{3,4}$ & Louis Taillefer$^{1,4}$

Twofold twisted Fermi surface from staggered order in an underdoped high $T_c$ superconductor

Suchitra E. Sebastian,¹* N. Harrison,² F. F. Balakirev,² M. M. Altarawneh,²,³ Ruixing Liang,⁴,⁵ D. A. Bonn,⁴,⁵ W. N. Hardy,⁴,⁵ G. G. Lonzarich,¹

APS March meeting 2013
B2.00004
Magnetic-field-induced charge-stripe order in the high-temperature superconductor YBa$_{2}$Cu$_{3}$O$_{y}$

Tao Wu$^{1}$, Hadrien Mayaffre$^{1}$, Steffen Krämer$^{1}$, Mladen Horvatić$^{1}$, Claude Berthier$^{1}$, W. N. Hardy$^{2,3}$, Ruixing Liang$^{2,3}$, D. A. Bonn$^{2,3}$ & Marc-Henri Julien$^{1}$
The comparison of different acoustic modes indicates that the charge modulation is biaxial, which differs from a uniaxial stripe charge order.
A Four Unit Cell Periodic Pattern of Quasi-Particle States Surrounding Vortex Cores in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$

J. E. Hoffman,¹ E. W. Hudson,¹,²* K. M. Lang,¹ V. Madhavan,¹ H. Eisaki,³† S. Uchida,³ J. C. Davis¹,²‡

SCIENCE VOL 295 18 JANUARY 2002
Local Ordering in the Pseudogap State of the High-$T_c$ Superconductor $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$

Michael Vershinin,¹* Shashank Misra,¹* S. Ono,² Y. Abe,²† Yoichi Ando,² Ali Yazdani¹†§

SCIENCE VOL 303 26 MARCH 2004
An Intrinsic Bond-Centered Electronic Glass with Unidirectional Domains in Underdoped Cuprates


9 MARCH 2007 VOL 315 SCIENCE
Charge-density-wave origin of cuprate checkerboard visualized by scanning tunnelling microscopy

W. D. Wise, M. C. Boyer, Kamalesh Chatterjee, Takeshi Kondo, Takeuchi, H. Ikuta, Yayu Wang and E. W. Hudson

nature physics | VOL 4 | SEPTEMBER 2008 |
Long-Range Incommensurate Charge Fluctuations in (Y,Nd)Ba$_2$Cu$_3$O$_{6+x}$


SCIENCE VOL 337 17 AUGUST 2012

**Fig. 3.** Dependence of the CDW signal at 15 K on the hole doping level $p$. The CDW signal is present in several YBa$_2$Cu$_3$O$_{6+x}$ and Nd$_{1+y}$Ba$_2$-$y$Cu$_3$O$_7$ samples, but only for 0.09 $\leq p \leq$ 0.13. In this doping range (shaded in the central panel), the $T_c$-versus-$p$ relation exhibits a plateau. The CDW peak position does not change with $p$ outside of the experimental error, but its intensity is maximum at $p \approx 0.11$. 

resonant soft x-ray scattering
Direct observation of competition between superconductivity and charge density wave order in YBa$_2$Cu$_3$O$_{6.67}$

J. Chang$^{1,2}$*, E. Blackburn$^3$, A. T. Holmes$^3$, N. B. Christensen$^4$, J. Larsen$^{4,5}$, J. Mesot$^{1,2}$, Ruixing Liang$^{6,7}$, D. A. Bonn$^{6,7}$, W. N. Hardy$^{6,7}$, A. Watenphul$^8$, M. v. Zimmermann$^8$, E. M. Forgan$^3$ and S. M. Hayden$^9$

**Figure 2**

![Graph showing CDW Bragg peak intensity](image)

- In zero field, the intensity of the CDW Bragg peak (Fig. 2) grows linearly with magnetic field up to the highest applied field.
- The peak width we estimate that the modulation has an in-plane length of approximately 0.6 lattice units.
- Below a field of 95 T, the peak intensity grows approximately linearly with magnetic field up to the highest applied field.

**Figure 1**

- Scan through the (2 0 0) peak.
- The cusp of the peak occurs at $T = 66$ K.
- The modulation of the peak width is strongest, but also becomes more coherent, down to a temperature where the peak width we estimate that the modulation has an in-plane length of approximately 0.6 lattice units.

**Figure 1a**

- Scan through the (2 0 0) peak.
- The cusp of the peak occurs at $T = 66$ K.
- The modulation of the peak width is strongest, but also becomes more coherent, down to a temperature where the peak width we estimate that the modulation has an in-plane length of approximately 0.6 lattice units.

**Figure 1b**

- Scan through the (2 0 0) peak.
- The cusp of the peak occurs at $T = 66$ K.
- The modulation of the peak width is strongest, but also becomes more coherent, down to a temperature where the peak width we estimate that the modulation has an in-plane length of approximately 0.6 lattice units.

**Figure 1c**

- Scan through the (2 0 0) peak.
- The cusp of the peak occurs at $T = 66$ K.
- The modulation of the peak width is strongest, but also becomes more coherent, down to a temperature where the peak width we estimate that the modulation has an in-plane length of approximately 0.6 lattice units.
Moreover, the energy dependence of the CDW order in the planes is shown to result from a spatial modulation of energies of the Cu 2$p$ to $3d_{x^2−y^2}$ transition, similar to stripe-ordered 214 cuprates.
These energy shifts are interpreted as a spatial modulation of the electronic structure and may point to a valence-bond-solid interpretation of the stripe phase.
Do we finally have a resolution to the low energy electronic structure of underdoped YBCO?

N. Harrison and S. E. Sebastian
Low T phase diagram of a doped antiferromagnet

Main results

There is an approximate pseudospin symmetry in metals with antiferromagnetic spin correlations.

The pseudospin partner of $d$-wave superconductivity is an incommensurate $d$-wave bond order

S. Sachdev and R. La Placa, arXiv:1303.2114
Outline

1. Stability of metal in Hartree-Fock-BCS theory

2. Emergent pseudospin symmetry in low energy theory of metal with antiferromagnetic interactions

3. Quantum Monte Carlo without the sign problem
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1. Stability of metal in Hartree-Fock-BCS theory

2. Emergent pseudospin symmetry in low energy theory of metal with antiferromagnetic interactions

3. Quantum Monte Carlo without the sign problem
\[ H_{tJ} = -\sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Optimize the free energy w.r.t. a mean field Hamiltonian which allows for spin-singlet superconductivity (pairing order \( \Delta_S(k) \)) and spin-singlet “charge” order (\( \Delta_Q(k) \)):

\[ H_{MF} = \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_k \Delta_S(k) \epsilon_{\alpha\beta} c_{\mathbf{k},\alpha} c_{\mathbf{-k},\beta} + \text{H.c.} \]

\[ + \sum_{\mathbf{k},Q} \Delta_Q(k) c_{\mathbf{k}+Q/2,\alpha}^\dagger c_{\mathbf{k}-Q/2,\alpha} \]
\[ H_{tJ} = - \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

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\[ + \sum_{\mathbf{k},Q} \Delta_Q(\mathbf{k}) c_{\mathbf{k}+Q/2,\alpha}^\dagger c_{\mathbf{k}-Q/2,\alpha} \]

In real space, the “charge” order \( \Delta_Q(\mathbf{k}) \) corresponds to a modulation in local and non-local “density” variables:

\[ \left\langle c_{i\alpha}^\dagger c_{j\alpha} \right\rangle \sim \left[ \sum_{\mathbf{k}} \Delta_Q(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{i\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2} \]

We obtain purely “bond” order with no modulation in the site density, \( \left\langle c_{i\alpha}^\dagger c_{i\alpha} \right\rangle \), if \( \sum_{\mathbf{k}} \Delta_Q(\mathbf{k}) = 0 \).
\[ H_{tJ} = - \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \]

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+ \sum_{k,Q} \Delta_Q(k) c_{k+Q/2,\alpha}^\dagger c_{k-Q/2,\alpha}
\]

Expanding the free energy in powers of the order parameters we obtain

\[
F = F_0 + \sum_k \Delta_S^*(k) M_S(k, k') \Delta_S(k') \\
+ \sum_{k,Q} \Delta_Q^*(k) M_Q(k, k') \Delta_Q(k')
\]
Due to pseudospin symmetry, the kernels have identical forms:

\[
\mathcal{M}_S(k, k') = \Pi_S(k) \left[ \delta_{k, k'} - \frac{3}{V} J(k - k') \Pi_S(k') \right]
\]

\[
\mathcal{M}_Q(k, k') = \Pi_Q(k) \left[ \delta_{k, k'} - \frac{3}{V} J(k - k') \Pi_Q(k') \right]
\]

The pseudospin symmetry is broken only by the differing forms of the polarizabilities of fermions with dispersion \( \varepsilon(k) \).

\[
\Pi_S(k) = \frac{1 - 2f(\varepsilon(k))}{2\varepsilon(k)}
\]

\[
\Pi_Q(k) = \frac{f(\varepsilon(k + Q/2)) - f(\varepsilon(k - Q/2))}{\varepsilon(k - Q/2) - \varepsilon(k + Q/2)},
\]

\[= \cos(kx) \cos(ky) + \ldots \]
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We diagonalized \( J(k - k') \Pi_S(k') \) and \( J(k - k') \Pi_Q(k') \) and obtained the lowest eigenvalues \( \lambda_S \) and \( \lambda_Q \) and the corresponding right eigenvectors \( \Delta_S(k) \) and \( \Delta_Q(k) \).
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The smallest eigenvalue was always \(\lambda_S\) with eigenvector \(\Delta_S(k) = \Delta_0(\cos k_x - \cos k_y) + \ldots\).
Charge-ordering eigenvalue $\lambda_Q/J_0$. 

S. Sachdev and R. La Placa, arXiv:1303.2114
\[ \Delta_Q(k) = \sum_{\gamma} c_{Q,\gamma} \psi_{\gamma}(k) \]

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<tr>
<th>( \gamma )</th>
<th>( \psi_{\gamma}(k) )</th>
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**Charge-ordering eigenvector**

S. Sachdev and R. La Placa, arXiv:1303.2114
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**Charge-ordering eigenvector**

S. Sachdev and R. La Placa, arXiv:1303.2114
Incommensurate $d$-wave bond order

Charge-ordering eigenvalue $\lambda_{\mathbf{Q}}/J_0$. 

Incommensurate $d$-wave bond order

"Bond density" measures amplitude for electrons to be in spin-singlet valence bond.

\[
\langle c_{r\alpha}^\dagger c_{s\alpha} \rangle = \sum_Q \sum_k e^{iQ \cdot (r+s)/2} e^{-i k \cdot (r-s)} \langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle
\]

where $Q$ extends over $Q = (\pm Q_0, \pm Q_0)$ with $Q_0 = 2\pi/(7.3)$ and

\[
\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = \Delta_Q (\cos k_x - \cos k_y)
\]

Note $\langle c_{r\alpha}^\dagger c_{s\alpha} \rangle$ is non-zero only when $r, s$ are nearest neighbors.

\[
\Delta Q(k) = \sum_{\gamma} c_{Q,\gamma} \psi_\gamma(k)
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S. Sachdev and R. La Placa, arXiv:1303.2114

Ising-nematic order

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<tr>
<td>(g)</td>
<td>((\cos k_x - \cos k_y) \times \sqrt{8} \sin k_x \sin k_y)</td>
<td>-0.009</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Charge-ordering eigenvector**

S. Sachdev and R. La Placa, arXiv:1303.2114
Staggered orbital currents


Charge-ordering eigenvalue $\lambda_Q / J_0$. 

S. Sachdev and R. La Placa, arXiv:1303.2114
\[ \Delta Q(k) = \sum_{\gamma} c_{Q,\gamma} \psi_{\gamma}(k) \]

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\psi_{\gamma}(k)$</th>
<th>$Q = (1.15,1.15)$</th>
<th>$Q = (1.15,0)$</th>
<th>$Q = (0,0)$</th>
<th>$Q = (\pi,\pi)$</th>
<th>$\Delta S(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>1</td>
<td>0</td>
<td>-0.231</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s'$</td>
<td>$\cos k_x + \cos k_y$</td>
<td>0</td>
<td>0.044</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s''$</td>
<td>$\cos(2k_x) + \cos(2k_y)$</td>
<td>0</td>
<td>-0.046</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$d$</td>
<td>$\cos k_x - \cos k_y$</td>
<td>0.993</td>
<td>0.963</td>
<td>0.997</td>
<td>0</td>
<td>0.997</td>
</tr>
<tr>
<td>$d'$</td>
<td>$\cos(2k_x) - \cos(2k_y)$</td>
<td>-0.069</td>
<td>-0.067</td>
<td>-0.057</td>
<td>0</td>
<td>-0.056</td>
</tr>
<tr>
<td>$d''$</td>
<td>$2 \sin k_x \sin k_y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_x$</td>
<td>$\sqrt{2} \sin k_x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.706</td>
<td>0</td>
</tr>
<tr>
<td>$p_y$</td>
<td>$\sqrt{2} \sin k_y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.706</td>
<td>0</td>
</tr>
<tr>
<td>$g$</td>
<td>$(\cos k_x - \cos k_y) \times \sqrt{8} \sin k_x \sin k_y$</td>
<td>-0.009</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Charge-ordering eigenvector**

S. Sachdev and R. La Placa, arXiv:1303.2114
Incommensurate \( d(+s) \) wave bond order

Charge-ordering eigenvalue \( \lambda_\mathbf{Q}/J_0 \).
Charge-ordering eigenvalue $\lambda_Q/J_0$.
What determines the $\mathbf{Q}$ at which $\lambda_{\mathbf{Q}}$ is a minimum?
Outline

1. Stability of metal in Hartree-Fock-BCS theory

2. Emergent pseudospin symmetry in low energy theory of metal with antiferromagnetic interactions

3. Quantum Monte Carlo without the sign problem
1. Stability of metal in Hartree-Fock-BCS theory

2. Emergent pseudospin symmetry in low energy theory of metal with antiferromagnetic interactions

3. Quantum Monte Carlo without the sign problem
Pseudospin symmetry of the exchange interaction

\[ H_J = \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

with \( \vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta} \) is the antiferromagnetic exchange interaction. Introduce the Nambu spinor

\[
\Psi_{i\uparrow} = \begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix}, \quad \Psi_{i\downarrow} = \begin{pmatrix} c_{i\downarrow} \\ -c_{i\uparrow} \end{pmatrix}
\]

Then we can write

\[
H_J = \frac{1}{8} \sum_{i<j} J_{ij} \left( \Psi_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \Psi_{i\beta} \right) \cdot \left( \Psi_{j\gamma}^\dagger \vec{\sigma}_{\gamma\delta} \Psi_{i\delta} \right)
\]

which is invariant under the SU(2) pseudospin transformations

\[
\Psi_{i\alpha} \rightarrow U_i \Psi_{i\alpha}
\]

This pseudospin symmetry is important in classifying spin liquid ground states of \( H_J \).

Pseudospin symmetry of the exchange interaction

\[
H_{tJ} = - \sum_{i,j} t_{ij} c_i^{\uparrow \alpha} c_j^{\downarrow \alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

with \( \vec{S}_i = \frac{1}{2} c_i^{\uparrow \alpha} \vec{\sigma}^{\alpha \beta} c_i^{\downarrow \beta} \) is the antiferromagnetic exchange interaction. Introduce the Nambu spinor

\[
\Psi_{i \uparrow} = \begin{pmatrix} c_i^{\uparrow \alpha} \\ c_i^{\downarrow \alpha} \end{pmatrix}, \quad \Psi_{i \downarrow} = \begin{pmatrix} c_i^{\downarrow \alpha} \\ -c_i^{\uparrow \alpha} \end{pmatrix}
\]

Then we can write

\[
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\]

which is invariant under the SU(2) pseudospin transformations

\[
\Psi_{i \alpha} \rightarrow U_i \Psi_{i \alpha}
\]

This pseudospin symmetry is important in classifying spin liquid ground states of \( H_{tJ} \). It is fully broken by the electron hopping \( t_{ij} \) but does have remnant consequences in doped spin liquid states.

Pseudospin symmetry of the exchange interaction

\[ H_{tJ} = - \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

with \( \vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta} \) is the antiferromagnetic exchange interaction. Introduce the Nambu spinor

\[
\Psi_{i\uparrow} = \left( \begin{array}{c} c_{i\uparrow} \\ \overline{c}_{i\downarrow} \end{array} \right), \quad \Psi_{i\downarrow} = \left( \begin{array}{c} c_{i\downarrow} \\ -c_{i\uparrow} \end{array} \right)
\]

Then we can write

\[
H_{J} = \frac{1}{8} \sum_{i<j} J_{ij} \left( \Psi_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \Psi_{i\beta} \right) \cdot \left( \Psi_{j\gamma}^\dagger \vec{\sigma}_{\gamma\delta} \Psi_{i\delta} \right)
\]

which is invariant under the SU(2) pseudospin transformations

\[
\Psi_{i\alpha} \rightarrow U_i \Psi_{i\alpha}
\]

We will start with the \textbf{Néel state}, and find important consequences of the pseudospin symmetry in metals with antiferromagnetic correlations.

The electron spin polarization obeys
\[ \langle \vec{S}(\mathbf{r}, \tau) \rangle = \varphi(\mathbf{r}, \tau)e^{i\mathbf{K}\cdot\mathbf{r}} \]
where \( \mathbf{K} \) is the ordering wavevector.
Fermi surface + antiferromagnetism

Metal with “large” Fermi surface
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$. 
Fermi surface + antiferromagnetism

“Hot” spots
Fermi surface + antiferromagnetism

Electron and hole pockets in antiferromagnetic phase with antiferromagnetic order parameter $\langle \varphi \rangle \neq 0$
Fermi surface + antiferromagnetism

$\langle \varphi \rangle \neq 0$

Metal with electron and hole pockets

$\langle \varphi \rangle = 0$

Metal with "large" Fermi surface

$\mathcal{R}$
**Fermi surface + antiferromagnetism**

- Metal with electron and hole pockets
  - $\langle \varphi \rangle \neq 0$

- Metal with “large” Fermi surface
  - $\langle \varphi \rangle = 0$

Focus of this talk
Fermi surface + antiferromagnetism

“Hot” spots
Low energy theory for critical point near hot spots
Fermi surface + antiferromagnetism

Low energy theory for critical point near hot spots
Theory has fermions $\psi_{1,2}$ (with Fermi velocities $v_{1,2}$) and boson order parameter $\vec{\varphi}$, interacting with coupling $\lambda$.

$$S = \int d^2 r d\tau \left[ \psi_{1\alpha}^\dagger (\partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \right. \\
\left. + \frac{1}{2} (\nabla_r \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 - \lambda \varphi \cdot \left( \psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \right]$$

This low-energy theory is invariant under independent SU(2) pseudospin rotations on each pair of hot-spots: there is a global SU(2)⁴ symmetry, and \( \Pi_S(k) = \Pi_Q(k) \) near the hot spots.
Unconventional pairing at and near hot spots
\[ \langle c^{\dagger}_{k\alpha} c^{\dagger}_{-k\beta} \rangle = \varepsilon_{\alpha\beta} \Delta_S (\cos k_x - \cos k_y) \]

Unconventional pairing at and near hot spots

After pseudospin rotation on half the hot-spots

M. A. Metlitski and S. Sachdev, 

Unconventional particle-hole pairing at *and near* hot spots

\[
\langle c_{k+Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = \Delta_Q (\cos k_x - \cos k_y)
\]
Incommensurate $d$-wave bond order


$$\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = \Delta_Q (\cos k_x - \cos k_y)$$
Incommensurate $d$-wave bond order

\[ \langle c_{k-Q/2,\alpha}^{\dagger} c_{k+Q/2,\alpha} \rangle = \Delta_Q (\cos k_x - \cos k_y) \]

Incommensurate $d$-wave bond order

$\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = \Delta_Q (\cos k_x - \cos k_y)$
Charge-ordering eigenvalue $\lambda_Q/J_0$. 

S. Sachdev and R. La Placa, arXiv:1303.2114
Incommensurate $d$-wave bond order

High $T$ pseudogap: Fluctuating composite order parameter of nearly degenerate $d$-wave pairing and incommensurate $d$-wave bond order. (Approximate) SU(2) symmetry of composite order prevents long-range order $T > 0$.


$$\left\langle c_{k-\frac{Q}{2},\alpha}^\dagger c_{k+\frac{Q}{2},\alpha} \right\rangle = \Delta_Q (\cos k_x - \cos k_y)$$
Observed low $T$ ordering
Charge-ordering eigenvalue $\lambda_Q / J_0$. 

S. Sachdev and R. La Placa, arXiv:1303.2114
Evidence bond order is along (1,0), (0,1) directions in low $T$ superconducting phase.
Evidence bond order is along (1,0), (0,1) directions in low T superconducting phase

PHYSICAL REVIEW B 77, 094504 (2008)

Superconducting d-wave stripes in cuprates: Valence bond order coexisting with nodal quasiparticles

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(Received 8 January 2008; revised manuscript received 10 January 2008; published 6 March 2008)

We point out that unidirectional bond-centered charge-density-wave states in cuprates involve electronic order in both s- and d-wave channels, with nonlocal Coulomb repulsion suppressing the s-wave component. The resulting bond-charge-density wave, coexisting with superconductivity, is compatible with recent photoemission and tunneling data and as well as neutron-scattering measurements, once long-range order is destroyed by slow fluctuations or glassy disorder. In particular, the real-space structure of d-wave stripes is consistent with the scanning-tunneling-microscopy measurements on both underdoped Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and Ca$_{2-x}$Na$_x$CuO$_2$Cl$_2$ of Kohsaka et al. [Science 315, 1380 (2007)].
Electron spectral function

\[ \text{Im} G(k, \omega + i\eta) \quad \text{log \, Im} G(k, \omega + i\eta) \]

\[ \langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle \propto \Delta_Q(k) = \begin{cases} 
\Delta_s + \Delta_d (\cos k_x - \cos k_y) & , \quad Q = (\pm Q_0, 0) \\
\Delta_s - \Delta_d (\cos k_x - \cos k_y) & , \quad Q = (0, \pm Q_0) 
\end{cases} \]

with \( \Delta_s / \Delta_d = -0.234 \).

S. Sachdev and R. La Placa, arXiv:1303.2114
Outline

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3. Quantum Monte Carlo without the sign problem
Outline

1. Stability of metal in Hartree-Fock-BCS theory

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3. Quantum Monte Carlo without the sign problem
Low energy theory for critical point near hot spots
QMC for the onset of antiferromagnetism

Hot spots in a single band model
QMC for the onset of antiferromagnetism


Hot spots in a two band model
Faithful realization of the generic universal low energy theory for the onset of antiferromagnetism.

Hot spots in a two band model

QMC for the onset of antiferromagnetism

Requires only time-reversal symmetry. Particle-hole or point-group symmetries or commensurate densities **not** required!


Sign problem is absent as long as $K$ connects hotspots in distinct bands.

Hot spots in a two band model.
QMC for the onset of antiferromagnetism

Electrons with dispersion $\varepsilon_k$
interacting with fluctuations of the
antiferromagnetic order parameter $\bar{\varphi}$.

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\bar{\varphi} \exp (-S)$$

$$S = \int d\tau \sum_k c_\alpha^\dagger \left( \frac{\partial}{\partial \tau} - \varepsilon_k \right) c_\alpha$$

$$+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \bar{\varphi})^2 + \frac{r}{2} \bar{\varphi}^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \bar{\varphi}_i \cdot (-1)^{x_i} c_i^\dagger \bar{\sigma}_{\alpha\beta} c_{i\beta}$$
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon_{\mathbf{k}}^{(x)}$ and $\varepsilon_{\mathbf{k}}^{(y)}$
interacting with fluctuations of the antiferromagnetic order parameter $\bar{\varphi}$.

\[
\mathcal{Z} = \int \mathcal{D}c_{\alpha}^{(x)} \mathcal{D}c_{\alpha}^{(y)} \mathcal{D}\bar{\varphi} \exp (-S)
\]

\[
S = \int d\tau \sum_{\mathbf{k}} c_{\mathbf{k}\alpha}^{(x)} \left( \frac{\partial}{\partial \tau} - \varepsilon_{\mathbf{k}}^{(x)} \right) c_{\mathbf{k}\alpha}^{(x)}
\]

\[
+ \int d\tau \sum_{\mathbf{k}} c_{\mathbf{k}\alpha}^{(y)} \left( \frac{\partial}{\partial \tau} - \varepsilon_{\mathbf{k}}^{(y)} \right) c_{\mathbf{k}\alpha}^{(y)}
\]

\[
+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \bar{\varphi})^2 + \frac{r}{2} \bar{\varphi}^2 + \ldots \right]
\]

\[- \lambda \int d\tau \sum_{i} \bar{\varphi}_i \cdot (-1)^{\mathbf{x}_i} c_{i\alpha}^{(x)} \bar{\sigma}_{\alpha\beta} c_{i\beta}^{(y)} + \text{H.c.}\]
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon^{(x)}_k$ and $\varepsilon^{(y)}_k$ interacting with fluctuations of the antiferromagnetic order parameter $\bar{\varphi}$.

$$Z = \int \mathcal{D}c^{(x)}_\alpha \mathcal{D}c^{(y)}_\alpha \mathcal{D}\bar{\varphi} \exp (-S)$$

$$S = \int d\tau \sum_k c^{(x)\dagger}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_k \right) c^{(x)}_k + \int d\tau \sum_k c^{(y)\dagger}_k \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_k \right) c^{(y)}_k + \int d\tau d^2 x \left[ \frac{1}{2} (\nabla_x \bar{\varphi})^2 + \frac{r}{2} \bar{\varphi}^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \bar{\varphi}_i \cdot (-1)^{\mathbf{x}_i} c^{(x)\dagger}_{i\alpha} \bar{\sigma}_{\alpha\beta} c^{(y)}_{i\beta} + \text{H.c.}$$


No sign problem!
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon_k^{(x)}$ and $\varepsilon_k^{(y)}$
interacting with fluctuations of the
antiferromagnetic order parameter $\vec{\varphi}$.

$$Z = \int \mathcal{D}c_\alpha^{(x)} \mathcal{D}c_\alpha^{(y)} \mathcal{D}\vec{\varphi} \exp (-S)$$

$$S = \int d\tau \sum_k c_{k\alpha}^{(x)} \left( \frac{\partial}{\partial \tau} - \varepsilon_k^{(x)} \right) c_{k\alpha}^{(x)}$$

$$+ \int d\tau \sum_k c_{k\alpha}^{(y)} \left( \frac{\partial}{\partial \tau} - \varepsilon_k^{(y)} \right) c_{k\alpha}^{(y)}$$

$$+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \vec{\varphi})^2 + \frac{r}{2} \vec{\varphi}^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \vec{\varphi}_i \cdot (-1)^{x_i} c_{i\alpha}^{(x)} \bar{\sigma}_{\alpha\beta} c_{i\beta}^{(y)} + \text{H.c.}$$

E. Berg, M. Metlitski, and S. Sachdev,

Applies without changes to the microscopic band structure in the iron-based superconductors.
QMC for the onset of antiferromagnetism

Electrons with dispersions $\varepsilon^{(x)}_k$ and $\varepsilon^{(y)}_k$ interacting with fluctuations of the antiferromagnetic order parameter $\vec{\varphi}$.

$$Z = \int \mathcal{D}c^{(x)}_\alpha \mathcal{D}c^{(y)}_\alpha \mathcal{D}\varphi \exp (-S)$$

$$S = \int d\tau \sum_k c^{(x)}_{k\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(x)}_k \right) c^{(x)}_{k\alpha}$$

$$+ \int d\tau \sum_k c^{(y)}_{k\alpha} \left( \frac{\partial}{\partial \tau} - \varepsilon^{(y)}_k \right) c^{(y)}_{k\alpha}$$

$$+ \int d\tau d^2x \left[ \frac{1}{2} (\nabla_x \varphi)^2 + \frac{r}{2} \varphi^2 + \ldots \right]$$

$$- \lambda \int d\tau \sum_i \vec{\varphi}_i \cdot (-1)^x_i c^{(x)}_{i\alpha} \bar{\sigma}_{\alpha\beta} c^{(y)}_{i\beta} + \text{H.c.}$$

Can integrate out $\varphi$ to obtain an extended Hubbard model. The interactions in this model only couple electrons in separate bands.

QMC for the onset of antiferromagnetism

Hot spots in a two band model

E. Berg, M. Metlitski, and S. Sachdev, 
QMC for the onset of antiferromagnetism

QMC for the onset of antiferromagnetism


Move one of the Fermi surface by $(\pi, \pi,)$
QMC for the onset of antiferromagnetism

Now hot spots are at Fermi surface intersections

QMC for the onset of antiferromagnetism

Expected Fermi surfaces in the AFM ordered phase

QMC for the onset of antiferromagnetism

Electron occupation number $n_k$ as a function of the tuning parameter $r$

QMC for the onset of antiferromagnetism

AF susceptibility, $\chi_\varphi$, and Binder cumulant as a function of the tuning parameter $r$

QMC for the onset of antiferromagnetism

\[ \bar{P}(\tilde{x}_{\text{max}}) \]

\[ P_+ \quad P_- \]

\[ L = 10 \]
\[ L = 12 \]
\[ L = 14 \]

\( r_c \)

\( s/d \) pairing amplitudes \( P_+/P_- \)
as a function of the tuning parameter \( r \)

Conclusions

Metals with antiferromagnetic spin correlations have nearly degenerate instabilities: to $d$-wave superconductivity, and to a charge density wave with a $d$-wave form factor.
Metals with antiferromagnetic spin correlations have nearly degenerate instabilities: to $d$-wave superconductivity, and to a charge density wave with a $d$-wave form factor.

New sign-problem-free quantum Monte Carlo for studying such metals. Obtained (first ?) convincing evidence for unconventional superconductivity at strong coupling.
Metals with antiferromagnetic spin correlations have nearly degenerate instabilities: to $d$-wave superconductivity, and to a charge density wave with a $d$-wave form factor.

New sign-problem-free quantum Monte Carlo for studying such metals. Obtained (first ?) convincing evidence for unconventional superconductivity at strong coupling.

Good prospects for studying competing charge orders, and non-Fermi liquid physics at non-zero temperature.