Fermi surfaces and gauge-gravity duality

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Lecture notes
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Compressible quantum matter

- Consider a continuum quantum system with a globally conserved U(1) charge $Q$ (the “electron density”) in spatial dimension $d > 1$. 

Tuesday, March 1, 2011
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There are only a few established examples of such phases in condensed matter physics. However, they appear naturally as duals of gravitational theories, and we want to interpret them in the gauge theory.
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All known examples of such phases have a **Fermi Surface**

(even in systems with only bosons in the Hamiltonian)
The Fermi surface

This is the locus of zero energy singularities in momentum space in the two-point correlator of fermions carrying charge $Q$.

$$G^{-1}_{\text{fermion}}(k = k_F, \omega = 0) = 0.$$  

**Luttinger relation:** The total “volume (area)” $A$ enclosed by Fermi surfaces of fermions carrying charge $Q$ is equal to $\langle Q \rangle$. This is a key constraint which allows extrapolation from weak to strong coupling.
Examples of compressible phases
and their Fermi surfaces

1. The Fermi liquid (FL)
2. Fermions coupled to gauge fields
3. Fermion-boson mixtures
4. The fractionalized Fermi liquid (FL*)
5. Theories similar to ABJM
6. Theories similar to $\mathcal{N} = 4$ SYM
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The Fermi Liquid (FL)

Most common example: electrons with short-range interactions (or screened long-range interactions), which are adiabatically connected to the non-interacting limit. The electron Green’s function $G_f$ has a pole which crosses zero energy at $k = k_F$, and the Fermi surface has the same area as the non-interacting case.

$$\mathcal{L} = \bar{f} (\partial_a - \mu \delta_{at}) \gamma^a f + 4 \text{ Fermi terms}$$
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$$\mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f + 4 \text{ Fermi terms}$$

$$\mathcal{A} = \langle f^\dagger f \rangle = \langle Q \rangle$$

$$G_f = \frac{1}{\omega - v_F(k - k_F) + i\omega^2}$$
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• Couple fermions to a dynamical gauge field $A_a$.

\[ \mathcal{L} = f^\dagger \left( \partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right) f \]
• Couple fermions to a dynamical gauge field $A_a$.

$$\mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - i\sigma A_\tau - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right) f_\sigma \quad ; \quad \sigma = \pm 1$$
• Couple fermions to a dynamical gauge field $A_\alpha$.

• Longitudinal gauge fluctuations are screened by the fermions. But transverse gauge fluctuations remain unscreened, and are Landau-damped by excitations near the Fermi surface. The theory of a Fermi surface coupled to transverse gauge fluctuations is strongly coupled in two spatial dimensions.


\[
\mathcal{L} = f^\dagger_\sigma \left( \partial_\tau - i\sigma A_\tau - \frac{(\nabla - i\sigma \mathbf{A})^2}{2m} - \mu \right) f_\sigma \quad ; \quad \sigma = \pm 1
\]
Couple fermions to a dynamical gauge field $A_a$.

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The overdamped transverse gauge modes lead to “non-Fermi liquid” broadening of the fermion pole near the Fermi surface.

$$\mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - i\sigma A_\tau - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right) f_\sigma ; \quad \sigma = \pm 1$$
\[ \mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]

\[ \mathcal{A} = \langle f^\dagger f \rangle = \langle \mathcal{Q} \rangle \]
The location of the Fermi surfaces is well defined, and the Luttinger relation applies as before.

Fluctuations near the Fermi surface are described by a strongly-coupled two-patch theory. Ward identities allow consistent matching of the patches, and patches along different directions decouple in the low energy limit.

The singularity of the Green's function upon approaching the Fermi surface is described by the scaling form

\[ G^{-1} f = q_1^{-\eta} F \left( \frac{\omega}{q \frac{z}{2}} \right) \]

where \( q_x = k_x - k_F \), \( q_y = k_y \), and \( q = q_x + q_y \frac{2}{2k_F} \) and \( \eta \) and \( z \) are anomalous exponents.

\[ 2A = \langle f_\sigma^\dagger f_\sigma \rangle = \langle Q \rangle \]

\[ \mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - i\sigma A_\tau - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right) f_\sigma ; \quad \sigma = \pm 1 \]
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• The singularity of the Green's function upon approaching the Fermi surface is described by the scaling form

\[ G_f^{-1} = q^{1-\eta} F\left(\frac{\omega}{q^z/2}\right) \]

where \( q_x = k_x - k_F \), \( q_y = k_y \), and \( q = q_x + \frac{q_y^2}{2k_F} \), and \( \eta \) and \( z \) are anomalous exponents.

\[ 2\mathcal{A} = \langle f_\sigma^\dagger f_\sigma \rangle = \langle Q \rangle \]

\[ \mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - i\sigma A_\tau - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right) f_\sigma \; ; \; \sigma = \pm 1 \]

Examples of compressible phases and their Fermi surfaces

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Consider mixture of fermions $f$ and bosons $b$. 

\[ \mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \\
+ b^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m_b} - \mu_b \right) b + s|b|^2 + -gb^\dagger f^\dagger fb + \ldots \]
Consider mixture of fermions $f$ and bosons $b$. There is a $U(1) \times U_b(1)$ symmetry and 2 conserved charges:

\[
Q = f^\dagger f \\
Q_b = b^\dagger b
\]

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\]
Consider mixture of fermions $f$ and bosons $b$. There is a $U(1) \times U_b(1)$ symmetry and 2 conserved charges:

The 2 symmetries imply 2 Luttinger constraints. However, bosons at non-zero density invariably Bose condense at $T = 0$, and so $U_b(1)$ is broken. So there is only the single constraint on the $f$ Fermi surface. This describes mixtures of $^3$He and $^4$He.

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\]
Consider mixture of fermions \( f \) and bosons \( b \). There is a \( \text{U}(1) \times \text{U}_b(1) \) symmetry and 2 conserved charges:

\[
A = \langle Q \rangle
\]

\[
Q = f^\dagger f
\]
\[
Q_b = b^\dagger b
\]

Superfluid: \( \langle b \rangle \neq 0 \)
\( \text{U}_b(1) \) broken; \( \text{U}(1) \) unbroken

\[
\mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f
\]
\[
+ \ b^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m_b} - \mu_b \right) b + s|b|^2 + -gb^\dagger f^\dagger fb + \ldots
\]
Increase the coupling $g$ until the boson, $b$, and fermion, $f$, can bind into a ‘molecule’, the fermion $c$.

\[ \mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]
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\[ Q = f^\dagger f \]
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Increase the coupling $g$ until the boson, $b$, and fermion, $f$, can bind into a ‘molecule’, the fermion $c$. Decouple the interaction between $b$ and $f$ by a fermion $c$

\[
\mathcal{L} = f\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \\
+ b\dagger \left( \partial_\tau - \frac{\nabla^2}{2m_b} - \mu_b \right) b + s|b|^2 + \frac{1}{g} c\dagger c - c\dagger f b - c b\dagger f\dagger + \ldots
\]
In a phase with $U_b(1)$ unbroken, there is a Luttinger relation for each conserved $U(1)$ charge. However, the boson, $b$ cannot have a Fermi surface in its Green’s function, and so there is no area associated with it, although the boson density is included in the Luttinger relation.

\[
\mathcal{A}_c + \mathcal{A}_f = \langle f^\dagger f \rangle = \langle Q \rangle
\]
\[
\mathcal{A}_c = \langle b^\dagger b \rangle = \langle Q_b \rangle
\]

The $b$ bosons have bound with $f$ fermions to form $c$ “molecules”

Phase diagram of boson-fermion mixture

\[ \mathcal{A} = \langle Q \rangle \]

Superfluid: \( \langle b \rangle \neq 0 \)
U_b(1) broken; U(1) unbroken

\[ \mathcal{A}_c = \langle Q_b \rangle \]

Normal: \( \langle b \rangle = 0 \)
U(1)\( \times \)U_b(1) unbroken

\[ \mathcal{A}_f = \langle Q - Q_b \rangle \]

\[ \mathcal{L} = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]

\[ + b^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m_b} - \mu_b \right) b + s|b|^2 + -gb^\dagger f^\dagger fb + \ldots \]
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• Now gauge $Q - Q_b$ by a dynamic gauge field $A_a$. This leaves fermion $c$ gauge-invariant

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\[ \mathcal{L} = f^\dagger \left( \partial_\tau - iA_\tau - \frac{(\nabla - iA)^2}{2m} - \mu \right) f + \ b^\dagger \left( \partial_\tau + iA_\tau - \frac{(\nabla + iA)^2}{2m_b} - \mu_b \right) b + s|b|^2 + -gb^\dagger f^\dagger fb + \ldots \]
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\[ L = f^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f \]

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Phase diagram of $\text{U}(1)$ gauge theory

$A_c = \langle Q_b \rangle$

$A_f = \langle Q - Q_b \rangle$

Higgs/confining phase:
Fermi liquid (FL)

Deconfined phase:
Fractionalized Fermi liquid (FL*)

$\mathcal{L} = f^\dagger \left( \partial_\tau - i A_\tau - \frac{(\nabla - i A)^2}{2m} - \mu \right) f$

$+ \ b^\dagger \left( \partial_\tau + i A_\tau - \frac{(\nabla + i A)^2}{2m_b} - \mu_b \right) b + s |b|^2 + - g b^\dagger f^\dagger f b + \ldots$
Phase diagram of $U(1)$ gauge theory

Higgs/confining phase: Fermi liquid (FL)

Deconfined phase: Fractionalized Fermi liquid (FL*)

\[ \mathcal{L} = \bar{f} \left( \partial_\tau - i A_\tau - \frac{(\nabla - iA)^2}{2m} - \mu \right) f + \bar{b} \left( \partial_\tau + i A_\tau - \frac{(\nabla + iA)^2}{2m_b} - \mu_b \right) b + s|b|^2 + -\frac{g}{2}b^\dagger f^\dagger fb + \ldots \]
Phase diagram of U(1) gauge theory

- **FL phase**: Fermi surface of gauge-neutral fermions encloses total global charge $Q$

- **FL* phase**: Fermi surface of gauge-neutral fermions encloses only part of the global charge $Q$

**Higgs/confining phase:**
Fermi liquid (FL)

**Fractionalized Fermi liquid (FL*)**

\[
\mathcal{L} = f^+ \left( \partial_\tau - iA_\tau - \frac{(\nabla - iA)^2}{2m} - \mu \right) f + b^+ \left( \partial_\tau + iA_\tau - \frac{(\nabla + iA)^2}{2m_b} - \mu_b \right) b + s|b|^2 + -gb^+f^+fb + \ldots
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ABJM theory in D=2+1 dimensions

- $4N^2$ Weyl fermions carrying fundamental charges of $U(N) \times U(N) \times SU(4)_R$.
- $4N^2$ complex bosons carrying fundamental charges of $U(N) \times U(N) \times SU(4)_R$.
- $\mathcal{N} = 6$ supersymmetry
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- $\mathcal{N} = 6$ supersymmetry

Adding a chemical potential coupling to a SU(4) charge breaks supersymmetry and SU(4) invariance
• U(1) gauge invariance and U(1) global symmetry

• Fermions, $f_+$ and $f_-$, carry U(1) gauge charges $\pm 1$, and global U(1) charge 1.

• Bosons, $b_+$ and $b_-$, carry U(1) gauge charges $\pm 1$, and global U(1) charge 1.

• No supersymmetry
Theory similar to ABJM

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• No supersymmetry

• Fermions, $c$, gauge-invariant bound states of fermions and bosons carrying global U(1) charge 2.
Theory similar to ABJM

\[ \mathcal{L} = f^{\dagger}_{\sigma} \left[ (\partial_\tau - i\sigma A_\tau) - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right] f_\sigma \\
+ b^{\dagger}_{\sigma} \left[ (\partial_\tau - i\sigma A_\tau) - \frac{(\nabla - i\sigma A)^2}{2m_b} + \epsilon_1 - \mu \right] b_\sigma \\
+ \frac{u}{2} (b^{\dagger}_{\sigma} b_\sigma)^2 - g_1 \left( b^{\dagger}_+ b^{\dagger}_- f_- f_+ + \text{H.c.} \right) \]

The index \( \sigma = \pm 1 \)
Theory similar to ABJM

\[ \mathcal{L} = f^\dagger_\sigma \left[ (\partial_\tau - i\sigma A_\tau) - \frac{(\nabla - i\sigma A)^2}{2m} - \mu \right] f_\sigma \]

\[ + b^\dagger_\sigma \left[ (\partial_\tau - i\sigma A_\tau) - \frac{(\nabla - i\sigma A)^2}{2m_b} + \epsilon_1 - \mu \right] b_\sigma \]

\[ + \frac{u}{2} (b^\dagger_\sigma b_\sigma)^2 - g_1 \left( b^\dagger_+ b^\dagger_- f_- f_+ + \text{H.c.} \right) \]

\[ + c^\dagger \left[ \partial_\tau - \frac{\nabla^2}{2m_c} + \epsilon_2 - 2\mu \right] c \]

\[ - g_2 \left[ c^\dagger (f_+ b_- + f_- b_+) + \text{H.c.} \right] \]

The index \( \sigma = \pm 1, \) and \( \epsilon_1,2 \) are tuning parameters of phase diagram

\[ Q = f^\dagger_\sigma f_\sigma + b^\dagger_\sigma b_\sigma + 2c^\dagger c \]
Phases of ABJM-like theories

\[ \langle b_\pm \rangle = 0 \]

\[ 2A_c = \langle Q \rangle \]

Fermi liquid (FL) of gauge-neutral particles

U(1) gauge theory is in confining phase
Phases of ABJM-like theories

\[ \langle b_\pm \rangle = 0 \]

\[ 2A_c + 2A_f = \langle Q \rangle \]

Fractionalized Fermi liquid (FL*)

U(1) gauge theory is in deconfined phase
Phases of ABJM-like theories

\[ \langle b_\pm \rangle \neq 0 \]
\[ \langle b_+ b_- \rangle \neq 0 \]

No constraint on Fermi surface area, which can be zero

Superconductor
U(1) gauge theory is in Higgs phase and global U(1) is broken
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\[ \mathcal{N} = 4 \text{ SYM in } D=3+1 \text{ dimensions} \]

- SU(\(N\)) gauge invariance and SO(6) global symmetry

- Fermions carry adjoint gauge charges and are SO(6) spinors

- Bosons carry adjoint gauge charges and are SO(6) fundamentals. Bosons are paired fermions.

- \( \mathcal{N} = 4 \) supersymmetry
\[ \mathcal{N} = 4 \text{ SYM in D}=3+1 \text{ dimensions} \]

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Adding a chemical potential coupling to a SO(6) charge breaks supersymmetry and SO(6) invariance
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

- SU($N$) gauge invariance and U(1) global symmetry
- Fermions, $f_\alpha$, ($\alpha = 1 \ldots N^2 - 1$) carry adjoint gauge charges and U(1) charge 1.
- Bosons, $b_\alpha$, carry adjoint gauge charges and U(1) charge 2. Bosons are paired $f_\alpha$ fermions.
- No supersymmetry
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

- **SU($N$) gauge invariance and U(1) global symmetry**

- **Fermions**, $f_\alpha$, ($\alpha = 1 \ldots N^2 - 1$) carry adjoint gauge charges and U(1) charge 1.

- **Bosons**, $b_\alpha$, carry adjoint gauge charges and U(1) charge 2. Bosons are paired $f_\alpha$ fermions.

- **No supersymmetry**

- **Fermions**, $c$, (analog of baryons), gauge-invariant bound states of $b$ and $f$, carry U(1) charge 3.

\[
Q = f_\alpha^\dagger f_\alpha + 2b_\alpha^\dagger b_\alpha + 3c^\dagger c
\]
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

\[ H_f = \sum_{k, \alpha} \frac{k^2}{2m_3} f^\dagger_\alpha f_\alpha - \mu \sum_k \left( \sum_\alpha f^\dagger_\alpha f_\alpha + 2 \sum_\alpha b^\dagger_\alpha b_\alpha + 3c^\dagger c \right) \]

\[ Q = f^\dagger_\alpha f_\alpha + 2b^\dagger_\alpha b_\alpha + 3c^\dagger c \]
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

\[ H_f = \sum_{k,\alpha} \frac{k^2}{2m_3} f^\dagger_\alpha f_\alpha - \mu \sum_k \left( \sum_{\alpha} f^\dagger_\alpha f_\alpha + 2 \sum_{\alpha} b^\dagger_\alpha b_\alpha + 3c^\dagger c \right) \]

\[ H_b = \sum_{k,\alpha} \left( \frac{k^2}{2m_1} + \varepsilon_1 \right) b^\dagger_\alpha b_\alpha + u \int d^d x \left( b^\dagger_\alpha b_\alpha \right)^2 \]

\[ Q = f^\dagger_\alpha f_\alpha + 2b^\dagger_\alpha b_\alpha + 3c^\dagger c \]
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

\[
H_f = \sum_{k,\alpha} \frac{k^2}{2m_3} f^\dagger_\alpha f_\alpha - \mu \sum_k \left( \sum_\alpha f^\dagger_\alpha f_\alpha + 2 \sum_\alpha b^\dagger_\alpha b_\alpha + 3 c^\dagger c \right)
\]

\[
H_b = \sum_{k,\alpha} \left( \frac{k^2}{2m_1} + \varepsilon_1 \right) b^\dagger_\alpha b_\alpha + u \int d^d x \left( b^\dagger_\alpha b_\alpha \right)^2
\]

\[
H_c = \sum_k \left( \frac{k^2}{2m_2} + \varepsilon_2 \right) c^\dagger c
\]

\[
Q = f^\dagger_\alpha f_\alpha + 2 b^\dagger_\alpha b_\alpha + 3 c^\dagger c
\]
Theory similar to $\mathcal{N} = 4$ SYM in a chemical potential

\[ H_f = \sum_{k,\alpha} \frac{k^2}{2m_3} f_\alpha^\dagger f_\alpha - \mu \sum_k \left( \sum_\alpha f_\alpha^\dagger f_\alpha + 2 \sum_\alpha b_\alpha^\dagger b_\alpha + 3 c^\dagger c \right) \]

\[ H_b = \sum_{k,\alpha} \left( \frac{k^2}{2m_1} + \varepsilon_1 \right) b_\alpha^\dagger b_\alpha + u \int d^d x (b_\alpha^\dagger b_\alpha)^2 \]

\[ H_c = \sum_k \left( \frac{k^2}{2m_2} + \varepsilon_2 \right) c^\dagger c \]

\[ H_{\text{int}} = g \int d^d x (\epsilon_{\alpha\beta\gamma} b_\alpha^\dagger f_\beta f_\gamma + \text{c.c.}) + \lambda \int d^d x (c^\dagger b_\alpha f_\alpha + \text{c.c.}) \]

The indices, $\alpha, \beta, \gamma = 1 \ldots N^2 - 1$, the structure constants of SU($N$) are $\epsilon_{\alpha\beta\gamma}$, and $\varepsilon_{1,2}$ are parameters tuning between possible phases. The SU($N$) gauge fields are not shown, and are included as usual by covariantizing derivatives.

\[ Q = f_\alpha^\dagger f_\alpha + 2 b_\alpha^\dagger b_\alpha + 3 c^\dagger c \]
Phases of SYM-like theories

\[ \langle b_\alpha \rangle = 0 \]

\[ 3A_c = \langle Q \rangle \]

Fermi liquid (FL) of baryon-like particles

SU(\(N\)) gauge theory is in confining phase
Phases of SYM-like theories

\[ \langle b_\alpha \rangle = 0 \]

\[ 3A_c + (N^2 - 1)A_f = \langle Q \rangle \]

Fractionalized Fermi liquid (FL*)

SU(N) gauge theory is in deconfined phase
Phases of SYM-like theories

\[ \langle b_\alpha \rangle \neq 0 \]

\[ A_c \]

\[ A_f \]

No constraint on Fermi surface areas

Color Superconductor

SU(\(N\)) gauge theory is in Higgs phase
Phases of SYM-like theories

\[ \langle b_\alpha \rangle = 0 \]

\[ 3A_c + (N^2 - 1)A_f = \langle Q \rangle \]

Fractionalized Fermi liquid (FL*)

SU(N) gauge theory is in deconfined phase
Phases of ABJM-like theories

\[ \langle b_\pm \rangle = 0 \]

\[ 2A_c + 2A_f = \langle Q \rangle \]

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Fractionalized Fermi liquid (FL*)

$\langle b_\pm \rangle = 0$

$2A_c + 2A_f = \langle Q \rangle$

U(1) gauge theory is in deconfined phase

Claim: this is the phase underlying recent holographic theories of compressible metallic states.

However, a number of artifacts appear in the classical gravity approximation.
Gauge-gravity duality
and
impurity mean-field theories
Begin with a CFT e.g. the SYM theory with a SO(6) global symmetry

The CFT is dual to a gravity theory on AdS$_5 \times S^5$
Begin with a CFT e.g. the SYM theory with a SO(6) global symmetry
Add some SO(6) charge by turning on a chemical potential (this breaks the SO(6) symmetry)

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In the Einstein-Maxwell theory, the chemical potential leads at T=0 to an extremal Reissner-Nordstrom black hole in the AdS$_5$ spacetime.
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In the Einstein-Maxwell theory, the chemical potential leads at T=0 to an extremal Reissner-Nordstrom black hole in the AdS$_5$ spacetime.
The near-horizon geometry of the RN black hole is AdS$_2 \times \mathbb{R}^3$. This factorization leads to finite ground state entropy density
AdS theory of finite density quantum matter

Features of AdS Einstein-Maxwell theory of non-zero density quantum matter, not expected in the final theory:

- Non-zero ground state entropy density.
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Features of AdS Einstein-Maxwell theory of non-zero density quantum matter, not expected in the final theory:

- Non-zero ground state entropy density.
- Single fermion self energies are momentum independent, and their singular behavior is the same on and off the Fermi surface.
- Low energy singularities are described by “conformal quantum mechanics”: a 0+1 dimensional defect in a d+1 dimensional CFT. This is linked to the factorization of the near-horizon metric to $\text{AdS}_2 \times \mathbb{R}^d$. 

Tuesday, March 1, 2011
Solution of lattice models

Place U(1) gauge theory theory on a lattice, integrate out $b$ and $A_a$, to obtain Kondo lattice Hamiltonian

$$H = -\sum_{i<j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + \sum_{i<j} J_H(i, j) \vec{S}_i \cdot \vec{S}_j + J_K \sum_i \vec{S}_i \cdot c^\dagger_{i\alpha} \sigma_{\alpha\beta} c_{i\beta}$$

where $\vec{S}_i = f^\dagger_{i\alpha} \sigma_{\alpha\beta} f_{\beta}$

Solution of lattice models

\[ \mathcal{L} = \mathcal{L}_{\text{imp}}[c_0, f_0] + c_0^\dagger F_{\text{bulk}} + F_{\text{bulk}}^\dagger c_0 + \mathcal{L}_{\text{bulk}} \]

Has to be combined with a \textit{self-consistency condition} between correlators on the impurity and the bulk.

Obtain both FL and FL* phases;

**properties of the FL* phase:**

- The ground state has a non-zero entropy density

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- The correlations of $F_{\text{bulk}}$ are local ($z = \infty$)
- The correlations $F_{\text{bulk}}$ in time have a conformal structure with scaling dimension $\Delta$ (as in the boundary of AdS$_2$)
- Imposition of the self-consistency condition between impurity and boundary yields the scaling dimension $\Delta = 1$, the ‘marginal Fermi liquid’ value.

Obtain both FL and FL* phases;

properties of the FL* phase:

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These features, and the resulting fermion correla-
tor and transport properties, co-incide with those
obtained (for general $\Delta$) using the holographic
$\text{AdS}_2 \times \mathbb{R}^d$ theory defined on the extremal horizon
of the Reissner-Nordstrom black hole (T. Faulkner,


between impurity and boundary yields the
scaling dimension $\Delta = 1$, the ‘marginal Fermi
liquid’ value.

Compressible quantum matter is characterized by Fermi surfaces.

Fermi surfaces can be removed from the Luttinger count if the fermions acquire gauge charges.

Phases of a strongly-coupled gauge theory: Fermi liquids (FL) and fractionalized Fermi liquids (FL*)
Mean field Kondo lattice models capture the physics of holographic metals with a AdS$_2 \times R^d$ geometry

Needed: Holographic theory for FL$^*$ or related compressible phases, without a factorized geometry. Challenge: detect Fermi surfaces of fermions with gauge charges