Quantum criticality of metals and high temperature superconductivity

50th Karpacz Winter School of Theoretical Physics
Quantum Criticality in Condensed Matter: Phenomena, Materials and Ideas in Theory and Experiment
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Talk online: sachdev.physics.harvard.edu
High temperature superconductors

$\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$
Smaller hole Fermi-pockets

K.M. Shen et al., Science 2005

Large hole Fermi surface

M. Platé et al., PRL 2005
Direct observation of competition between superconductivity and charge density wave order in YBa$_2$Cu$_3$O$_{6.67}$

J. Chang$^{1,2}$*, E. Blackburn$^3$, A. T. Holmes$^3$, N. B. Christensen$^4$, J. Larsen$^{4,5}$, J. Mesot$^{1,2}$, Ruixing Liang$^{6,7}$, D. A. Bonn$^{6,7}$, W. N. Hardy$^{6,7}$, A. Watenphul$^8$, M. v. Zimmermann$^8$, E. M. Forgan$^3$ and S. M. Hayden$^9$

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**Figure 1**

(a) Intensity (cnts s$^{-1}$) against $T$ (K) for different magnetic fields.

- $Q = (1.695, 0, 0.5)$
- $Q = (0, 3.691, 0.5)$ (x4)

(b) Temperature (K) against doping ($\rho$ (holes/Cu)) for YBCO.

- $T_N$, $T_H$, $T_{CDW}$, $T_{SDW}$, $T_{NMR}$

(c) Intensity (cnts s$^{-1}$) against $h$ (r.l.u.) for different $T$.

- $T = 2$ K
- $T = 66$ K
- $T = 150$ K

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**Figure 2**

Illustrates the magnetic field dependence of charge density wave (CDW) and superconductivity.

- $T_C$, $T_{CDW}$
- $Q = (1.695, 0, 0.5)$
- $Q = (0, 3.691, 0.5)$ (x4)
Strange Metal

K.M. Shen et al., Science 2005

M. Platé et al., PRL 2005

Small hole Fermi-pockets

Large hole Fermi surface
Magnetic-field-induced charge-stripe order in the high-temperature superconductor YBa$_2$Cu$_3$O$_y$

Tao Wu$^1$, Hadrien Mayaffre$^1$, Steffen Krämer$^1$, Mladen Horvatić$^1$, Claude Berthier$^1$, W. N. Hardy$^{2,3}$, Ruixing Liang$^{2,3}$, D. A. Bonn$^{2,3}$ & Marc-Henri Julien$^f$

8 September 2011 | Vol 477 | Nature | 191

![Diagram](image.png)

**Figure 3**

- **Phase diagram of underdoped YBa$_2$Cu$_3$O$_y$**

- **Temperature** ($T$) vs. **doping** ($p$) with regions marked for **Spin order**, **Field-induced charge order**, and **Superconducting**

- **Color Coding**:
  - Green: Spin order
  - Red: Field-induced charge order
  - Gray: Superconducting

- **Crystal Structure**
  - **Cu$^{2+}$**, **Cu$^{3+}$**
  - **O$^{2-}$**
  - **Y$^{3+}$**
  - **Ba$^{2+}$**

- **Lattice Parameters**
  - $c = 11.6802$ Å
  - $a = 3.8227$ Å
  - $b = 3.8872$ Å

- **Additional Information**
  - The discovery of our work is that, on cooling in a field, magnetic order in field-dependent and temperature-dependent orbital occupancy (for $p < 0.108$, where the magnetic order is accompanied by a crossover of the time decay occurring at a slightly higher ($\sim 0.04$ T) and slightly lower than ($\sim 0.06$ T) remaining magnetic order). This is fully consistent with the remarkable similarity of transport data in high-$T_c$ superconductors (an extremely clean copper oxide in which charge modulation is unlikely: to the best of our knowledge, no such modulation has ever been observed).

- **Discussion**
  - The main result of our work is that, on cooling in a field, magnetic order in field-dependent and temperature-dependent orbital occupancy (for $p < 0.108$, where the magnetic order is accompanied by a crossover of the time decay occurring at a slightly higher ($\sim 0.04$ T) and slightly lower than ($\sim 0.06$ T) remaining magnetic order). This is fully consistent with the remarkable similarity of transport data in high-$T_c$ superconductors (an extremely clean copper oxide in which charge modulation is unlikely: to the best of our knowledge, no such modulation has ever been observed).

- **Conclusion**
  - The discovery of our work is that, on cooling in a field, magnetic order in field-dependent and temperature-dependent orbital occupancy (for $p < 0.108$, where the magnetic order is accompanied by a crossover of the time decay occurring at a slightly higher ($\sim 0.04$ T) and slightly lower than ($\sim 0.06$ T) remaining magnetic order). This is fully consistent with the remarkable similarity of transport data in high-$T_c$ superconductors (an extremely clean copper oxide in which charge modulation is unlikely: to the best of our knowledge, no such modulation has ever been observed).
J. D. Sau and S. Sachdev, Physical Review B 89, 075129 (2014)
The absence of any intensity loss at low temperatures also rules out the presence of magnetic order with any significant moment. Error bars represent the added uncertainty.

The magnetic transition temperature probably signifies that the condition is always accompanied by a crossover of the time decay from the high-temperature Gaussian form to an exponential form \( \exp(-t/T) \). The magnetic transition temperature is already present at 15 T, whereas no line splitting is detected at this field. The field therefore affects the spin fluctuations quantitatively but not qualitatively.

The increase of the magnetic fluctuations strongly enhances the spin–lattice (1/T) relaxation rates between 0.12 (33.5 T) and 0.06 (15 T) for both samples in the same temperature range in YBa\(_2\)Cu\(_3\)O\(_y\) for both samples in the same temperature range in YBa\(_2\)Cu\(_3\)O\(_y\). In Supplementary Information we provide evidence (explaining the rotational symmetry breaking) over a broad spin-lattice relaxation rate 1/T dependence of the planar spin order (the smectic phase of an electronic liquid crystal, partly fluctuating (that is, nematic). Therefore, instead of a crossover from a Gaussian to an exponential decay to a stretched form (hole/Cu), the transition to 0.12 (33.5 T). Slowing down of the spin fluctuations provides evidence (explaining the rotational symmetry breaking) over a broad temperature range in YBa\(_2\)Cu\(_3\)O\(_y\).
Outline

1. Antiferromagnetism in metals and $d$-wave superconductivity

2. Competing order: $d$-wave bond order

3. Nematic quantum criticality and the strange metal

4. The pseudogap regime of the hole-doped cuprate superconductors

   Angular fluctuations of a multicomponent order
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   *Angular fluctuations of a multicomponent order*
The electron spin polarization obeys

\[ \langle \vec{S}(r, \tau) \rangle = \vec{\varphi}(r, \tau)e^{iK \cdot r} \]

where \( K \) is the ordering wavevector.
The Hubbard Model

\[ H = - \sum_{i<j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c^\dagger_{i\alpha} c_{i\alpha} \]

\( t_{ij} \to \) “hopping”. \( U \to \) local repulsion, \( \mu \to \) chemical potential

Spin index \( \alpha = \uparrow, \downarrow \)

\[ n_{i\alpha} = c^\dagger_{i\alpha} c_{i\alpha} \]

\[ c^\dagger_{i\alpha} c_{j\beta} + c_{j\beta} c^\dagger_{i\alpha} = \delta_{ij} \delta_{\alpha\beta} \]

\[ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \]
The Hubbard Model

Decouple $U$ term by a Hubbard-Stratanovich transformation

\[ S = \int d^2 r d\tau \left[ \mathcal{L}_c + \mathcal{L}_\phi + \mathcal{L}_{c\phi} \right] \]

\[ \mathcal{L}_c = c_a^\dagger \varepsilon (-i \nabla) c_a \]

\[ \mathcal{L}_\phi = \frac{1}{2} ( \nabla \phi_\alpha )^2 + \frac{r}{2} \phi_\alpha^2 + \frac{u}{4} ( \phi_\alpha^2 )^2 \]

\[ \mathcal{L}_{c\phi} = \lambda \phi_\alpha e^{i \mathbf{K} \cdot \mathbf{r}} c_\alpha^\dagger \sigma_{ab}^\alpha c_b. \]

“Yukawa” coupling between fermions and antiferromagnetic order:

$\lambda^2 \sim U$, the Hubbard repulsion
Fermi surface + antiferromagnetism

Metal with “large” Fermi surface
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$. 

Fermi surface+antiferromagnetism
Fermi surface + antiferromagnetism

“Hot” spots
Electron and hole pockets in antiferromagnetic phase with $\langle \bar{\phi} \rangle \neq 0$
Quantum phase transition with onset of antiferromagnetism in a metal

\[ \langle \bar{\phi} \rangle \neq 0 \]

Metal with electron and hole pockets

\[ \langle \bar{\phi} \rangle = 0 \]

Metal with “large” Fermi surface
Quantum phase transition with onset of antiferromagnetism in a metal

Metal with electron and hole pockets

\[ \langle \phi \rangle \neq 0 \]

Metal with “large” Fermi surface

Find new instabilities upon approaching critical point
Pairing by SDW fluctuation exchange

We now allow the SDW field \( \vec{\varphi} \) to be dynamical, coupling to electrons as

\[
H_{sdw} = - \sum_{\mathbf{k}, \mathbf{q}, \alpha, \beta} \vec{\varphi}_\mathbf{q} \cdot \mathbf{c}_{\mathbf{k},\alpha}^\dagger \mathbf{\bar{\sigma}}_{\alpha\beta} \mathbf{c}_{\mathbf{k}+\mathbf{K}+\mathbf{q},\beta}.
\]

Exchange of a \( \vec{\varphi} \) quantum leads to the effective interaction

\[
H_{ee} = -\frac{1}{2} \sum_{\mathbf{q}} \sum_{\mathbf{p}, \gamma, \delta} \sum_{\mathbf{k}, \alpha, \beta} V_{\alpha\beta,\gamma\delta}(\mathbf{q}) \mathbf{c}_{\mathbf{k},\alpha}^\dagger \mathbf{c}_{\mathbf{k}+\mathbf{q},\beta}^\dagger \mathbf{c}_{\mathbf{p},\gamma} \mathbf{c}_{\mathbf{p}-\mathbf{q},\delta},
\]

where the pairing interaction is

\[
V_{\alpha\beta,\gamma\delta}(\mathbf{q}) = \mathbf{\bar{\sigma}}_{\alpha\beta} \cdot \mathbf{\bar{\sigma}}_{\gamma\delta} \frac{\chi_0}{\xi^{-2} + (\mathbf{q} - \mathbf{K})^2},
\]

with \( \chi_0 \xi^2 \) the SDW susceptibility and \( \xi \) the SDW correlation length.
In BCS theory, this interaction leads to the ‘gap equation’ for the pairing gap $\Delta_k \propto \langle c_{k\uparrow} c_{-k\downarrow} \rangle$.

$$\Delta_k = -\sum_p \left( \frac{3\chi_0}{\xi^{-2} + (p - k - K)^2} \right) \frac{\Delta_p}{2\sqrt{\varepsilon_p^2 + \Delta_p^2}}$$

Non-zero solutions of this equation require that $\Delta_k$ and $\Delta_p$ have opposite signs when $p - k \approx K$. 
Pairing “glue” from antiferromagnetic fluctuations

\[ \langle c_{k\alpha}^\dagger c_{-k\beta}^\dagger \rangle = \varepsilon_{\alpha\beta} \Delta_S (\cos k_x - \cos k_y) \]

d-wave superconductor: particle-particle pairing at and near hot spots, with sign-changing pairing amplitude

Near the antiferromagnetic critical point, the coupling becomes infinitely strong:
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Near the antiferromagnetic critical point, the coupling becomes infinitely strong:

- Pairing glue becomes stronger.

- There is stronger fermion-boson scattering, and fermionic quasi-particles lose their integrity.

- An instability to charge-density-wave/bond order can become nearly degenerate with superconductivity if the Fermi-surface is not too curved.

M.A. Metlitski and S. Sachdev, 
QMC for the onset of antiferromagnetism

Hot spots in a single band model
QMC for the onset of antiferromagnetism


Hot spots in a two band model
No sign problem in fermion determinant Monte Carlo!

Determinant is positive because of Kramer’s degeneracy, and no additional symmetries are needed; holds for arbitrary band structure and band filling, provided $K$ only connects hot spots in distinct bands.

Sign-problem-free Quantum Monte Carlo for antiferromagnetism in metals

\[ \bar{P}(\hat{r}_{\text{max}}) \]

\[ \bar{P}_+ - \bar{P}_- \]

\[ L = 10 \]

\[ L = 12 \]

\[ L = 14 \]

\[ s/d \text{ pairing amplitudes } P_+/P_- \]

as a function of the tuning parameter \( r \)

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   *Angular fluctuations of a multicomponent order*
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\textit{Angular fluctuations of a multicomponent order}
Pseudospin symmetry of the exchange interaction

\[ H_J = \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

with \( \vec{S}_i = \frac{1}{2} c_{i\alpha} \bar{\sigma}_{\alpha\beta} c_{i\beta} \) is the antiferromagnetic exchange interaction. Introduce the Nambu spinor

\[ \Psi_{i\uparrow} = \begin{pmatrix} c_{i\uparrow} \\ c_{i\uparrow}^\dagger \end{pmatrix}, \quad \Psi_{i\downarrow} = \begin{pmatrix} c_{i\downarrow} \\ -c_{i\uparrow}^\dagger \end{pmatrix} \]

Then we can write

\[ H_J = \frac{1}{8} \sum_{i<j} J_{ij} \left( \Psi_{i\alpha a}^\dagger \bar{\sigma}_{\alpha\beta} \Psi_{i\beta a} \right) \cdot \left( \Psi_{j\gamma b}^\dagger \bar{\sigma}_{\gamma\delta} \Psi_{j\delta b} \right) \]

where \( a, b \) are the Nambu indices. This form makes explicit the symmetry under independent SU(2) pseudospin transformations on each site

\[ \Psi_{i\alpha a} \rightarrow U_{i,ab} \Psi_{i\alpha b} \]

This pseudospin (gauge) symmetry is important in classifying spin liquid ground states of \( H_J \).

Pseudospin symmetry of the exchange interaction

\[ H_{tJ} = - \sum_{i,j} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j + V \sum_{\langle ij \rangle} n_i n_j \]

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This pseudospin (gauge) symmetry is important in classifying spin liquid ground states of \( H_J \). It is fully broken by the electron hopping \( t_{ij} \) but does have remnant consequences in doped spin liquid states.

Pseudospin symmetry of the exchange interaction

\[ H_{tJ} = - \sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j + V \sum_{\langle ij \rangle} n_i n_j \]

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\[ \Psi_{i\alpha a} \rightarrow U_{i,ab} \Psi_{i\alpha b} \]

We will find important consequences of the pseudospin symmetry in ordinary metals with antiferromagnetic correlations.

Pairing “glue” from antiferromagnetic fluctuations

Same “glue” leads to particle-hole pairing

\[ \left\langle c_{\mathbf{k}\alpha}^\dagger c_{-\mathbf{k}\beta}^\dagger \right\rangle = \varepsilon_{\alpha\beta} \Delta_S (\cos k_x - \cos k_y) \]


\textbf{d-wave superconductor: particle-particle pairing at and near hot spots, with sign-changing pairing amplitude}
\[
\left\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \right\rangle = P_Q (\cos k_x - \cos k_y)
\]

Incommensurate d-wave bond order: particle-hole pairing at \textit{and near} hot spots, with sign-changing pairing amplitude

After pseudospin rotation on half the hot-spots


Q is ‘2k\textsubscript{F}’ wavevector
Incommensurate $d$-wave bond order

\[
\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = P_Q (\cos k_x - \cos k_y)
\]
Incommensurate $d$-wave bond order

$\langle c_{k-Q/2,\alpha}^\dagger c_{k+Q/2,\alpha} \rangle = P_Q (\cos k_x - \cos k_y)$
**Bond, charge, and current order**

Consider modulation in a bilocal variable at the Cu sites $r_i$ and $r_j$

\[
\langle c_{i\alpha}^\dagger c_{j\alpha} \rangle \sim \left[ \sum_k P_Q(k) e^{i k \cdot (r_i - r_j)} \right] e^{i Q \cdot (r_i + r_j)/2}
\]

The wavevector $Q$ is associated with a modulation in the *average* co-ordinate $(r_i + r_j)/2$: this determines the wavevector of the X-ray scattering peak.
Bond, charge, and current order

Consider modulation in a bilocal variable at the Cu sites $r_i$ and $r_j$

$$\left\langle c_{i\alpha}^\dagger c_{j\alpha} \right\rangle \sim \left[ \sum_k P_Q(k) e^{ik \cdot (r_i - r_j)} \right] e^{iQ \cdot (r_i + r_j)/2}$$

The wavevector $Q$ is associated with a modulation in the average co-ordinate $(r_i + r_j)/2$: this determines the wavevector of the X-ray scattering peak. The interesting part is the dependence on the relative co-ordinate $r_i - r_j$. The order parameter $P_Q(k)$ can always be expanded as

$$P_Q(k) = \sum_\ell \mathcal{P}_\ell \phi_\ell(k)$$

$$\phi_\ell(k) = \{1, \cos k_x + \cos k_y, \cos k_x - \cos k_y, \sin k_x - \sin k_y, \ldots\}$$
Consider modulation in a bilocal variable at the Cu sites $r_i$ and $r_j$

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The wavevector $\mathbf{Q}$ is associated with a modulation in the *average* co-ordinate $(r_i + r_j)/2$: this determines the wavevector of the X-ray scattering peak. The interesting part is the dependence on the *relative* co-ordinate $r_i - r_j$. The order parameter $P_Q(k)$ can always be expanded as

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The usual charge-density-wave has only $\mathcal{P}_s \neq 0$, and so the density wave is non-zero only if $r_i = r_j$. 
Bond, charge, and current order

Consider modulation in a bilocal variable at the Cu sites $\mathbf{r}_i$ and $\mathbf{r}_j$

$$\langle c_{i\alpha}^\dagger c_{j\alpha} \rangle \sim \left[ \sum_{\mathbf{k}} P_{Q}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{i\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2}$$

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The bond-ordered state has predominantly $\mathcal{P}_{s'}, \mathcal{P}_d$ non-zero: in this case the density wave is non-zero only if $\mathbf{r}_i$ and $\mathbf{r}_j$ are nearest neighbors.
Bond, charge, and current order

Consider modulation in a bilocal variable at the Cu sites \( r_i \) and \( r_j \)

\[
\langle c_i^{\dagger} c_j \rangle \sim \sum_{\mathbf{k}} P_{\mathbf{Q}}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} e^{i\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2}
\]

The wavevector \( \mathbf{Q} \) is associated with a modulation in the average co-ordinate \( (\mathbf{r}_i + \mathbf{r}_j)/2 \): this determines the wavevector of the X-ray scattering peak.

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The bond-ordered state has predominantly \( \mathcal{P}_{s'}, \mathcal{P}_d \) non-zero: in this case the density wave is non-zero only if \( r_i \) and \( r_j \) are nearest neighbors.

States with spontaneous currents have \( \mathcal{P}_p \) non-zero: they break time-reversal
Bond, charge, and current order

Plot of $P_{ij} = \langle c_{i\alpha}^\dagger c_{j\alpha} \rangle$ for $i = j$, and $i, j$ nearest neighbors.

Predominantly $d$-wave bond order at $Q = (\pi/4, 0)$
Bond, charge, and current order

Plot of $P_{ij} = \langle c^\dagger_{i\alpha} c_{j\alpha} \rangle$ for $i = j$, and $i, j$ nearest neighbors.

$d$-wave bond order at $Q = (\pi/4, \pi/4)$
Bond, charge, and current order

Plot of $P_{ij} = \langle c_{i\alpha}^\dagger c_{j\alpha} \rangle$ for $i = j$, and $i, j$ nearest neighbors.

$p$-wave current order at $Q = (\pi, \pi)$;
This state is also known as “$d$-density wave” (unfortunately!), and “staggered-flux (SF)”.

The main change is that the eigenvalues near purely are shown in Figs. 3 and 4 for the metallic state with the Fermi surface in Fig. 1. and plays no special role in the Hamiltonian. shown with boundary, as shown in Fig. 1. The hot spots for this dispersion are separated by the vectors.

We choose the dispersion

![Diagram](computation_of_P_l.png)

Find the lowest eigenvalues \( \lambda(Q) \), and corresponding eigenvectors, of the matrix

\[
\delta_{\ell m} - \frac{1}{2} \left( \frac{3}{4} J_{\ell} + V_{\ell} \right) \Pi_{\ell m}(Q) + \delta_{\ell,0} W(Q) \Pi_{0m}(Q), \quad \text{where}
\]

\[
\Pi_{\ell m}(Q) = 2 \sum_{k} \phi_{\ell}(k) \phi_{m}(k) \frac{f(\varepsilon(k - Q/2)) - f(\varepsilon(k + Q/2))}{\varepsilon(k + Q/2) - \varepsilon(k - Q/2)}
\]

and

\[
W(Q) = \sum_{\ell} V_{\ell} \phi_{\ell}(0) \phi_{\ell}(Q)
\]
Eigenvalues, $\lambda(Q)$, of the spin-singlet, particle-hole propagator. The corresponding eigenvector is $P_Q(k)$ and this leads to the order

$$\langle c_{i\alpha}^{\dagger} c_{j\alpha} \rangle = \left[ \sum_k P_Q(k) e^{i k \cdot (r_i - r_j)} \right] e^{i Q \cdot (r_i + r_j)/2}$$
Eigenvalues, $\lambda(Q)$, of the spin-singlet, particle-hole propagator. The corresponding eigenvector is $P_Q(k)$ and this leads to the order

$$
\langle c_{i\alpha}^\dagger c_{j\alpha} \rangle = \left[ \sum_k P_Q(k)e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{i\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2}
$$

Minimum at $Q = (Q_m, Q_m)$ with

$$
P_Q(k) = 0.996(\cos k_x - \cos k_y)
+ 0.087(\cos(2k_x) - \cos(2k_y))
$$

Incommensurate $d$-wave bond order
Remarkable agreement between the value of $Q_m$ from Hartree-Fock in a metal with short-range incommensurate spin correlations, and the value of $Q_0$ from hot spots of commensurate antiferromagnetism.
Eigenvalues, $\lambda(Q)$, of the spin-singlet, particle-hole propagator. The corresponding eigenvector is $P_Q(k)$ and this leads to the order

$$\langle c_{i\alpha}^\dagger c_{j\alpha} \rangle = \left[ \sum_k P_Q(k)e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{iQ \cdot (\mathbf{r}_i + \mathbf{r}_j)/2}$$

$Q = (\pi, \pi)$ with

$$P_Q(k) = i(\sin k_x - \sin k_y)$$

Staggered orbital currents
Eigenvalues, \( \lambda(Q) \), of the spin-singlet, particle-hole propagator. The corresponding eigenvector is \( P_Q(k) \) and this leads to the order
\[
\left\langle c_{i\alpha}^\dagger c_{j\alpha} \right\rangle = \left[ \sum_k P_Q(k) e^{ik \cdot (r_i - r_j)} \right] e^{iQ \cdot (r_i + r_j)/2}
\]
Eigenvalues, $\lambda(Q)$, of the spin-singlet, particle-hole propagator. The corresponding eigenvector is $P_Q(k)$ and this leads to the order

$$\langle c_{i\alpha}^\dagger c_{j\alpha} \rangle = \left[ \sum_k P_Q(k) e^{i k \cdot (r_i - r_j)} \right] e^{i Q \cdot (r_i + r_j)/2}$$

$$Q = (0, Q_m) \text{ with}$$

$$P_Q(k) = \begin{pmatrix} -0.931 \cos k_x - \cos k_y \\ -0.352 \\ -0.028 \cos(2k_x) - \cos(2k_y) \\ -0.168 \cos k_x \cos k_y \\ -0.017 \cos k_x + \cos k_y \\ +0.029 \cos(2k_x) + \cos(2k_y) \end{pmatrix}$$

**Incommensurate $d_+s$-wave bond order**
Results of a variational Monte Carlo computation on a wavefunction with double-occupancy projected out.

The ordering at \((Q_0, 0)\) is predominantly \(d\)-wave, while that at \((Q_0, Q_0)\) is purely \(d\)-wave.
In addition, by adopting a special experimental geometry, we also resolve the \textit{intra-unit-cell} symmetry of the charge ordered state, which is revealed to be a \textbf{\textit{d-wave bond-order}}. These results represent a fundamental advancement in our microscopic description of charge order in cuprates, and provide crucial insights for the understanding of its origin and interplay with superconductivity and magnetism.

This type of \textit{d-wave bond order} at $\mathbf{Q} = (Q_0, 0)$ was first predicted in S. Sachdev and R. LaPlaca, Phys. Rev. Lett. \textbf{111}, 027202 (2013).
The symmetry of charge order in the cuprates

This type of \( d \)-wave bond order at \( Q = (Q_0, 0) \) was first predicted in S. Sachdev and R. LaPlaca, Phys. Rev. Lett. 111, 027202 (2013).

<table>
<thead>
<tr>
<th>( \Delta_{\text{CDW}} )</th>
<th>Probability levels ( P ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_s )</td>
<td>Bidirectional 30.3 Unidirectional 38.8</td>
</tr>
<tr>
<td>( \Delta_{s'} (\cos k_x + \cos k_y) )</td>
<td>Bidirectional 12.0 Unidirectional 6.0</td>
</tr>
<tr>
<td>( \Delta_d (\cos k_x - \cos k_y) )</td>
<td>Bidirectional 81.8 Unidirectional 87.6</td>
</tr>
</tbody>
</table>

Figure 4a compares the REXS profiles from Fig. 4a, as evaluated by the O-2\( _{\text{REXS}} \) (15.0.15 Figure 4b compares the theoretical predictions for the hierarchy in the likelihood of the various terms for unidirectional and bi-directional order describe the data equally well, as it greater likelihood of describing the experimental data. A uniformly model outperforms the other models in that it has a substantially greater probability level for all the experimental points (inclusive of YBCO-Ortho II), YBCO-Ortho III, and YBCO-Ortho IV).
1. Antiferromagnetism in metals and $d$-wave superconductivity

2. Competing order: $d$-wave bond order

3. Nematic quantum criticality and the strange metal

4. The pseudogap regime of the hole-doped cuprate superconductors

**Angular fluctuations of a multicomponent order**
1. Antiferromagnetism in metals and $d$-wave superconductivity

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*Angular fluctuations of a multicomponent order*
Multi-component order parameter for the pseudogap

Superconducting order $\Psi(\mathbf{r})$:

$$\left\langle c_{i\alpha}^{\dagger} c_{j\beta}^{\dagger} \right\rangle = \varepsilon_{\alpha\beta} \left[ \sum_{\mathbf{k}} \Delta_{S}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] \Psi((\mathbf{r}_i + \mathbf{r}_j)/2)$$

Charge/bond order $\Phi_{x,y}(\mathbf{r})$ at wavevectors $\mathbf{Q}_{x,y}$:

$$\left\langle c_{i\alpha}^{\dagger} c_{j\beta}^{\dagger} \right\rangle = \delta_{\alpha\beta} \left[ \sum_{\mathbf{k}} P_{Q_x}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{iQ_x \cdot (\mathbf{r}_i + \mathbf{r}_j)/2} \Phi_x((\mathbf{r}_i + \mathbf{r}_j)/2)$$

$$+ \delta_{\alpha\beta} \left[ \sum_{\mathbf{k}} P_{Q_y}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{iQ_y \cdot (\mathbf{r}_i + \mathbf{r}_j)/2} \Phi_y((\mathbf{r}_i + \mathbf{r}_j)/2)$$
Associated with the bond orders $\Phi_x(r)$ and $\Phi_y(r)$, we can define the Ising-nematic order parameter $\phi(r) = |\Phi_x(r)|^2 - |\Phi_y(r)|^2$. We can imagine a state with only Ising-nematic order $\langle \phi \rangle \neq 0$, but no bond order $\langle \Phi_x \rangle = \langle \Phi_y \rangle = 0$.

![Diagram of the phase diagram of underdoped YBa$_2$Cu$_3$O$_y$](image)

**Nematic quantum critical point at optimal doping?**
Visualization of the emergence of the pseudogap state and the evolution to superconductivity in a lightly hole-doped Mott insulator

Y. Kohsaka, T. Hanaguri, M. Azuma, M. Takano, J. C. Davis, and H. Takagi

Nature Physics, 8, 534 (2012).

Evidence for “nematic” order (i.e. breaking of 90° rotation symmetry) in Ca_{1.88}Na_{0.12}CuO_2Cl_2.
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Evidence for “nematic” order (i.e. breaking of 90° rotation symmetry) in Ca$_{1.88}$Na$_{0.12}$CuO$_2$Cl$_2$. 
Nematic order as a function of hole density, $p$

Quantum criticality of Ising-nematic ordering in a metal

A metal with a **Fermi surface** with full square lattice symmetry
Quantum criticality of Ising-nematic ordering in a metal

Pomeranchuk instability as a function of coupling $\lambda$

or $\langle \phi \rangle \neq 0$

or $\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Quantum critical

$T_{l-n}$

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

Fermi liquid

Fermi liquid

$\lambda_c$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Classical $d=2$ Ising criticality

Fermi liquid

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

$T_{l-n}$

Fermi liquid

$\langle \phi \rangle \neq 0$

Fermi liquid

$\langle \phi \rangle = 0$

$D=2+1$ Ising criticality?

Quantum critical
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

Fermi liquid

Fermi liquid
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles
Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u\phi^4 \right] \]
Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \]

\[ \equiv \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]
Quantum criticality of Ising-nematic ordering in a metal

“Yukawa” coupling between Ising order and electrons

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]

for spatially dependent \( \phi \)

\[ \langle \phi \rangle > 0 \]

\[ \langle \phi \rangle < 0 \]
Quantum criticality of Ising-nematic ordering in a metal

The “standard model”:

\[ S_\phi = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \epsilon_k) c_{k\alpha} \]

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]
Quantum criticality of Ising-nematic ordering in a metal

- \( \phi \) fluctuation at wavevector \( \vec{q} \) couples most efficiently to fermions near \( \pm \vec{k}_0 \).
Quantum criticality of Ising-nematic ordering in a metal

- \( \phi \) fluctuation at wavevector \( \vec{q} \) couples most efficiently to fermions near \( \pm \vec{k}_0 \).
- Expand fermion kinetic energy at wavevectors about \( \pm \vec{k}_0 \) and boson (\( \phi \)) kinetic energy about \( \vec{q} = 0 \).
\[ \mathcal{L}[\psi_\pm, \phi] = \]
\[ \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \]
\[ -\phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i \partial_x - \partial_y^2) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

One loop \( \phi \) self-energy with \( N_f \) fermion flavors:

\[
\Sigma_\phi(\bar{q}, \omega) = N_f \int \frac{d^2 k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{\left[ -i(\Omega + \omega) + k_x + q_x + (k_y + q_y)^2 \right] \left[ -i\Omega - k_x + k_y^2 \right]} \\
= \frac{N_f}{4\pi} \frac{|\omega|}{|q_y|} \]

Landau-damping
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi^\dagger_+ (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi^\dagger_- (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \\
- \phi \left( \psi^\dagger_+ \psi_+ + \psi^\dagger_- \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Electron self-energy at order \(1/N_f\):

\[ \Sigma(\vec{k}, \Omega) = -\frac{1}{N_f} \int \frac{d^2q}{4\pi^2} \frac{d\omega}{2\pi} \frac{1}{\left[ -i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2 \right]} \left[ \frac{q_y^2}{g^2} + \frac{|\omega|}{|q_y|} \right] \]

\[ = -i \frac{2}{\sqrt{3}N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega)|\Omega|^{2/3} \sim |\Omega|^{d/3} \text{ in dimension } d. \]
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- 
\]
\[ - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Schematic form of \( \phi \) and fermion Green’s functions in \( d \) dimensions

\[ D(\vec{q}, \omega) = \frac{1/N_f}{q_y^2 + \frac{\omega}{|q_y|}} \]
\[ G_f(\vec{q}, \omega) = \frac{1}{q_x + q_y^2 - i\text{sgn}(\omega)\omega^{2/3}/N_f} \]

In the boson case, \( q_y^2 \sim \omega^{1/z_b} \) with \( z_b = 3/2 \).
In the fermion case, \( q_x \sim q_y^2 \sim \omega^{1/z_f} \) with \( z_f = 3/d \).

Note \( z_f < z_b \) for \( d > 2 \) \( \Rightarrow \) Fermions have higher energy than bosons, and perturbation theory in \( g \) is OK.

Strongly-coupled theory in \( d = 2 \) without quasiparticles.
Quantum criticality of Ising-nematic ordering in a metal

\[ \mathcal{L} = \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Schematic form of \( \phi \) and fermion Green's functions in \( d \) dimensions

\[ D(\vec{q}, \omega) = \frac{1/N_f}{q_y^2 + \frac{|\omega|}{|q_y|}} \quad \text{,} \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_y^2 - i \text{sgn}(\omega)|\omega|^{2/3}/N_f} \]

In the boson case, \( q_y^2 \sim \omega^{1/z_b} \) with \( z_b = 3/2 \).
In the fermion case, \( q_x \sim q_y^2 \sim \omega^{1/z_f} \) with \( z_f = 3/d \).

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Strongly-coupled theory in \( d = 2 \) without quasiparticles.
Quantum criticality of Ising-nematic ordering in a metal

Phase diagram as a function of $T$ and $\lambda$
Quantum criticality of Ising-nematic ordering in a metal

\[ T \]

\[ T_{I-n} \]

Strange Metal

Fermi liquid $\langle \phi \rangle \neq 0$

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles

Fermi liquid $\langle \phi \rangle = 0$

Common theoretical belief from an analysis of scattering of charged electronic quasiparticles off bosonic $\phi$ fluctuations: resistivity of strange metal $\rho(T) \sim T^{4/3}$. 
Quantum criticality of Ising-nematic ordering in a metal

The “standard model”:

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^{\dagger} \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]

\[ S_{\phi c} = -g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^{\dagger} c_{k-q/2,\alpha} \]
**Quantum criticality of Ising-nematic ordering in a metal**

The “standard model”:

\[
S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]
\]

\[
S_c = \sum_{\alpha=1}^{N_f} \int d^2r d\tau c_\alpha^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) c_\alpha
\]

\[
S_{\phi c} = -g \int d^2r d\tau \sum_{\alpha=1}^{N_f} \phi \left[ c_\alpha^\dagger \left\{ (\partial_x^2 - \partial_y^2) c_\alpha \right\} + \left\{ (\partial_x^2 - \partial_y^2) c_\alpha^\dagger \right\} c_\alpha \right]
\]

This continuum theory has a conserved momentum \( \mathbf{P} \), and \( \chi_{J,\mathbf{P}} \neq 0 \), and so the resistivity \( \rho(T) = 0 \).
Quantum criticality of Ising-nematic ordering in a metal

The “standard model”:

\[ S_\phi = \int d^2r d\tau [(\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \int d^2r d\tau c_\alpha^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) c_\alpha \]

\[ S_{\phi c} = -g \int d^2r d\tau \sum_{\alpha=1}^{N_f} \phi \left[ c_\alpha^\dagger \left\{ (\partial_x^2 - \partial_y^2) c_\alpha \right\} + \left\{ (\partial_x^2 - \partial_y^2) c_\alpha^\dagger \right\} c_\alpha \right] \]

This continuum theory has a conserved momentum \( P \), and \( \chi_{J, P} \neq 0 \), and so the resistivity \( \rho(T) = 0 \).

The resistivity of the strange metal is *not* determined by the scattering rate of charged excitations near the Fermi surface, but by the dominant rate of momentum loss by *any* excitation, whether neutral or charged, or fermionic or bosonic.
Resistivity of strange metal

In the presence of weak disorder of quenched Gaussian random fields

\[ S_{\text{dis}} = \int d^2rd\tau \left[ V(\mathbf{r}) c^\dagger c + h(\mathbf{r}) \phi \right], \]

\[ V(\mathbf{r}) = 0 \quad ; \quad V(\mathbf{r})V(\mathbf{r'}) = V_0^2 \delta(\mathbf{r} - \mathbf{r'}), \]

\[ h(\mathbf{r}) = 0 \quad ; \quad h(\mathbf{r})h(\mathbf{r'}) = h_0^2 \delta(\mathbf{r} - \mathbf{r'}), \]

we use the memory-function approach to obtain the resistivity for current along angle \( \vartheta \)

\[ \rho(T) = \frac{1}{\chi_{J,P}^2} \lim_{\omega \to 0} \int \frac{d^2k}{(2\pi)^2} k^2 \cos^2(\theta_k - \vartheta) \left( V_0^2 \frac{\text{Im} \Pi_{c^\dagger c}^R(\omega, \mathbf{k})}{\omega} + h_0^2 \frac{\text{Im} D_{\phi}^R(\omega, \mathbf{k})}{\omega} \right). \]
**Resistivity of strange metal**

In the presence of weak disorder of quenched Gaussian random fields

\[
S_{\text{dis}} = \int d^2r d\tau \left[ V(\mathbf{r}) c^\dagger c + h(\mathbf{r}) \phi \right],
\]

\[
\overline{V(\mathbf{r})} = 0 ; \quad \overline{V(\mathbf{r})V(\mathbf{r}')} = V_0^2 \delta(\mathbf{r} - \mathbf{r}'),
\]

\[
\overline{h(\mathbf{r})} = 0 ; \quad \overline{h(\mathbf{r})h(\mathbf{r}')} = h_0^2 \delta(\mathbf{r} - \mathbf{r}'),
\]

we use the memory-function approach to obtain the **resistivity** for current along angle \( \vartheta \)

\[
\rho(T) = \frac{1}{\chi_{J,\mathbf{P}}^2} \lim_{\omega \to 0} \int \frac{d^2k}{(2\pi)^2} k^2 \cos^2(\theta_k - \vartheta) \left( V_0^2 \frac{\text{Im} \Pi_{c^\dagger c}^R(\omega, \mathbf{k})}{\omega} + h_0^2 \frac{\text{Im} D_\phi^R(\omega, \mathbf{k})}{\omega} \right).
\]

**Fermi surface term:** Obtain \( T \)-dependent corrections to residual resistivity similar to earlier work


Resistivity of strange metal

In the presence of weak disorder of quenched Gaussian random fields

\[ S_{\text{dis}} = \int d^2 r d\tau \left[ V(\mathbf{r}) \, c^\dagger c + h(\mathbf{r}) \, \phi \right] , \]

\[ \overline{V(\mathbf{r})} = 0 ; \quad \overline{V(\mathbf{r})V(\mathbf{r}')} = V_0^2 \, \delta(\mathbf{r} - \mathbf{r}') , \]

\[ \overline{h(\mathbf{r})} = 0 ; \quad \overline{h(\mathbf{r})h(\mathbf{r}')} = h_0^2 \, \delta(\mathbf{r} - \mathbf{r}') , \]

we use the memory-function approach to obtain the resistivity for current along angle \( \theta \)

\[ \rho(T) = \frac{1}{\chi_{J,P}^2} \lim_{\omega \to 0} \int \frac{d^2 k}{(2\pi)^2} \frac{k^2 \cos^2(\theta_k - \vartheta)}{k^2} \left( V_0^2 \, \frac{\text{Im} \, \Pi_{c^\dagger c}^R(\omega, \mathbf{k})}{\omega} + h_0^2 \, \frac{\text{Im} \, D_{\phi}^R(\omega, \mathbf{k})}{\omega} \right) . \]

**Bosonic term**: Dominant contribution:

\[ \rho(T) \sim T^{(d-z+\eta)/z} \]

Crosses over from the “relativistic” form \( (z = 1, \eta \approx 0) \) with \( \rho(T) \sim T \) at higher \( T \),

to the “Landau-damped” form \( (z = 3, \eta = 0) \) with \( \rho(T) \sim (T \ln(1/T))^{-1/2} \) at lower \( T \) (subtle corrections to scaling specific to this field theory).

Quantum criticality of Ising-nematic ordering in a metal

Strongly-coupled “non-Fermi liquid” metal with no quasiparticles
Quantum criticality of Ising-nematic ordering in a metal

Strange metal regime with $\rho(T) \sim T$ due to scattering of $z = 1$ neutral bosons off a random field.
1. Antiferromagnetism in metals and $d$-wave superconductivity

2. Competing order: $d$-wave bond order

3. Nematic quantum criticality and the strange metal

4. The pseudogap regime of the hole-doped cuprate superconductors

*Angular fluctuations of a multicomponent order*
Outline

1. Antiferromagnetism in metals and $d$-wave superconductivity

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   Angular fluctuations of a multicomponent order
Figure 3

All measurements are with...
Direct observation of competition between superconductivity and charge density wave order in YBa$_2$Cu$_3$O$_{6.67}$

J. Chang$^{1,2,*}$, E. Blackburn$^3$, A. T. Holmes$^3$, N. B. Christensen$^4$, J. Larsen$^{4,5}$, J. Mesot$^{1,2}$, Ruixing Liang$^{6,7}$, D. A. Bonn$^{6,7}$, W. N. Hardy$^{6,7}$, A. Watenphul$^8$, M. v. Zimmermann$^8$, E. M. Forgan$^3$ and S. M. Hayden$^9$

![Graph showing intensity vs. temperature](image1)

![Graph showing temperature vs. doping](image2)

![Graph showing intensity vs. doping](image3)
CDW Scattering Intensity (arbitrary units)

$T$ (K)

$YBa_2Cu_3O_{6.67}$

$Q = (-0.31 \ 0 \ 1.48)$

Onset is unlike an arrested ordering transition, or precursor critical fluctuations.
Key idea: analogy with the onset of antiferromagnetism in the insulator La$_2$CuO$_4$

Gradual onset of intensity over a wide range of $T$ is a consequence of angular thermal fluctuations of an order parameter with 3 or more components in 2 spatial dimensions

$T_{\text{Néel}} = 325$K


Above $T_{\text{Néel}}$

Polyakov, 1975
Chakravarty, Halperin, Nelson 1989
O(3) non-linear sigma model

\[ Z = \int \mathcal{D}\vec{n}(x) \delta (\vec{n}^2(x) - 1) \exp \left( -\frac{\rho_s}{2T} \int d^2x (\nabla_x \vec{n})^2 \right) \]
Generalize \( \vec{n} \) to a \( N \)-component vector \( n_\alpha, \alpha = 1 \ldots N \), and take the \( N \to \infty \) limit while taking \( \rho_s \propto N \). This is implemented by a Lagrange multiplier \( \lambda \)

\[
Z = \int \mathcal{D}\lambda(x)\mathcal{D}n_\alpha(x) \exp \left( -\frac{\rho_s}{2T} \int d^2x \left( \nabla_x n_\alpha \right)^2 + i\lambda(n_\alpha^2 - 1) \right)
\]
**O(3) non-linear sigma model**

\[ Z = \int \mathcal{D}\vec{n}(x) \delta(\vec{n}^2(x) - 1) \exp \left( -\frac{\rho_s}{2T} \int d^2x (\nabla_x \vec{n})^2 \right) \]

Generalize \( \vec{n} \) to a \( N \)-component vector \( n_\alpha, \alpha = 1 \ldots N \), and take the \( N \to \infty \) limit while taking \( \rho_s \propto N \). This is implemented by a Lagrange multiplier \( \lambda \)

\[ Z = \int \mathcal{D}\lambda(x) \mathcal{D}n_\alpha(x) \exp \left( -\frac{\rho_s}{2T} \int d^2x \left[ (\nabla_x n_\alpha)^2 + i\lambda(n_\alpha^2 - 1) \right] \right) \]

We can now perform the Gaussian integral over \( n_\alpha \)

\[ Z = \int \mathcal{D}\lambda(x) \exp \left( -\frac{N}{2} \text{Tr} \ln (-\nabla_x^2 + i\lambda) + \frac{\rho_s}{2T} \int d^2x i\lambda \right) \]

Because \( \rho_s \propto N \), in the \( N \to \infty \) limit the partition function is dominated by the saddle point.
At the saddle point, we set $i\lambda(x) = \xi^{-1}$, and then the “structure factor” $S(k)$ of the order parameter is

$$S(k) = \int d^2x \langle n_\alpha(x)n_\alpha(0) \rangle e^{ikx} = \frac{NT}{\rho_s} \frac{1}{(k^2 + \xi^{-2})}$$

This identifies $\xi$ as the correlation length.
O(3) non-linear sigma model

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\]

This identifies \( \xi \) as the correlation length. The value of \( \xi \) is determined by the saddle-point equation, which simply enforces the constraint \( n_\alpha^2(x) = 1 \). So we have

\[
\frac{NT}{\rho_s} \int \frac{d^2k}{4\pi^2} \frac{1}{k^2 + \xi^{-2}} = 1
\]

Performing the \( k \) integral with a momentum cutoff \( \Lambda \) we obtain

\[
\frac{NT}{4\pi \rho_s} \ln \left(1 + \Lambda^2 \xi^2 \right) = 1 \quad \Rightarrow \quad \xi = \Lambda^{-1} \exp \left( \frac{2\pi \rho_s}{NT} \right)
\]

So \( \xi \) is finite at all non-zero \( T \) (no LRO), and diverges exponentially as \( T \to 0 \) (consistent with Mermin-Wagner theorem).
O(3) non-linear sigma model

The exact result (for the exponential) at finite $N$ is

$$\xi = \Lambda^{-1} \exp \left( \frac{2\pi \rho_s}{(N - 2)T} \right)$$
O(3) non-linear sigma model

The *exact* result (for the exponential) at finite $N$ is

$$\xi = \Lambda^{-1} \exp \left( \frac{2\pi \rho_s}{(N - 2)T} \right)$$

Neutron scattering measures the structure factor, and the peak value is $S(0)$

$$S(0) = \frac{NT}{\rho_s} \xi^2 = \frac{NT}{\Lambda^2 \rho_s} \exp \left( \frac{4\pi \rho_s}{(N - 2)T} \right)$$

So there is no Bragg peak at the ordering wavevector for any two-dimensional antiferromagnet.

La$_2$CuO$_4$ has a non-zero ordering temperature $T_N = 325$K, and this arises solely from the *inter-layer* coupling.
Key idea: analogy with the onset of antiferromagnetism in the insulator $La_2CuO_4$

Gradual onset of intensity over a wide range of $T$ is a consequence of angular thermal fluctuations of an order parameter with 3 or more components in 2 spatial dimensions.

$T_{\text{Néel}} = 325$K


Polyakov, 1975
Chakravarty, Halperin, Nelson 1989
Multi-component order parameter for the pseudogap

Superconducting order $\Psi(r)$:

$$\langle c_{i\alpha}^\dagger c_{j\beta}^\dagger \rangle = \varepsilon_{\alpha\beta} \left[ \sum_k \Delta_S(k) e^{i \mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] \Psi((\mathbf{r}_i + \mathbf{r}_j)/2)$$

Charge/bond order $\Phi_{x,y}(r)$ at wavevectors $\mathbf{Q}_{x,y}$:

$$\langle c_{i\alpha}^\dagger c_{j\beta}^\dagger \rangle = \delta_{\alpha\beta} \left[ \sum_k P_{Q_x}(k) e^{i \mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{i \mathbf{Q}_x \cdot (\mathbf{r}_i + \mathbf{r}_j)/2} \Phi_x((\mathbf{r}_i + \mathbf{r}_j)/2)$$

$$+ \delta_{\alpha\beta} \left[ \sum_k P_{Q_y}(k) e^{i \mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \right] e^{i \mathbf{Q}_y \cdot (\mathbf{r}_i + \mathbf{r}_j)/2} \Phi_y((\mathbf{r}_i + \mathbf{r}_j)/2)$$
Multi-component order parameter
Multi-component order parameter

Support from theory of antiferromagnetic quantum criticality
Multi-component order parameter

Excluded region

Preferred configurations

Support from theory of antiferromagnetic quantum criticality
Multi-component order parameter

Label order parameter by a 6-component unit vector $n_\alpha$ with $\sum_\alpha n_\alpha^2 = 1$
\[
Z = \int \mathcal{D}n_\alpha(r) \delta \left( \sum_{\alpha=1}^{6} n_\alpha^2(r) - 1 \right) \exp \left( -\frac{\rho_s}{2T} \int d^2r \left[ \sum_{\alpha=1}^{2} (\nabla n_\alpha)^2 + \lambda \sum_{\alpha=3}^{6} (\nabla n_\alpha)^2 + g \sum_{\alpha=3}^{6} n_\alpha^2 + w \left( (n_3^2 + n_4^2)^2 + (n_5^2 + n_6^2)^2 \right) \right] \right).
\]

where \( \Psi \propto n_1 + in_2, \Phi_x \propto n_3 + in_4, \Phi_y \propto n_5 + in_6. \)

Describes \( O(6) \Rightarrow O(2) \times O(2) \times O(2) \times \mathbb{Z}_2. \) The coupling \( g \) determines the anisotropy between superconductivity and charge order.

Solve by cluster Monte Carlo and \( 1/N \) expansion.

Comparison of Monte Carlo with experiments

Charge order structure factor $S_{\Phi_x}$

For $ga^2 = 0.30$ and $wa^2 = 0.00$ we have $\rho_s = 160$K.
The height was also rescaled to make the peak heights match.

Comparison of Monte Carlo with experiments

\[ S_{\Phi_x} = \int d^2r \langle \Phi_x(r)\Phi_x(0) \rangle \]

Charge order structure factor \( S_{\Phi_x} \)

For \( ga^2 = 0.30 \) and \( wa^2 = 0.00 \) we have \( \rho_s = 160K \).
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Diamagnetism in the pseudogap

Diamagnetism of YBa$_2$Cu$_3$O$_{6+x}$ crystals above $T_c$: Evidence for Gaussian fluctuations

I. Kokanović, 1,2,* D. J. Hills, 1 M. L. Sutherland, 1 R. Liang, 3 and J. R. Cooper 1

PHYSICAL REVIEW B 88, 060505(R) (2013)
Diamagnetism in the pseudogap

- The same set of parameters used to describe X-ray scattering, also predict the strength of superconducting fluctuations above $T_c$. We characterize the diamagnetism by computing a dimensionless ratio, $R(T)$, between the diamagnetic susceptibility, $\chi_d$, and the charge order correlation length:

$$R(T) \equiv \frac{12\pi \chi_d(T)}{k_B T \xi_{\text{cdw}}^2}$$

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1. Antiferromagnetism in metals and $d$-wave superconductivity

2. Competing order: $d$-wave bond order

3. Nematic quantum criticality and the strange metal

4. The pseudogap regime of the hole-doped cuprate superconductors

*Angular fluctuations of a multicomponent order*