Low energy theory of a single vortex and electronic quasiparticles in a $d$-wave superconductor
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Talk online at http://sachdev.physics.harvard.edu
BCS theory for local density of states (LDOS) at the center of a vortex in a $d$-wave superconductor

Prominent feature: large peak at zero bias

STM around vortices induced by a magnetic field in the superconducting state


Local density of states (LDOS)

1Å spatial resolution image of integrated LDOS of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (1meV to 12 meV) at B=5 Tesla.

Outline

1. Our model
2. Influence of electronic quasiparticles on vortex motion
3. Influence of vortex motion on electronic quasiparticles
4. Aharonov-Bohm phases in vortex motion and “checkerboard” modulations in LDOS
I. The model
Degrees of freedom

- We consider a point vortex (with vanishing core radius) whose (first-quantized) position is $r_v(\tau)$. The $\tau$ dependence represents the zero-point quantum motion of this vortex.

- The Bogoliubov quasiparticles are represented at low energies by the (second-quantized) Dirac field $\Psi(r, \tau)$. 
A single vortex in a $d$-wave superconductor.

Effective low energy action

After the Franz-Tesanovic gauge transformation, this vortex appears as a $\pi$ flux tube to the fermionic quasiparticles. The complete low energy theory for the vortex and the fermionic “Dirac” quasiparticles is then

$$Z = \int \mathcal{D}\Psi(r, \tau) \mathcal{D}r_v(\tau) \exp(-S)$$

$$S = \int d^2r d\tau \bar{\Psi} \gamma^\mu (\partial_\mu - ia_\mu) \Psi$$

$$+ \text{ additional terms from the “Doppler shift”}$$

where

$$\vec{\nabla} \times \dot{\vec{a}} = \pi \delta(\vec{r} - \vec{r}_v(\tau))$$
A single vortex in a $d$-wave superconductor.

Effective low energy action

After the Franz-Tesanovic gauge transformation, this vortex appears as a $\pi$ flux tube to the fermionic quasiparticles. The complete low energy theory for the vortex and the fermionic “Dirac” quasiparticles is then

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\mathcal{Z} = \int \mathcal{D}\Psi(r, \tau) \mathcal{D}r_v(\tau) \exp(-\mathcal{S})
$$

$$
\mathcal{S} = \int d^2r d\tau \bar{\Psi} \gamma^\mu (\partial_\mu - ia_\mu) \Psi
$$

+ additional terms from the “Doppler shift”

where

$$
\vec{\nabla} \times \vec{a} = \pi \delta(r - r_v(\tau))
$$

Note: The action is has no coupling constants, and much can be deduced simply by a $z = 1$ scaling analysis.
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II. Influence of electronic quasiparticles on vortex motion
Integrate out the nodal quasiparticles and expand the resulting action in powers of $d\mathbf{r}_v/d\tau$. This gives a result of the form

$$S [\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 K(\omega)$$

Translational invariance implies $K(0) = 0$. The scaling dimension of $K(\omega)$ is 3, and this allows us to deduce its functional form.
Integrate out the nodal quasiparticles and expand the resulting action in powers of $d\mathbf{r}_v/d\tau$. We obtained:

$$S[\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 \left[ \frac{m_v\omega^2}{2} + C_1|\omega|^3 + C_2T^2|\omega| \right] + \ldots$$
Integrate out the nodal quasiparticles and expand the resulting action in powers of $d\mathbf{r}_v/d\tau$. We obtained:

$$S[\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 \left[ \frac{m_v\omega^2}{2} + C_1|\omega|^3 + C_2T^2|\omega| \right] + \ldots$$

A finite effective mass

$$m_v \sim \frac{\Lambda}{v_F^2}$$

where $\Lambda \sim \Delta$ is a high energy cutoff.

By power-counting, there are no infra-red singularities to this order, and hence only an analytic dependence on $\omega$ is possible.
Integrate out the nodal quasiparticles and expand the resulting action in powers of $d\mathbf{r}_v/d\tau$. We obtained:

$$S[\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 \left[ \frac{m_v \omega^2}{2} + C_1 |\omega|^3 + C_2 T^2 |\omega| \right] + \ldots$$

sub-Ohmic damping with

$$C_1 = v_F^{-2} \times \left( \text{Universal function of } \frac{v_\Delta}{v_F} \right)$$
Integrate out the nodal quasiparticles and expand the resulting action in powers of $d\mathbf{r}_v/d\tau$. We obtained:

$$\mathcal{S}[\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 \left[ \frac{m_v \omega^2}{2} + C_1 |\omega|^3 + C_2 T^2 |\omega| \right] + \ldots$$

**Bardeen-Stephen viscous drag with**

$$C_2 = v_F^{-2} \times \left( \text{Universal function of } \frac{\nu_\Delta}{v_F} \right)$$
Integrate out the nodal quasiparticles and expand the resulting action in powers of \( d\mathbf{r}_v/d\tau \). We obtained:

\[
S[\mathbf{r}_v(\tau)] = \int \frac{d\omega}{2\pi} |\mathbf{r}_v(\omega)|^2 \left[ \frac{m_v\omega^2}{2} + C_1|\omega|^3 + C_2T^2|\omega| \right] + \ldots
\]

**Bardeen-Stephen viscous drag with**

\[
C_2 = v_F^{-2} \times \left( \frac{\nu}{v_F} \right)
\]

**Negligible damping of vortex from nodal quasiparticles at \( T=0 \). Damping increases as \( T^2 \) at higher \( T \)**
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III. Influence of vortex motion on electronic quasiparticles
A single vortex in a $d$-wave superconductor.
Effective low energy action for electronic quasiparticles

Add a harmonic pinning potential to the vortex, and ignore damping of vortex motion

\[
\mathcal{Z} = \int \mathcal{D}\Psi(\mathbf{r}, \tau)\mathcal{D}\mathbf{r}_v(\tau) \exp(-\mathcal{S})
\]

\[
\mathcal{S} = \int d^2r d\tau \bar{\Psi} \gamma^\mu (\partial_\mu - ia_\mu) \Psi
\]

\[
+ \frac{1}{2} m_v \left( \frac{d\mathbf{r}_v}{d\tau} \right)^2 + \frac{1}{2} m_v \omega_v^2 \mathbf{r}_v^2
\]

where

\[
\vec{\nabla} \times \vec{a} = \pi \delta(\mathbf{r} - \mathbf{r}_v(\tau))
\]

Now integrate out $\mathbf{r}_v$ and determine change in electronic LDOS.
Influence of the quantum oscillating vortex on the LDOS

$\omega / \omega_v$

$\alpha^2 = \frac{m v_F^2}{\omega_v} = 1$

Resonant feature near the vortex oscillation frequency and no zero-bias peak
Influence of the quantum oscillating vortex on the LDOS

Resonant feature near the vortex oscillation frequency and no zero-bias peak

\[ \frac{\omega}{\omega_v} \]

\[ \alpha^2 = \frac{mv_F^2}{\omega_v} = 1 \]

Influence of the quantum oscillating vortex on the LDOS

Resonant feature near the vortex oscillation frequency and no zero-bias peak


Is there an independent way to determine \( m_v \) and \( \omega_v \)?
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IV. Aharonov-Bohm phases in vortex motion and “checkerboard” modulations in LDOS


In ordinary fluids, vortices experience the Magnus Force

\[ F_M = (\text{mass density of air}) \cdot g(\text{velocity of ball}) \cdot g(\text{circulation}) \]
For a vortex in a superfluid, this is

\[ \mathbf{F}_M = (m \rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right) \]

\[ = n \hbar \rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \]

where \( \rho \) = number density of bosons

\( \mathbf{v}_s \) = local velocity of superfluid

\( \mathbf{r}_v \) = position of vortex
For a vortex in a superfluid, this is

\[
F_M = (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \int \mathbf{v}_s \cdot d\mathbf{r} \right)
\]

\[
= n\hbar\rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z}
\]

\[
= n \left( \mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right)
\]

where \( \mathbf{E} = \rho\mathbf{v}_s \times \hat{z} \) and \( \mathbf{B} = -\hbar\rho\hat{z} \)

**Dual picture:**

The vortex is a quantum particle with dual “electric” charge \( n \), moving in a dual “magnetic” field of strength \( = \hbar \times \) (number density of Bose particles)

Let the Hamiltonian of a single vortex be $\mathcal{H}_v$.

In general, this is a very complicated object, but we can obtain all needed information by symmetry considerations.

The Hamiltonian $\mathcal{H}_v$ should commute with $T_x$, the operator which translates the square lattice by one site in the $x$ direction (and similarly for $T_y$):

\[
[T_x, \mathcal{H}_v] = 0 \\
[T_y, \mathcal{H}_v] = 0
\]
However, $T_x$ and $T_y$ do not commute with each other.

Under translation along a distance $\mathbf{s}$, a vortex picks up a Aharanov-Bohm phase factor $\exp\left(i \int_0^s d\mathbf{r} \cdot \mathbf{A}\right)$.

Consequently

$$T_x T_y = \exp\left(i\phi\right) T_y T_x$$

where $\phi$ is the dual “flux” through a unit cell, This “flux” has the value

$$\phi = 2\pi f$$

where $f$ is the filling fraction of bosons (Cooper pairs). We will consider the case of rational filling fraction $f = p/q$, where $p$, $q$ are relatively prime integers.
Bosons on the square lattice at filling fraction $f=p/q$

$$[T_x, \mathcal{H}_v] = 0$$

$$[T_y, \mathcal{H}_v] = 0$$

$$T_x T_y = \exp \left(2\pi i p/q \right) T_y T_x$$
Bosons on the square lattice at filling fraction \( f = p/q \)

\[
\begin{align*}
[T_x, \mathcal{H}_v] &= 0 \\
[T_y, \mathcal{H}_v] &= 0 \\
T_x T_y &= \exp\left(2\pi i p/q\right) T_y T_x
\end{align*}
\]

Theorem:
The ground state of \( \mathcal{H}_v \) is at least \( q \)-fold degenerate. We can choose a basis, \( |m\rangle \) (\( m = 0 \ldots (q - 1) \)), for the ground states such that

\[
\begin{align*}
T_x |m\rangle &= |m + 1\rangle \\
T_y |m\rangle &= e^{2\pi i m p/q} |m\rangle
\end{align*}
\]
Properties of a quantum-fluctuating vortex weakly pinned by an impurity.

- Any impurity breaks translational invariance, and so chooses a preferred orientation in vortex “flavor space”. This chooses some linear combination among the ground states: \( |G\rangle = \sum_{m=0}^{q-1} c_m |m\rangle \)
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- Any impurity breaks translational invariance, and so chooses a preferred orientation in vortex “flavor space”. This chooses some linear combination among the ground states: \( |G\rangle = \sum_{m=0}^{q-1} c_m |m\rangle \)

- The expectation value of any observable \( \mathcal{O} \), \( \langle G|\mathcal{O}|G\rangle \) can be related to the matrix of overlaps \( \langle m|n\rangle \) which, in turn, are linearly related to quantities \( \rho_{mn} \) which transform under \( T_x, T_y \) like the Fourier components of a density \( \rho_Q \) at the wavevectors \( Q = 2\pi f(m, n) \):

\[
T_x : \rho_Q \rightarrow e^{iQ \cdot \hat{x}} \rho_Q \quad T_y : \rho_Q \rightarrow e^{iQ \cdot \hat{y}} \rho_Q
\]
Properties of a quantum-fluctuating vortex weakly pinned by an impurity.

- Any impurity breaks translational invariance, and so chooses a prefered orientation in vortex “flavor space”. This chooses some linear combination among the ground states: $|G\rangle = \sum_{m=0}^{q-1} c_m |m\rangle$

- The expectation value of any observable $\mathcal{O}$, $\langle G|\mathcal{O}|G\rangle$, can be written in general, and is determined by

$$\rho_{mn} = e^{i\pi mnp/q} \sum_{\ell=0}^{q-1} e^{2\pi i\ell m} \langle \ell | \ell + n \rangle$$

$$T_x : \rho_Q \rightarrow e^{iQ \cdot \hat{x}} \rho_Q$$

$$T_y : \rho_Q \rightarrow e^{iQ \cdot \hat{y}} \rho_Q$$
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T_x : \rho_Q \rightarrow e^{iQ \cdot \hat{x}} \rho_Q \quad T_y : \rho_Q \rightarrow e^{iQ \cdot \hat{y}} \rho_Q
\]

- It can be shown that there is no linear combination \( |G\rangle \) for which all the \( \rho_{mn} \) are zero.
Properties of a quantum-fluctuating vortex weakly pinned by an impurity.

- Any pinned vortex exhibits modulations in “density”-like observables at the wavevectors $Q$ over the region in which the vortex executes its quantum zero-point motion.
Vortex-induced LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ integrated from 1meV to 12meV at 4K

Vortices have halos with LDOS modulations at a period $\approx 4$ lattice spacings


Using as input (i) the size of the “checkerboard halo” in STM as a measure of the zero-point motion radius of the vortex, and (ii) the forces between the vortices as determined from an estimate of the superfluid stiffness, we obtain as output an estimate of \( m_v \approx 2 - 9m_e \) and the vortex oscillation frequency \( \omega_v \approx 2 - 7 \text{ meV} \).
Influence of the quantum oscillating vortex on the LDOS

Resonant feature near the vortex oscillation frequency and no zero-bias peak.


Independent estimate of $\omega_v$, gives a consistency check.
Conclusions

• Evidence that vortices in the cuprate superconductors carry a “flavor” index which encodes the spatial modulations of a proximate insulator. Quantum zero point motion of the vortex provides a natural explanation for LDOS modulations observed in STM experiments.

• Size of modulation halo allows estimate of the inertial mass of a vortex

• Direct detection of vortex zero-point motion may be possible in inelastic neutron or light-scattering experiments

• The quantum zero-point motion of the vortices influences the spectrum of the electronic quasiparticles, in a manner consistent with LDOS spectrum

• “Aharanov-Bohm” or “Berry” phases lead to surprising kinematic duality relations between seemingly distinct orders. These phase factors allow for continuous quantum phase transitions in situations where such transitions are forbidden by Landau-Ginzburg-Wilson theory.