Quantum phase transitions
of correlated electrons and atoms

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Course at Harvard University:
Physics 268r
Classical and Quantum Phase Transitions.
MWF 10 in Jefferson 256
First meeting: Feb 1.
Outline

A. Magnetic quantum phase transitions in “dimerized” Mott insulators
   Landau-Ginzburg-Wilson (LGW) theory

B. Mott insulators with spin $S=1/2$ per unit cell
   1. Berry phases and the mapping to a compact $U(1)$ gauge theory
   2. Valence-bond-solid (VBS) order in the paramagnet
   3. Mapping to hard-core bosons at half-filling

C. The superfluid-insulator transition of bosons in lattices
   Multiple order parameters in quantum systems

D. Boson-vortex duality
   Breakdown of the LGW paradigm
A. Magnetic quantum phase transitions in “dimerized” Mott insulators:

Landau-Ginzburg-Wilson (LGW) theory:

Second-order phase transitions described by fluctuations of an order parameter associated with a broken symmetry
M. Matsumoto, B. Normand, T.M. Rice, and M. Sigrist, cond-mat/0309440.
**Coupled Dimer Antiferromagnet**


$S=1/2$ spins on coupled dimers

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

$0 \leq \lambda \leq 1$
Weakly coupled dimers

\[ \lambda \text{ close to 0} \]
$\lambda$ close to 0

Weakly coupled dimers

Paramagnetic ground state

$\langle \vec{S}_i \rangle = 0$, $\langle \vec{\phi} \rangle = 0$

$\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$
\( \lambda \) close to 0

Weakly coupled dimers

\[
\begin{aligned}
\text{Excitation: } S &= 1 \text{ triplon} \\
\mathcal{O} &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\end{aligned}
\]
\[ \lambda \text{ close to } 0 \]

Weakly coupled dimers

Excitation: \( S=1 \) *triplon*
\( \lambda \) close to 0

Weakly coupled dimers

\[ \begin{array}{c}
\text{Excitation: } S=1 \text{ triplon}
\end{array} \]
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Excitation: $S=1$ triplon

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$\lambda$ close to 0

Weakly coupled dimers

Excitation: $S=1$ triplon

$\frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$
Weakly coupled dimers

Excitation: $S=1$ triplon (exciton, spin collective mode)

Energy dispersion away from antiferromagnetic wavevector

$$\varepsilon_p = \Delta + \frac{c_x^2 p_x^2 + c_y^2 p_y^2}{2\Delta}$$

$\Delta \to$ spin gap
For quasi-one-dimensional systems, the triplon linewidth takes the exact universal value $1.20k_BT e^{-\Delta/k_BT}$ at low $T$


Coupled Dimer Antiferromagnet
$\lambda$ close to 1

Weakly dimerized square lattice
Weakly dimerized square lattice

Excitations:
2 spin waves \((\text{magnons})\)

Ground state has long-range spin density wave (Néel) order at wavevector \(K = (\pi, \pi)\)

spin density wave order parameter: \(\bar{\phi} = \frac{\eta_i \tilde{S}_i}{S} ; \eta_i = \pm 1\) on two sublattices
Neutron Diffraction Study of the Pressure-Induced Magnetic Ordering in the Spin Gap System TlCuCl$_3$

Akira OOSAWA*, Masashi FUJISAWA$^1$, Toyotaka OSKABE, Kazuhisa KAKURAI and Hidekazu TANAKA$^2$

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**Fig. 3.** Temperature dependence of the magnetic Bragg peak intensity for $Q = (1,0,-3)$ reflection measured at $P = 1.48$ GPa in TlCuCl$_3$. 

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\[ \lambda_c = 0.52337(3) \]


Néel state
\[ \langle \phi \rangle \neq 0 \]

Quantum paramagnet
\[ \langle \phi \rangle = 0 \]
LGW theory for quantum criticality

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\bar{\phi}$ by expanding in powers of $\bar{\phi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian

$$S_\phi = \int d^2 x d\tau \left[ \frac{1}{2} \left( (\nabla_x \bar{\phi})^2 + \frac{1}{c^2} (\partial_\tau \bar{\phi})^2 + (\lambda_c - \lambda) \bar{\phi}^2 \right) + \frac{u}{4!} (\bar{\phi}^2)^2 \right]$$
Quantum field theory for critical point

$\lambda$ close to $\lambda_c$: use “soft spin” field

$$S_b = \int d^2x d\tau \left[ \frac{1}{2} \left( (\nabla_x \phi_\alpha)^2 + c^2 (\partial_\tau \phi_\alpha)^2 + (\lambda_c - \lambda) \phi_\alpha^2 \right) + \frac{u}{4!} \left( \phi_\alpha^2 \right)^2 \right]$$

$\phi_\alpha \rightarrow$ 3-component antiferromagnetic order parameter

Oscillations of $\phi_\alpha$ about zero (for $\lambda < \lambda_c$)

$\rightarrow$ spin-1 collective mode

$T=0$ spectrum

$$\epsilon_p = \Delta + \frac{c^2 p^2}{2\Delta}$$

$$\Delta = \sqrt{\lambda_c - \lambda}/c$$
Critical coupling \( \lambda = \lambda_c \)

Dynamic spectrum at the critical point

\[ \text{Im} \chi(p, \omega) \approx \left( \omega^2 - c^2 p^2 \right)^{-(2-\eta)/2} \]

No quasiparticles --- dissipative critical continuum
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B. Mott insulators with spin $S=1/2$ per unit cell:

1. *Berry phases and the mapping to a compact $U(1)$ gauge theory.*
Recall: dimerized Mott insulators

\[ \lambda_c = 0.52337(3) \]

M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama,

\[ \langle \tilde{\phi} \rangle \neq 0 \]

\[ \langle \tilde{\phi} \rangle = 0 \]

Pressure in TlCuCl$_3$
Mott insulator with two $S=1/2$ spins per unit cell
Mott insulator with one $S=1/2$ spin per unit cell
Mott insulator with one $S=1/2$ spin per unit cell

Ground state has Neel order with $\bar{\varphi} \neq 0$
Mott insulator with one $S=1/2$ spin per unit cell

Destroy Neel order by perturbations which preserve full square lattice symmetry e.g. second-neighbor or ring exchange. The strength of this perturbation is measured by a coupling $g$.

Small $g \Rightarrow$ ground state has Neel order with $\langle \phi \rangle \neq 0$

Large $g \Rightarrow$ paramagnetic ground state with $\langle \phi \rangle = 0
Mott insulator with one $S=1/2$ spin per unit cell

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Large $g \Rightarrow$ paramagnetic ground state with $\langle \phi \rangle = 0$
LGW theory for such a quantum transition

\[ S_\varphi = \int d^2xd\tau \left[ \frac{1}{2} \left( (\nabla_x \bar{\varphi})^2 + \frac{1}{c^2} (\partial_\tau \bar{\varphi})^2 + r\bar{\varphi}^2 \right) + \frac{u}{4!}(\bar{\varphi}^2)^2 \right] \]

What is the state with \( \langle \bar{\varphi} \rangle = 0 \) ?

The field theory predicts that this state has no broken symmetries and has a stable \( S=1 \) quasiparticle excitation (the triplon)
Problem: there is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries
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“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.
Quantum theory for destruction of Neel order

Ingredient missing from LGW theory:

Spin Berry Phases

Coherent state path integral for a single spin

\[
Z = \text{Tr} \left( e^{-H[S]/T} \right) \\
= \int \mathcal{D}N(\tau) \delta\left( N^2 - 1 \right) \exp \left( -i \int A_\tau(\tau) d\tau - \int d\tau H \left[ SN(\tau) \right] \right)
\]

\[ A_\tau(\tau) d\tau = \text{Oriented area of triangle on surface of unit sphere bounded by } N(\tau), N(\tau + d\tau), \text{ and a fixed reference } N_0 \]

Quantum theory for destruction of Neel order

**Ingredient missing from LGW theory:**
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Quantum theory for destruction of Neel order

Ingredient missing from LGW theory:
Spin Berry Phases

\[ e^{iSA} \]
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Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points $a$
Quantum theory for destruction of Neel order

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points $a$

Recall $\vec{\phi}_a = 2\eta_a \vec{S}_a \rightarrow \vec{\phi}_a = (0,0,1)$ in classical Neel state;

$\eta_a \rightarrow \pm 1$ on two square sublattices.
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formed by $\vec{\phi}_a$, $\vec{\phi}_{a+\mu}$, and an arbitrary reference point $\vec{\phi}_0$

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\( \eta_a \rightarrow \pm 1 \) on two square sublattices;
\( A_{a\mu} \rightarrow \text{half} \) oriented area of spherical triangle formed by \( \vec{\phi}_a, \vec{\phi}_{a+\mu} \), and an arbitrary reference point \( \vec{\phi}_0 \).

\[
2A_{a\mu} \rightarrow 2A_{a\mu} - \gamma_{a+\mu} + \gamma_a
\]

Change in choice of \( \vec{\phi}_0 \) is like a “gauge transformation”

Quantum theory for destruction of Neel order

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points $a$

Recall $\vec{\phi}_a = 2\eta_a \vec{S}_a \to \vec{\phi}_a = (0,0,1)$ in classical Neel state;

$\eta_a \to \pm 1$ on two square sublattices ;

$A_{a\mu} \to \text{half}$ oriented area of spherical triangle

formed by $\vec{\phi}_a$, $\vec{\phi}_{a+\mu}$, and an arbitrary reference point $\vec{\phi}_0$

\[
2A_{a\mu} \to 2A_{a\mu} - \gamma_{a+\mu} + \gamma_a
\]

Change in choice of $\vec{\phi}_0$ is like a “gauge transformation”

The area of the triangle is uncertain modulo $4\pi$, and the action has to be invariant under $A_{a\mu} \to A_{a\mu} + 2\pi$

Quantum theory for destruction of Neel order

**Ingredient missing from LGW theory:**

**Spin Berry Phases**

\[ \exp \left( i \sum_a \eta_a A_{a\tau} \right) \]

Sum of Berry phases of all spins on the square lattice.

\[ = \exp \left( i \sum_{a,\mu} J_{a\mu} A_{a\mu} \right) \]

with "current" \( J_{a\mu} \) of static charges \( \pm 1 \) on sublattices.
Quantum theory for destruction of Neel order

Partition function on cubic lattice

\[ Z = \prod_a \int d\bar{\phi}_a \delta(\bar{\phi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \bar{\phi}_a \cdot \bar{\phi}_{a+\mu} + i \sum_a \eta_a A_{a\tau} \right) \]

Modulus of weights in partition function: those of a classical ferromagnet at a “temperature” \( g \)

Small \( g \) \( \Rightarrow \) ground state has Neel order with \( \langle \bar{\phi} \rangle \neq 0 \)

Large \( g \) \( \Rightarrow \) paramagnetic ground state with \( \langle \bar{\phi} \rangle = 0 \)

Berry phases lead to large cancellations between different time histories \( \Rightarrow \) need an effective action for \( A_{a\mu} \) at large \( g \)

Simplest large $g$ effective action for the $A_{a\mu}$

$$Z = \prod_{a,\mu} \int dA_{a\mu} \exp\left(\frac{1}{2e^2} \sum_{\square} \cos(\Delta_{\mu} A_{av} - \Delta_{\nu} A_{a\mu}) + i \sum_a \eta_a A_{a\tau}\right)$$

with $e^2 \sim g^2$

This is compact QED in 3 spacetime dimensions with static charges $\pm 1$ on two sublattices.

Analysis by a duality mapping shows that this gauge theory has *valence bond solid* (VBS) order in the ground state for all $e$

B. Mott insulators with spin $S=1/2$ per unit cell:

1. *Berry phases and the mapping to a compact $U(1)$ gauge theory.*

2. *Valence bond solid (VBS) order in the paramagnet.*
Another possible state, with $\langle \tilde{\phi} \rangle = 0$, is the valence bond solid (VBS)
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Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
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\Psi_{\text{vbs}} (i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}
$$
Another possible state, with $\langle \tilde{\phi} \rangle = 0$, is the valence bond solid (VBS)

$$\Psi = \sum G_{\text{vbs}}(i)$$

Such a state breaks the symmetry of rotations by $n\pi/2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}}(i) = \sum_{\langle j \rangle} \vec{S}_i \cdot \vec{S}_j e^{i\arctan(r_j - r_i)}$$
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\Psi_{vbs}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i\arctan(r_j - r_i)}
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The VBS state does have a stable $S=1$ quasiparticle excitation.

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Ordering by quantum fluctuations
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\[ Z = \prod_a \int d\bar{\phi}_a \delta(\bar{\phi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \bar{\phi}_a \cdot \vec{\phi}_{a+\mu} + i \sum_a \eta_a A_{a\tau} \right) \]
LGW theory of multiple order parameters

\[ F = F_{\text{vbs}} \left[ \Psi_{\text{vbs}} \right] + F_\phi \left[ \phi \right] + F_{\text{int}} \]

\[ F_{\text{vbs}} \left[ \Psi_{\text{vbs}} \right] = r_1 \left| \Psi_{\text{vbs}} \right|^2 + u_1 \left| \Psi_{\text{vbs}} \right|^4 + \cdots \]

\[ F_\phi \left[ \phi \right] = r_2 \left| \phi \right|^2 + u_2 \left| \phi \right|^4 + \cdots \]

\[ F_{\text{int}} = v \left| \Psi_{\text{vbs}} \right|^2 \left| \phi \right|^2 + \cdots \]

Distinct symmetries of order parameters permit couplings only between their energy densities
LGW theory of multiple order parameters

First order transition

Neel order

VBS order

Coexistence

"disordered"

Neel order

VBS order
LGW theory of multiple order parameters

First order transition

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B. Mott insulators with spin $S=1/2$ per unit cell:

1. *Berry phases and the mapping to a compact U(1) gauge theory.*

2. *Valence bond solid (VBS) order in the paramagnet.*

At each site, identify the states $|\uparrow\rangle$, $|\downarrow\rangle$, with the occupation number of a hard-core boson:

\[
|\downarrow\rangle = |0\rangle \\
|\uparrow\rangle = b^\dagger |0\rangle
\]

Then the spin operators map as follows

\[
S_z = b^\dagger b - 1/2 \\
S_+ = b^\dagger \\
S_- = b
\]
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B. Superfluid-insulator transition

1. Bosons in a lattice at integer filling
Bose condensation
Velocity distribution function of ultracold $^{87}\text{Rb}$ atoms

Apply a periodic potential (standing laser beams) to trapped ultracold bosons ($^{87}\text{Rb}$)
Momentum distribution function of bosons

Bragg reflections of condensate at reciprocal lattice vectors

Superfluid-insulator quantum phase transition at $T=0$
Bosons at filling fraction $f = 1$

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$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity
Bosons at filling fraction $f = 1$

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Weak interactions: superfluidity
Bosons at filling fraction $f = 1$

Strong interactions: insulator

$\langle \Psi \rangle = 0$
The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, \( b_j^\dagger \), hopping between the sites, \( j \), of a lattice, with short-range repulsive interactions.

\[
H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots
\]

\[
 n_j \equiv b_j^\dagger b_j
\]


For small \( U/t \), ground state is a superfluid BEC with

superfluid density \( \approx \) density of bosons
What is the ground state for large $U/t$?

Typically, the ground state remains a superfluid, but with superfluid density $\ll$ density of bosons.

The superfluid density evolves smoothly from large values at small $U/t$, to small values at large $U/t$, and there is no quantum phase transition at any intermediate value of $U/t$.

(In systems with Galilean invariance and at zero temperature, superfluid density $=$ density of bosons always, independent of the strength of the interactions.)
What is the ground state for large $U/t$?

Incompressible, insulating ground states, with zero superfluid density, appear at special commensurately densities $3/n = t/2U$.

- Ground state has “density wave” order, which spontaneously breaks lattice symmetries.
LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x, \tau) \sim b_j^\dagger$
- Symmetries:
  
  Gauge invariance: $\Psi \rightarrow \Psi e^{i\theta}$
  Time reversal $\tau \rightarrow -\tau$ ; $\Psi \rightarrow \Psi^*$
  Spatial inversion $x \rightarrow -x$

- Write down most general Lagrangian consistent with symmetries

$$\mathcal{Z} = \int \mathcal{D}\Psi(x, \tau) \exp \left( -\int d^d x \int d\tau \mathcal{L}[\Psi] \right)$$

$$\mathcal{L}[\Psi] = K\Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r|\Psi|^2 + \frac{u}{2} |\Psi|^4 + \ldots$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.
- For $K \neq 0$, the particle and hole excitations have different energies.
Gauge-invariance of the underlying boson Hamiltonian shows that
\[ K = -\frac{\partial r}{\partial \mu} \]

In mean-field theory, the ground state energy, \( E \), across the superfluid-insulator transition has the non-analytic term
\[ E = E_0 - \frac{r^2}{2u} \theta(-r) \]

(Beyond mean-field theory, the non-analytic term is \( E \sim r^{(d+z)\nu} \)).

Because the density of bosons = \( -\partial E/\partial \mu \), this implies a change in the boson density across the transition unless \( \partial r/\partial \mu = 0 \).

A superfluid-insulator transition at fixed boson density must have
\[ K = 0 \]
B. Superfluid-insulator transition

2. *Bosons in a lattice at fractional filling*

Bosons at filling fraction \( f = \frac{1}{2} \)

\[ \langle \Psi \rangle \neq 0 \]

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

Weak interactions: superfluidity


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Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

Strong interactions: insulator

Bosons at filling fraction \( f = 1/2 \)

\[ \langle \Psi \rangle = 0 \]

**Strong interactions: insulator**

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ\cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

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**Insulating phases of bosons at filling fraction** $f = 1/2$

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All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Insulating phases of bosons at filling fraction $f = 1/2$

Seeking a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_\mathbf{Q} \rho_\mathbf{Q} e^{i\mathbf{Q}.r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_\mathbf{Q} \rangle \neq 0$ for certain $\mathbf{Q}$

Can define a common CDW/VBS order using a generalized "density" \( \rho(\mathbf{r}) = \sum \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{r}} \)

All insulators have \( \langle \Psi \rangle = 0 \) and \( \langle \rho_{\mathbf{Q}} \rangle \neq 0 \) for certain \( \mathbf{Q} \)

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All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Ginzburg-Landau-Wilson approach to multiple order parameters:

\[ F = F_{sc} \left[ \Psi_{sc} \right] + F_{\text{charge}} \left[ \rho_Q \right] + F_{\text{int}} \]

\[ F_{sc} \left[ \Psi_{sc} \right] = r_1 \left| \Psi_{sc} \right|^2 + u_1 \left| \Psi_{sc} \right|^4 + \cdots \]

\[ F_{\text{charge}} \left[ \rho_Q \right] = r_2 \left| \rho_Q \right|^2 + u_2 \left| \rho_Q \right|^4 + \cdots \]

\[ F_{\text{int}} = v \left| \Psi_{sc} \right|^2 \left| \rho_Q \right|^2 + \cdots \]

Distinct symmetries of order parameters permit couplings only between their energy densities

Predictions of LGW theory

First order transition

Coexistence
(Supersolid)

Disordered
(≠ topologically ordered)

\[ \left\langle \Psi_{sc} \right\rangle = 0, \left\langle \rho_Q \right\rangle = 0 \]
Predictions of LGW theory

First order transition

1. Superconductor

2. Charge-ordered insulator

Coexistence

(Supersolid)

1. Superconductor

2. Charge-ordered insulator

"Disordered"

\( \langle \Psi_{sc} \rangle = 0, \langle \rho_Q \rangle = 0 \)

Charge-ordered insulator
Outline

A. Magnetic quantum phase transitions in “dimerized” Mott insulators
   *Landau-Ginzburg-Wilson (LGW) theory*

B. Mott insulators with spin $S=1/2$ per unit cell
   1. Berry phases and the mapping to a compact $U(1)$ gauge theory
   2. Valence-bond-solid (VBS) order in the paramagnet;
   3. Mapping to hard-core bosons at half-filling

C. The superfluid-insulator transition of bosons in lattices
   *Multiple order parameters in quantum systems*

D. Boson-vortex duality
   *Breakdown of the LGW paradigm*
D. Boson-vortex duality

1. Bosons in a lattice at integer filling
Bosons at density $f = 1$

Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

**Approaching the transition from the insulator \((f=1)\)**

Excitations of the insulator:

Particles \(\sim \psi^\dagger\)

Holes \(\sim \psi\)

Density of particles = density of holes \(\Rightarrow\)

"relativistic" field theory for \(\psi\):

\[
S = \int d^2r d\tau \left[ |\partial_\tau \psi|^2 + |\nabla_r \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]
\]

Insulator \(\Leftrightarrow \langle \psi \rangle = 0\)

Superfluid \(\Leftrightarrow \langle \psi \rangle \neq 0\)
**Approaching the transition from the superfluid \((f=1)\)**

Excitations of the superfluid: (A) **Spin waves**

With \(\psi \sim e^{i\theta}\), action for spin waves is

\[
S_{sw} = \frac{\rho_s}{2} \int d^3x (\partial_\mu \theta)^2
\]

**Dual form:** After a Hubbard-Stratonovich transformation, write

\[
S_{sw} = \int d^3x \left[ \frac{1}{2\rho_s} J_\mu^2 + iJ_\mu \partial_\mu \theta \right]
\]

Integrating over \(\theta\) yields \(\partial_\mu J_\mu = 0\). Solve, by writing

\[
J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda
\]

leading to

\[
S_{sw} = \int d^3x \left[ \frac{1}{2\rho_s} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]
\]

Spin waves are dual to a \(U(1)\) gauge theory in 2+1 dimensions
Approaching the transition from the superfluid \((f=1)\)

Excitations of the superfluid: (B) Vortices

A vortex is a point-like object. We can therefore define a local field operator, \(\varphi\), which annihilates a vortex.
Approaching the transition from the superfluid \((f=1)\)

Excitations of the superfluid: (B) Vortices

A vortex is a point-like object. We can therefore define a local field operator, \(\varphi\), which annihilates a vortex.

Each vortex is the source of an ‘electric field’ \(\vec{E}\) associated with the U(1) gauge field \(A_\mu\).

Consequently, \(\varphi\) carries +1 U(1) gauge charge.
Approaching the transition from the superfluid ($f=1$)

Excitations of the superfluid: Spin wave and vortices

$\varphi$: vortex annihilation operator.

$\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$: boson current $\sim i\psi^*\partial_\mu \psi - i\partial_\mu \psi^*\psi$.

Density of vortices = density of antivortices $\Rightarrow$

"relativistic" field theory for $\varphi$:

$$S_{\text{dual}} = \int d^3x \left[ |(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2\rho_s}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

Superfluid $\Leftrightarrow \langle \varphi \rangle = 0$

Insulator $\Leftrightarrow \langle \varphi \rangle \neq 0$
Dual theories of the superfluid-insulator transition \((f=1)\)

Excitations of the superfluid: Spin wave and vortices

Using the boson quasiparticle excitations of the insulator \(\sim \psi\)

\[
S = \int d^2r d\tau \left[ |\partial_\tau \psi|^2 + |\nabla_r \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]
\]

Insulator \(\Leftrightarrow \langle \psi \rangle = 0\)

Superfluid \(\Leftrightarrow \langle \psi \rangle \neq 0\)

is dual to

Using the vortex quasiparticle excitations of the superfluid \(\sim \varphi\)

\[
S_{\text{dual}} = \int d^3x \left[ |(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2\rho_s} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]
\]

Superfluid \(\Leftrightarrow \langle \varphi \rangle = 0\)

Insulator \(\Leftrightarrow \langle \varphi \rangle \neq 0\)

A vortex in the vortex field is the original boson

A vortex in $\varphi$ carries $2\pi$ flux in the ‘magnetic field’ $B = \epsilon_{\tau\mu\nu} \partial_\mu A_\nu$. But this is just the original boson number operator. Consequently, in the path integral viewpoint, the world line of the vortex in $\varphi$ is just the world line of the original boson.
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A vortex in $\varphi$ carries $2\pi$ flux in the ‘magnetic field’ $B = \epsilon_{\tau \mu \nu} \partial_\mu A_\nu$. But this is just the original boson number operator. Consequently, in the path integral viewpoint, the world line of the vortex in $\varphi$ is just the world line of the original boson.

The wavefunction of a vortex acquires a phase of $2\pi$ each time the vortex encircles a boson.
D. Boson-vortex duality

2. Bosons in a lattice at fractional filling $f$

The wavefunction of a vortex acquires a phase of $2\pi$ each time the vortex encircles a boson.

Strength of “magnetic” field on vortex field $\varphi$

$= \text{density of bosons} = f \text{ flux quanta per plaquette}$

In ordinary fluids, vortices experience the Magnus Force.

\[ F_M = (\text{mass density of air}) \cdot (\text{velocity of ball}) \cdot (\text{circulation}) \]
For a vortex in a superfluid, this is

\[
\mathbf{F}_M = (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right) \\
= n\hbar \rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z}
\]

where \( \rho = \) number density of bosons
\( \mathbf{v}_s = \) local velocity of superfluid
\( \mathbf{r}_v = \) position of vortex
For a vortex in a superfluid, this is

$$F_M = (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right)$$

$$= n\hbar \rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z}$$

$$= n \left( \mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right)$$

where $\mathbf{E} = \rho \mathbf{v}_s \times \hat{z}$ and $\mathbf{B} = -\hbar \rho \hat{z}$

**Dual picture:**
The vortex is a quantum particle with dual “electric” charge $n$, moving in a dual “magnetic” field of strength $= h \times (\text{number density of Bose particles})$
• The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual “magnetic” field of strength $= h\rho$, where $\rho$ is the number density of bosons per unit cell.

• The vortex motion can be described by the effective Hofstadter Hamiltonian:

$$\mathcal{H}_v = -t \sum_{\langle ij \rangle} (e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.})$$

where $\varphi_i$ is an operator which annihilates a vortex particle at site $i$ of a square lattice.

$$A_1 + A_2 + A_3 + A_4 = 2\pi f$$

where $f$ is the boson filling fraction.
Bosons at filling fraction $f = 1$

- At $f=1$, the “magnetic” flux per unit cell is $2\pi$, and the vortex does not pick up any phase from the boson density.

- The effective dual “magnetic” field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.
Bosons at rational filling fraction $f = p/q$

Quantum mechanics of the vortex “particle” in a periodic potential with $f$ flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

$T_x, T_y$ : Translations by a lattice spacing in the $x, y$ directions

$R$ : Rotation by 90 degrees.

Magnetic space group:

$$T_x T_y = e^{2\pi i f} T_y T_x ;$$

$$R^{-1} T_y R = T_x ; \quad R^{-1} T_x R = T_y^{-1} ; \quad R^4 = 1$$

The low energy vortex states must form a representation of this algebra
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

Hofstadter spectrum of the quantum vortex “particle” with field operator $\varphi$

At filling $f=p/q$, there are $q$ species of vortices, $\varphi_\ell$ (with $\ell=1\ldots q$), associated with $q$ degenerate minima in the vortex spectrum. These vortices realize the smallest, $q$-dimensional, representation of the magnetic algebra.

The $q$ vortices form a projective representation of the space group

$$T_x : \varphi_\ell \to \varphi_{\ell+1} \; ; \; T_y : \varphi_\ell \to e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \to \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell mf}$$
Boson-vortex duality

The $q$ $\varphi_\ell$ vortices characterize both superconducting and density wave orders

Superconductor/insulator: $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$
The $q\,\varphi_\ell$ vortices characterize both superconducting and density wave orders.

Density wave order:
Status of space group symmetry determined by density operators $\rho_Q$ at wavevectors $Q_{mn} = \frac{2\pi p}{q}(m, n)$

$$
\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}
$$

$$
T_x : \rho_Q \rightarrow \rho_Q e^{i\hat{\rho} \cdot \hat{x}} ; \quad T_y : \rho_Q \rightarrow \rho_Q e^{i\hat{\rho} \cdot \hat{y}}
$$

$R : \rho(Q) \rightarrow \rho(RQ)$
Field theory with projective symmetry

Degrees of freedom:

- $q$ complex $\varphi_\ell$ vortex fields
- 1 non-compact U(1) gauge field $A_\mu$

$$S = \int d^2x dt \left[ \sum_\ell \left\{ \left| (\partial_\mu - iA_\mu) \varphi_\ell \right|^2 + s |\varphi_\ell|^2 \right\} 
+ \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 + \sum_{\ell mn} \gamma_{mn} \varphi_\ell^* \varphi_{\ell+m} \varphi_{\ell+n} \varphi_{\ell+m-n} \right]$$

The projective symmetries constrain the couplings $\gamma_{mn}$ to obey

$$\gamma_{mn} = \gamma_{-m,-n} ; \quad \gamma_{mn} = \gamma_{m,m-n} ; \quad \gamma_{mn} = \gamma_{m-2n,-n}$$

$$\gamma_{\bar{m}\bar{n}} = \frac{1}{q} \sum_{mn} \gamma_{mn} e^{-2\pi i f[n(\bar{m}-\bar{n})+\bar{n}(m-n)]}$$
Fluctuation-induced, weak, first order transition

\[ \langle \Psi_{sc} \rangle \]
Superconductor

\[ \langle \varphi_{\ell} \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

Field theory with projective symmetry

Fluctuation-induced, weak, first order transition

\[ \langle \rho_{Q} \rangle \]
Charge-ordered insulator

\[ \langle \varphi_{\ell} \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]

\[ r_1 - r_2 \]
Field theory with projective symmetry

\[ \langle \Psi_{sc} \rangle \]
Superconductor
\[ \langle \varphi \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

Charge-ordered insulator
\[ \langle \varphi \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]
\[ r_1 - r_2 \]

\[ \langle \Psi_{sc} \rangle \]
Supersolid
\[ \langle \varphi \rangle = 0, \langle \rho_{mn} \rangle \neq 0 \]

Charge-ordered insulator
\[ \langle \varphi \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]
\[ r_1 - r_2 \]
Field theory with projective symmetry

Fluctuation-induced, weak, first order transition

Superconductor

\[ \langle \Psi_{sc} \rangle \]

\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

Charge-ordered insulator

\[ \langle \rho_Q \rangle \]

\[ \langle \varphi_\ell \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]

Supersolid

\[ \langle \Psi_{sc} \rangle \]

\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle \neq 0 \]

Charge-ordered insulator

\[ \langle \rho_Q \rangle \]

\[ \langle \varphi_\ell \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]

Second order transition

\[ \langle \Psi_{sc} \rangle \]

\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

Charge-ordered insulator

\[ \langle \rho_Q \rangle \]

\[ \langle \varphi_\ell \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]
Field theory with projective symmetry

Spatial structure of insulators for $q=2 \ (f=1/2)$

All insulating phases have density-wave order $\rho(r) = \sum_q \rho_q e^{i\mathbf{q} \cdot \mathbf{r}}$ with $\langle \rho_q \rangle \neq 0$
Field theory with projective symmetry
Spatial structure of insulators for $q=4$ ($f=1/4$ or $3/4$)

$a \times b$ unit cells; $q/\lambda$, $q/\lambda'$, $a/b/q'$, all integers
Field theory with projective symmetry

Density operators $\rho_Q$ at wavevectors $Q_{mn} = \frac{2\pi p}{q}(m,n)$

$$\rho_{mn} = e^{i\pi m n f} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell m f}$$

Each pinned vortex in the superfluid has a halo of density wave order over a length scale $\approx$ the zero-point quantum motion of the vortex. This scale diverges upon approaching the insulator.
Vortex-induced LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ integrated from 1meV to 12meV at 4K

Vortices have halos with LDOS modulations at a period $\approx 4$ lattice spacings


Superfluids near Mott insulators

The Mott insulator has average Cooper pair density, \( f = p/q \) per site, while the density of the superfluid is close (but need not be identical) to this value

- Vortices with flux \( h/(2e) \) come in multiple (usually \( q \)) “flavors”
- The lattice space group acts in a projective representation on the vortex flavor space.
- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”
- Any pinned vortex must chose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.