Quantum phase transitions of insulators, superconductors and metals in two dimensions

Talk online: sachdev.physics.harvard.edu
Outline

1. Phenomenology of the cuprate superconductors (and other compounds)

2. QPT of antiferromagnetic insulators (and bosons at rational filling)

3. QPT of d-wave superconductors: Fermi points of massless Dirac fermions

4. QPT of Fermi surfaces:
   A. Finite wavevector ordering (SDW/CDW): “Hot spots” on Fermi surfaces
   B. Zero wavevector ordering (Nematic): “Hot Fermi surfaces”
Outline

1. Phenomenology of the cuprate superconductors (and other compounds)

2. QPT of antiferromagnetic insulators (and bosons at rational filling)

3. QPT of d-wave superconductors: 
   Fermi points of massless Dirac fermions

4. QPT of Fermi surfaces:
   A. Finite wavevector ordering (SDW/CDW): 
      “Hot spots” on Fermi surfaces
   B. Zero wavevector ordering (Nematic): 
      “Hot Fermi surfaces”
Max Metlitski
Strategy

1. Write down local field theory for order parameter and fermions

2. Apply renormalization group to field theory
1. Write down local field theory for order parameter and fermions

2. Apply renormalization group to field theory
Order parameter at a non-zero wavevector: “Hot spots” on the Fermi surface.
Strange Metal

Large Fermi surface

Small Fermi pockets with pairing fluctuations

Large Fermi surface

Thermally fluctuating SDW
d-wave SC

Magnetic quantum criticality

Spin gap

Fluctuating, paired Fermi pockets

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)
Theory of SDW quantum phase transition in metal
Theory of SDW quantum phase transition in metal
Hole-doped cuprates

Increasing SDW order

Large Fermi surface breaks up into electron and hole pockets

Hole-doped cuprates

Increasing SDW order

\( \phi \) fluctuations act on the large Fermi surface

Start from the “spin-fermion” model

\[ Z = \int \mathcal{D}c_\alpha \mathcal{D}\vec{\phi} \exp (-S) \]

\[ S = \int d\tau \sum_k c_k^\dagger (\frac{\partial}{\partial \tau} - \varepsilon_k) c_k + \lambda \int d\tau \sum_i c_{i \alpha}^\dagger \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha \beta} c_{i \beta} e^{iK \cdot r_i} \]

\[ + \int d\tau d^2r \left[ \frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right] \]
Low energy fermions
\[ \psi_{1\alpha}, \psi_{2\alpha} \]
\[ \ell = 1, \ldots, 4 \]

\[
\mathcal{L}_f = \psi_{1\alpha}^{\dagger} \left( \zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r \right) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\dagger} \left( \zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r \right) \psi_{2\alpha}^\ell
\]

\[
\mathbf{v}_1^{\ell=1} = (v_x, v_y), \quad \mathbf{v}_2^{\ell=1} = (-v_x, v_y)
\]
\[ \mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]
\[ \mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]
\[ L_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha}^{\dagger} + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha}^{\dagger} \]

Order parameter: \[ L_\phi = \frac{1}{2} (\nabla_r \bar{\phi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \bar{\phi})^2 + \frac{s}{2} \bar{\phi}^2 + \frac{u}{4} \bar{\phi}^4 \]
\[ L_f = \psi_{1\alpha}^{l\dagger} \left( \zeta \partial_\tau - i \mathbf{v}_1^l \cdot \nabla_r \right) \psi_{1\alpha}^l + \psi_{2\alpha}^{l\dagger} \left( \zeta \partial_\tau - i \mathbf{v}_2^l \cdot \nabla_r \right) \psi_{2\alpha}^l \]

Order parameter:

\[ L_\varphi = \frac{1}{2} \left( \nabla_r \varphi \right)^2 + \frac{\tilde{\zeta}}{2} \left( \partial_\tau \varphi \right)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

"Yukawa" coupling:

\[ L_c = -\lambda \varphi \cdot \left( \psi_{1\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^l + \psi_{2\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^l \right) \]
Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain non-local corrections to $\mathcal{L}_\varphi$

\[
\mathcal{L}_f = \psi_{1\alpha}^{l\dagger} (\zeta \partial_\tau - iv_1^l \cdot \nabla_r) \psi_{1\alpha}^{l} + \psi_{2\alpha}^{l\dagger} (\zeta \partial_\tau - iv_2^l \cdot \nabla_r) \psi_{2\alpha}^{l}
\]

Order parameter:

\[
\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4
\]

“Yukawa” coupling:

\[
\mathcal{L}_c = -\lambda \vec{\varphi} \cdot \left( \psi_{1\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^{l} + \psi_{2\alpha}^{l\dagger} \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^{l} \right)
\]

Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain non-local corrections to $\mathcal{L}_\varphi$

\[
\mathcal{L}_\varphi = \frac{1}{2} \vec{\varphi}^2 \left[ q^2 + \gamma |\omega| \right] / 2 ; \quad \gamma = \frac{2}{\pi v_x v_y}
\]

Exponent $z = 2$ and mean-field criticality (upto logarithms)
$$\mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha}$$

Order parameter: $$\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4$$

“Yukawa” coupling: $$\mathcal{L}_c = -\lambda \varphi \cdot (\psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^l + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^l)$$

Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain non-local corrections to $\mathcal{L}_\varphi$

$$\mathcal{L}_\varphi = \frac{1}{2} \varphi^2 \left[ q^2 + \gamma |\omega| \right] / 2 ; \quad \gamma = \frac{2}{\pi v_x v_y}$$

Exponent $z = 2$ and mean-field criticality (upto logarithms)

But, higher order terms contain an infinite number of marginal couplings . . . . . .

\[ L_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i v_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i v_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

Perform RG on both fermions and \( \bar{\varphi} \), using a local field theory.
Order parameter at zero wavevector: “Hot Fermi surface”.
Fluctuating, paired Fermi pockets

\( T^* \)

Strange Metal

Large Fermi surface

\( T_c \)

d-wave

SC

Fluctuating, paired Fermi pockets

\( T_{sdw} \)

VBS and/or Ising nematic order

SDW

Insulator

\( H_{c2} \)

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)
Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D. A. Bonn, W. N. Hardy, and Louis Taillefer

Quantum criticality of Pomeranchuk instability

Fermi surface with full square lattice symmetry
Spontaneous elongation along $x$ direction:
Ising order parameter $\phi > 0$. 

Quantum criticality of Pomeranchuk instability
Quantum criticality of Pomeranchuk instability

Spontaneous elongation along $y$ direction:
Ising order parameter $\phi < 0$. 
Quantum criticality of Pomeranchuk instability

Pomeranchuk instability as a function of coupling $\lambda$

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$\lambda_c$
Quantum criticality of Pomeranchuk instability

Phase diagram as a function of $T$ and $\lambda$

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

Quantum critical
Quantum criticality of Pomeranchuk instability

Phase diagram as a function of $T$ and $\lambda$

Classical $d=2$ Ising criticality

Quantum critical

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$T_c$

$\lambda_c$

Friday, April 13, 2012
Quantum criticality of Pomeranchuk instability

Phase diagram as a function of $T$ and $\lambda$

Quantum criticality?

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$T_c$
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

$$S_{\phi} = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_i^\dagger \partial_\tau c_i + \sum_{i<j} t_{ij} c_i^\dagger c_j + \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^{\dagger} \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \right] \]
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

\[ S_{\phi c} = -\gamma \int d\tau \phi \sum_{\alpha=1}^{N_f} \sum_{k} (\cos k_x - \cos k_y) c_{k\alpha}^{\dagger} c_{k\alpha} \]

for spatially independent \( \phi \)
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^{\dagger} c_{k-q/2,\alpha} \]

for spatially dependent \( \phi \)
Quantum criticality of Pomeranchuk instability

\[ S_\phi = \int d^2rd\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]
A $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$.

Expand fermion kinetic energy at wavevectors about $\vec{k}_0$. 
\[ \mathcal{L} = \psi_+^\dagger (\zeta \partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\zeta \partial_\tau + i \partial_x - \partial_y^2) \psi_- - \lambda \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g} (\partial_y \phi)^2 + \frac{r}{2} \phi^2 \]
Emergent “Galilean invariance” at low energy \( (s = \pm) \):

\[
\phi(x, y) \rightarrow \phi(x, y + \theta x), \quad \psi_s(x, y) \rightarrow e^{-is\left(\frac{\theta}{2}y + \frac{\theta^2}{4}x\right)}\psi_s(x, y + \theta x)
\]

which implies for the fermion Green’s function

\[
G(q_x, q_y) = G(sq_x + q_y^2).
\]
Emergent “Galilean invariance” at low energy \((s = \pm)\):

\[
\phi(x, y) \rightarrow \phi(x, y + \theta x), \quad \psi_s(x, y) \rightarrow e^{-is(\frac{y}{2} + \frac{\theta^2}{4} x)}\psi_s(x, y + \theta x)
\]

which implies for the fermion Green’s function

\[
G(q_x, q_y) = G(sq_x + q_y^2).
\]

Every point on the Fermi surface \(sq_x + q_y^2 = 0\) has the same singularity: “Hot Fermi surface”. 
Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain effective action for $\phi$

\[
\mathcal{L}_\phi = \frac{1}{2} \phi^2 \left[ \frac{q_y^2}{g} + \frac{|\omega|}{4\pi |q_y|} \right]
\]

Exponent $z = 3$ and mean-field criticality?
Strategy

1. Write down local field theory for order parameter and fermions

2. Apply renormalization group to field theory
1. Write down local field theory for order parameter and fermions

2. Apply renormalization group to field theory
Order parameter at a non-zero wavevector: “Hot spots” on the Fermi surface.
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi')^2 + \frac{\zeta'}{2} (\partial_\tau \varphi')^2 + \frac{s}{2} \varphi'^2 + \frac{u}{4} \varphi'^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi' \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta}^\l + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta}^\l \right) \]

Under the rescaling \( x' = xe^{-\ell}, \ \tau' = \tau e^{-z\ell} \), the spatial gradients are fixed if the fields transform as

\[ \varphi' = e^{(d+z-2)\ell/2} \varphi; \quad \psi' = e^{(d+z-1)\ell/2} \psi. \]

Then the Yukawa coupling transforms as

\[ \lambda' = e^{(4-d-z)\ell/2} \lambda \]

For \( d = 2 \), with \( z = 2 \) the Yukawa coupling is invariant, and the bare time-derivative terms \( \zeta, \zeta' \) are irrelevant.
\[ L_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ L_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

“Yukawa” coupling: \[ L_c = -\bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

With \( z = 2 \) scaling, \( \zeta \) is irrelevant. So we take \( \zeta \to 0 \)

(\( \forall \) watch for dangerous irrelevancy).
\[ \mathcal{L}_f = \psi^\dagger_{1\alpha} (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi^\dagger_{2\alpha} (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:
\[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling:
\[ \mathcal{L}_c = -\varphi \cdot \left( \psi^\dagger_{1\alpha} \sigma_{\alpha\beta} \psi_{2\beta} + \psi^\dagger_{2\alpha} \sigma_{\alpha\beta} \psi_{1\beta} \right) \]

Set \( \varphi \) wavefunction renormalization by keeping co-efficient of \((\nabla_r \varphi)^2\) fixed (as usual).
\[ \mathcal{L}_f = \psi_{1\alpha}^{l+} (\zeta \partial_\tau - i \mathbf{v}_1^l \cdot \nabla_r) \psi_{1\alpha}^l + \psi_{2\alpha}^{l+} (\zeta \partial_\tau - i \mathbf{v}_2^l \cdot \nabla_r) \psi_{2\alpha}^l \]

Order parameter:
\[ \mathcal{L}_{\varphi} = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling:
\[ \mathcal{L}_c = -\varphi \cdot \left( \psi_{1\alpha}^{l+} \vec{\sigma}_{\alpha\beta} \psi_{2\beta}^l + \psi_{2\alpha}^{l+} \vec{\sigma}_{\alpha\beta} \psi_{1\beta}^l \right) \]

Set fermion wavefunction renormalization by keeping Yukawa coupling fixed.

\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

"Yukawa" coupling: \[ \mathcal{L}_c = -\bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \bar{\sigma}^{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}^{\alpha\beta} \psi_{1\beta} \right) \]

We find consistent two-loop RG factors, as \( \zeta \to 0 \), for the velocities \( \nu_x, \nu_y \), and the wavefunction renormalizations.

**Consistency check:** the expression for the boson damping constant, \( \gamma = \frac{2}{\pi \nu_x \nu_y} \), is preserved under RG.
RG flow can be computed a $1/N$ expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{d\ell} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$
RG flow can be computed a $1/N$ expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{d\ell} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$

The velocities flow as

$$\frac{1}{v_x} \frac{dv_x}{d\ell} = \frac{\mathcal{A}(\alpha) + \mathcal{B}(\alpha)}{2}; \quad \frac{1}{v_y} \frac{dv_y}{d\ell} = -\frac{\mathcal{A}(\alpha) + \mathcal{B}(\alpha)}{2}$$

$$\mathcal{A}(\alpha) \equiv \frac{3}{\pi N} \frac{\alpha}{1 + \alpha^2}$$

$$\mathcal{B}(\alpha) \equiv \frac{1}{2\pi N} \left( \frac{1}{\alpha} - \alpha \right) \left( 1 + \left( \frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$
RG flow can be computed a $1/N$ expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{dl} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$

The anomalous dimensions of $\bar{\phi}$ and $\psi$ are

$$\eta_{\bar{\phi}} = \frac{1}{2\pi N} \left( \frac{1}{\alpha} - \alpha + \left( \frac{1}{\alpha^2} + \alpha^2 \right) \tan^{-1} \frac{1}{\alpha} \right)$$

$$\eta_{\psi} = -\frac{1}{4\pi N} \left( \frac{1}{\alpha} - \alpha \right) \left( 1 + \left( \frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \to 0 \text{ logarithmically in the infrared.} \]

Dynamical Nesting

Bare Fermi surface
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \to 0 \text{ logarithmically in the infrared.} \]

Dynamical Nesting

Dressed Fermi surface
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \to 0 \text{ logarithmically in the infrared.} \]

Dynamical Nesting

Bare Fermi surface
\[ \alpha = \frac{v_y}{v_x} \rightarrow 0 \text{ logarithmically in the infrared.} \]

**RG-improved Migdal-Eliashberg theory**

**Dynamical Nesting**

**Dressed Fermi surface**
\[
\alpha = \nu_y / \nu_x \to 0 \text{ logarithmically in the infrared.}
\]

In $\varphi$ SDW fluctuations, characteristic $q$ and $\omega$ scale as

\[
q \sim \omega^{1/2} \exp \left( -\frac{3}{64\pi^2} \left( \frac{\ln(1/\omega)}{N} \right)^3 \right).
\]

However, $1/N$ expansion cannot be trusted in the asymptotic regime.
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

$\bar{\phi}$ propagator

$$\frac{1}{N} \frac{1}{(q^2 + \gamma |\omega|)}$$

fermion propagator

$$\frac{1}{\mathbf{v} \cdot \mathbf{q} + i \zeta \omega + i \frac{1}{N \sqrt{\gamma v}} \sqrt{\omega F \left( \frac{v^2 q^2}{\omega} \right)}}$$
New infra-red singularities as $\zeta \to 0$ at higher loops
(Breakdown of Migdal-Eliashberg)

$\phi$ propagator

\[ \frac{1}{N} \frac{1}{(q^2 + \gamma|\omega|)} \]

fermion propagator

\[ v \cdot q + i\zeta \omega + i \frac{1}{N\sqrt{\gamma}v} \sqrt{\omega} F \left( \frac{v^2 q^2}{\omega} \right) \]

⚠️ Dangerous
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$

Actual order $\sim \frac{1}{N^0}$
Double line representation

- A way to compute the order of a diagram.
- Extra powers of N come from the Fermi-surface

\[ G(\omega, \vec{k}) = \frac{1}{-\Sigma_1(\omega, \vec{k}) - \vec{v} \cdot \vec{k}} \quad \Sigma_1 \sim \frac{1}{N} \]

- What are the conditions for all propagators to be on the Fermi surface?
- Concentrate on diagrams involving a single pair of hot-spots
- Any bosonic momentum may be (uniquely) written as

\[ \vec{q} = \vec{k}_1 - \vec{k}_2 \quad \vec{k}_1 \in \text{FS of } \psi_1 \quad \vec{k}_2 \in \text{FS of } \psi_2 \]

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Singularities as $\zeta \to 0$ appear when fermions in closed blue and red line loops are exactly on the Fermi surface

Actual order $\sim \frac{1}{N^0}$
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Graph is planar after turning fermion propagators also into double lines by drawing additional dotted single line loops for each fermion loop

Actual order $\sim \frac{1}{N^0}$

Graph is planar after turning fermion propagators also into double lines by drawing additional dotted single line loops for each fermion loop

Sung-Sik Lee, arXiv:0905.4532
New infra-red singularities as $\zeta \to 0$ at higher loops
(Breakdown of Migdal-Eliashberg)

Actual order $\sim \frac{1}{N^0}$

A consistent analysis requires resummation of all planar graphs
Theory for the onset of spin density wave order in metals is strongly coupled in two dimensions.