1. Z₂ spin liquid on the square lattice

- Start with the semiclassical ground state of the J₁-J₂-J₃ antiferromagnet on the square lattice which has planar spiral antiferromagnetic order at the wavevector (Q₀).
- Quantum fluctuations across a continuous phase transition lead to a spin liquid state with Z₂ topological order and long-range Ising-nematic order
- This state can be efficiently described by Schwinger boson mean field theory,

\[ H_{MF} = -\sum_{ij} (Q_{ij} c_{a\alpha} b_{i\alpha}^+ b_{j\beta} + h.c.) + \sum \lambda_i b_{i\alpha}^+ b_{i\alpha} \]

\[ \langle \Psi^b \rangle = C_P \text{exp} \left[ \sum_{ij} \xi_{ij} c_{a\alpha} b_{i\alpha}^+ b_{j\beta} \right] |0\rangle \]
- Excitations of the Z₂ spin liquid are:
  - (i) bosonic spinons \( z_0 \sim b_0^+ + b_1^+ \)
  - (ii) bosonic visons, which are Z₂ vortices in the \( Q_{ij} \).
- The bosonic spinons and visons are mutual semions.

2. From bosonic to fermionic spinons

Purely topological properties of Z₂ spin liquids:

- 4 kinds of excitations, \( e, m, \bar{e} \) and the trivial local excitation 1
- Have the following fusion rules:

\[
\begin{align*}
  e \times e &= m \times m = e \times \bar{e} = 1 \\
  1 \times 1 &= 1, e \times \bar{e} = e, m \times 1 = m, e \times e = e \\
  e \times m &= e \times \bar{e} = m \times e = e
\end{align*}
\]
- In the context of spin liquids, \( e \) and \( \bar{e} \) are bosonic and fermionic spinons, and \( m \) is the vison, 1 is a local excitation with integer spin

From the projective transformations of bosonic spinons (e particle) and the vison (the \( \bar{e} \) particle) under space-group symmetries of the antiferromagnet, we can determine the projective symmetry transformations of the fermionic spinons (\( e \) particle). Finally, we can determine the effective Hamiltonian of the fermionic spinons \( f_{ab} \).

Y.-M. Liu, G.Y. Cho, A. Vishwanath, 1403.0575

Symmetry relations for spin liquids on the rectangular lattice

<table>
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<th>Symmetry relation</th>
<th>Bosonic PSG</th>
<th>Fermionic PSG</th>
<th>Vison PSG</th>
<th>Unit factor</th>
<th>Relation</th>
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3. FL* metal from a Z₂ spin liquid

- Dopants in a FL* metal are fermions, \( e_{ab} \), with charge \( +e \) and spin \( S = 1/2 \) (the green dimers). So there need not be any low energy fractionalized excitations.
- The dopants form a Fermi surface of size equal to the dopant density \( \rho \).
- The emergent gauge excitations of the \( Z_2 \) spin liquid, i.e., visons, survive in the FL* metal. Note that the green and blue dimers have the same topological properties as the undoped dimer model.
- The violation of the Luttinger theorem in the FL* metal is justified by the presence of emergent gauge excitations (i.e., topological order).

Recent evidence for pseudogap metal as FL* in YBCO
Proest-Tailleur-Lebegue collaboration, Bedoux et al. arXiv:1511.08162

4. Confinement transition of a FL* metal

- Confinement can be induced by the condensation of the bosonic bilinear \( B \sim f_{ab} f_{bc} \) or \( e_{ab} e_{bc} \).
- This is a “Higgs” transition leading to confinement because \( B \) carries electric \( Z_2 \) charge.
- The \( B \)-condensed (Higgs) state is a superconductor because the pairing of the \( f_{ab} \) fermions in the \( Z_2 \) spin liquid now induces a pairing of the \( e_{ab} \) fermions. This pairing can have \( d_{x^2-y^2} + s \) symmetry.
- The \( e_{ab} \) fermions have trivial space group transformations, and so the projective space group transformations of \( B \) can be deduced from those of the fermionic spinons \( f_{ab} \).
- In many cases, the projective transformations of \( B \) also imply density-wave order in the superconductor. This implies there can be a direct second-order transition from the FL* metal to a confining Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state.