Planckian metals
and
SYK criticality

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Talk online: sachdev.physics.harvard.edu
1. Key puzzle in the cuprates

2. Numerical solution of the SU(2) random $t$-$J$ model

3. Large-$M$ SYK solution of the SU($M$) random $t$-$J$ model

Fractionalization and deconfined criticality
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*Fractionalization and deconfined criticality*
Momentum-space view at large $p$

Overdoped Tl$_2$Ba$_2$CuO$_{6+\delta}$

\[ T_c = 30 \text{K} \]

$1+p$ mobile holes in a filled band

Momentum-space view at small $p$

Ca$_{2-x}$Na$_x$CuO$_2$Cl$_2$

at $x = 0.10$

"Fermi arcs"

Fermi surface transformation at the pseudogap critical point of a cuprate superconductor


We use angle-dependent magnetoresistance (ADMR) to measure the Fermi surface of the cuprate La$_{1.6}$–$\chi$Nd$_0$.Sr$_x$CuO$_4$. Above the critical doping $p^*$—outside of the pseudogap phase—we find a Fermi surface that is in quantitative agreement with angle-resolved photoemission. Below $p^*$, however, the ADMR is qualitatively different, revealing a clear change in Fermi surface topology. We find that our data is most consistent with a Fermi surface that has been reconstructed by a $Q = (\pi,\pi)$ wavevector. While static $Q = (\pi,\pi)$ antiferromagnetism is not found at these dopings, our results suggest that this wavevector is a fundamental organizing principle of the pseudogap phase.

$p > p_c$ Large Fermi surface

$p < p_c$ Reconstructed Fermi surface
Angle-dependent magnetoresistance in Nd-LSCO near $p = p_c \approx 0.23$. 

$$\frac{1}{\tau} = \frac{1}{\tau_{\text{iso}}} + \frac{\alpha}{\hbar} \frac{k_B T}{\hbar}$$

$\alpha = 1.4 \pm 0.3$
Hole doped cuprates

The remarkable underlying ground states of cuprate superconductors

Hidden magnetism at the pseudogap critical point of a high temperature superconductor
Nature Physics doi: 10.1038/s41567-020-0950-5

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Quasi-static magnetism in the pseudogap state of La$_{2-x}$Sr$_x$CuO$_4$.
Temperature – doping phase diagram representing $T_{\text{min}}$, the temperature of the minimum in the sound velocity, at different fields. Since superconductivity precludes the observation of $T_{\text{min}}$ in zero-field, the dashed line (brown area) represents the extrapolated $T_{\text{min}}(B=0)$. While not exactly equal to the freezing temperature $T_f$ (see Fig. 2), $T_{\text{min}}$ is closely tied to $T_f$ and so is expected to have the same doping dependence, including a peak around $p = 0.12$ in zero/low fields (ref. 2). Onset temperatures of charge order are from ref. 33 (squares) and 35 (hexagons).
Needed: A small Fermi surface to large Fermi surface transition in a single band model with Planckian criticality.
1. Key puzzle in the cuprates

2. Numerical solution of the SU(2) random $t$-$J$ model

3. Large-$M$ SYK solution of the SU($M$) random $t$-$J$ model

*Fractionalization and deconfined criticality*
Antoine Georges

Henry Shackleton

arXiv:2012.06589

Antoine Georges

Alexander Wietek
Random $t$-$J$ model

\[
H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

We consider the hole-doped case, with no double occupancy.

\[
\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0
\]

\[
\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N} \sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p
\]

\[
J_{ij \text{ random}}, ~ J_{ij} = 0, ~ J_{ij}^2 = J^2
\]

\[
t_{ij \text{ random}}, ~ t_{ij} = 0, ~ t_{ij}^2 = t^2
\]
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

\[ H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\uparrow}^\dagger c_{j\uparrow} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.
Random *t-J* model

\[
H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_i^{\dagger} c_j + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j
\]

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

\[ H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} \, c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \, \vec{S}_i \cdot \vec{S}_j \]

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

\[ H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} \, c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \, \vec{S}_i \cdot \vec{S}_j \]

We consider the hole-doped case, with no double occupancy.
Why random and all-to-all couplings?

Randomness is present in the real system.
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- Randomness self-averages (except for certain correlators in spin-glass phase) — Green’s functions are the same on every site.
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The pseudogap-Fermi liquid transition is primarily a small-to-large Fermi surface transition: an analogous transition and a Luttinger theorem can also be defined with all-to-all randomness because the self-energy is local (in the non-spin-glass phase).
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Introducing randomness removes the “distractions” of multiple competing orders.
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Introducing randomness removes the “distractions” of multiple competing orders.

Averaging over many samples allows smoother and faster approach to the thermodynamic limit from finite size studies.
where $|\lambda\rangle$ are one-particle eigenstates of the $t_{ij}$. In a Fermi liquid, the Luttinger identity implies that $\mathcal{N}(\epsilon)$ has a discontinuity at the free particle Fermi energy $\epsilon_F$. ($D(\epsilon)$ is the Wigner semi-circle density of states.)
where $|\lambda\rangle$ are one-particle eigenstates of the $t_{ij}$. In a Fermi liquid, the Luttinger identity implies that $N(\epsilon)$ has a discontinuity at the free particle Fermi energy $\epsilon_F$. $(D(\epsilon)$ is the Wigner semi-circle density of states.)
Dynamic spin susceptibility

(1/\(n\))\(\chi''(\omega)\)

\(p = 0.56\)
\(p = 0.39\)
\(p = 0.17\)
\(p = 0.00\)

\(p\) values shown for several values of doping in Fig. 3. Using the Lanczos algorithm on an 18-site cluster, is

analyzed for this model at

Near criticality, the model is predicted to exhibit SYK-like criticality with a non-zero extensive entropy and a

behavior of

Note that the variance of the hump vanishes in the thermodynamic limit, the integrated

analysis of this hump, described

At low dopings, a sharp peak at low-frequency at

remains non-zero. Our analysis gives a large-

that the spectral function is described well by a combi-

tion of the SYK result and a low-frequency hump. A

large-N

A prominent

declines from its value at

that the spectral weight decreases from its value at

a disordered Fermi liquid phase for all non-zero values

of doping, up to a critical finite value of doping,

providing evidence that the spin glass phase shown to

that the exponents of these two leading SYK contri-

weight). This SYK spectral weight has a leading

Schwinger-Dyson equations of the

spectral weight obtained by rescaling the solution of the

near the critical point at low frequencies, we subtract a

subtract o
Dynamic spin susceptibility

\[ \chi''(\omega) = \sum_n |\langle 0 | S_{+i} | n \rangle|^2 \times \delta(h\omega - E_n + E_0), \quad \text{(at } T = 0) \]

Evidence for a quantum critical point at \( p = p_c \approx 0.3 \).
Spin glass order \( q \) non-zero for \( p < p_c \).
that a divergence of the large for a reliable estimate of the maximum. We note showing lowering the temperature for all values of doping. We attained between dopings approximately linear in temperature, with a maximal slope low temperatures we observe that the specific heat is ap-

\[ C/T \]

increase of the specific heat at higher temperatures. this maximum remains at previous results

\[ \text{Error estimates have been obtained from 1000} \]

Refs. \[ \text{hibits a broad maximum at} \]

\[ \text{where} \]

\[ \text{given by,} \]

\[ E \]

\[ \text{model for system sizes} \]

\[ E \]

\[ \text{corresponds to the maximal} \]

\[ \text{different temperatures and} \]

\[ \text{observed} \]

\[ \text{when estimated as above.} \]

\[ \text{denotes} \]

\[ \text{FIG. 2. Thermodynamics of the random} \]

\[ \text{Thermodynamics and Entanglement.} \]

\[ \text{FIG. 2. Thermodynamics of the random} \]

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We note that the specific heat increases with temperature, reaching a maximum. This maximum is gradually shifted towards higher temperatures as the system size and the temperature are increased. Results for system sizes similar to the finite-temperature Lanczos method are shown in Fig. 2. At small values of doping, the specific heat is increasing when lowering the temperature. However, we find that both increasing the specific heat at higher temperatures.

Error estimates have been obtained from 1000 previous results. A broad maximum at $T = 0$ is observed, which we obtain an (error) estimate of the parameters. The thermal entropy for different dopings as a function of temperature is shown in Fig. (d). Again we observe a broad maximum at $T = 0$. However, we find that both increasing the specific heat and thermal entropy correspond to the doping value with maximal entropy.

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The thermal entropy for di...
where the averaged quantity of a non-interacting system with fixed particle number is commonly used in systems with translational invariance. For this reason, the single-particle density of states and the total system size are shown in Fig. (a) for different temperatures at constant density, while the system size is varied. At temperatures close to the critical point of the one-particle energy distribution function, the system size plays a crucial role. However, at higher temperatures, the system size converges to a constant value. The Fermi energy is defined as the energy of the system where the density of states is equal to one. We find that the single-site and total system sizes of states for free fermions converge to the numerically-computed value given by the inflection point of the one-particle energy distribution function. For a comparison of the Fermi energy given by Luttinger's theorem, we find that the single-site density of states is well converged, suggesting a breakdown of Luttinger's theorem. (b) A dashed line. Black dots show the ansatz Eq. ..., and the numerically-computed value given by the inflection point of the one-particle energy distribution function.
Dynamic spin susceptibility

![Graph showing dynamic spin susceptibility](image)

- **Large-M**
  - $p = 0.56$
  - $p = 0.39$
  - $p = 0.17$
  - $p = 0.00$

**Legend**
- **Large-N**

**Data Points**
- $(\omega, (1/n)\chi''(\omega))$ for different values of $p$.
Critical spin susceptibility matches the large $M$ SU($M$) SYK model. $\chi''(\omega) \sim \text{sgn}(\omega) [1 - C\gamma|\omega| + \ldots]$ has the ‘marginal’ sgn(\omega) form, with a linear $\omega$ correction. Shown is the numerical solution of SYK equations (SY, PRL 1993), after rescaling $J$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{dynamic_spin_susceptibility.png}
\caption{Dynamic spin susceptibility}
\end{figure}
Consequences of 2D-gravity for the dynamic spin susceptibility of SYK model

\[ \chi_L(\omega) = \sum_n |\langle 0| X_i |n \rangle|^2 \delta(\hbar \omega - E_n + E_0), \text{ (at } T = 0) \]

\[ \text{Im} \chi_L(\omega) \sim \text{sgn}(\omega) \left[ 1 - C \gamma |\omega| - \frac{7}{16} (C \gamma)^2 |\omega|^2 - C' |\omega|^{2.77354\ldots} + \frac{37}{48} (C \gamma)^3 |\omega|^3 - \ldots \right] \]

Numerical solution of SYK equations (SY, PRL 1993), compared with conformal perturbation theory. \( C \) is the co-efficient of the action for the ‘boundary graviton’ in holographic dual.
Consequences of 2D-gravity for the dynamic spin susceptibility of SYK model

\[ \chi_L(\omega) = \sum_n |\langle 0| X_i |n\rangle|^2 \delta(\hbar \omega - E_n + E_0), \text{ (at } T = 0) \]

\[ \chi_L(\omega) \sim \tanh \left( \frac{\hbar \omega}{2k_B T} \right) \left[ 1 - C\gamma \omega \tanh \left( \frac{\hbar \omega}{2k_B T} \right) - \ldots \right] \]
Consequences of 2D-gravity for the dynamic spin susceptibility of SYK model

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Conformally (SL(2,R)) invariant result with characteristic dissipative time \( \sim \hbar/(k_B T) \)

A. Georges and O. Parcollet
PRB 59, 5341 (1999)
Consequences of 2D-gravity for the dynamic spin susceptibility of SYK model

\[ \chi_L(\omega) = \sum_n |\langle 0| X_i |n\rangle|^2 \delta(\hbar\omega - E_n + E_0), \text{ (at } T = 0) \]

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Correction from the boundary graviton

Doping the SU(2) t-J-U SY Model: An Intriguing Quantum Critical Point
Solving the EDMFT Equations

Dumitrescu, Wentzell, AG, Parcollet
Soon on arXiv…
Dumitrescu, Wentzell, AG, Parcollet, Soon on arXiv…

Scattering rate
Linear in $T$
at QCP

Luttinger volume of FS
Breaks down at QCP
The random $t$-$J$ model has

- Spin glass order for $p < p_c$.
- Fermi liquid with Luttinger volume Fermi surface for $p > p_c$.
- Maxima in entropy, specific heat, and entanglement entropy near $p = p_c$.
- SYK-Planckian criticality near $p_c$.
- ‘Marginal’ spin susceptibility near criticality, with boundary graviton correction ‘observed’ in SU(2) model.
1. Key puzzle in the cuprates

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*Fractionalization and deconfined criticality*
Random $t$-$J$ model

\[
H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

We consider the hole-doped case, with no double occupancy.
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Each site has 3 states which we map to the ‘superspin’ space of a boson $b$ (the holon) and a fermion $f_{\alpha}$ (the spinon):

$$\begin{align*}
 b^{\dagger} |v\rangle & \quad \uparrow \quad \uparrow \quad \uparrow \\
 f_{\uparrow}^{\dagger} |v\rangle & \quad f_{\downarrow}^{\dagger} |v\rangle & \quad f_{\uparrow}^{\dagger} |v\rangle
\end{align*}$$

- $c_\alpha = f_{\alpha} b^{\dagger}$
- $\vec{S} = \frac{1}{2} f_{\alpha}^{\dagger} \sigma_{\alpha\beta} f_{\beta}$
- $f_{\alpha}^{\dagger} f_{\alpha} + b^{\dagger} b = 1$

$U(1)$ gauge invariance,

$$b \rightarrow b e^{i\phi}, \quad f_{\alpha} \rightarrow f_{\alpha} e^{i\phi}$$

The physical electron ($c_\alpha$) and spin ($\vec{S}$) operators are rotations in this $SU(1|2)$ superspin space.
\[ H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Each site has 3 states which we map to the ‘superspin’ space of a fermion \( f \) (the holon) and a boson \( b_{\alpha} \) (the spinon):

\[
\begin{align*}
\hat{f}^\dagger |v\rangle & \quad \hat{b}_{\alpha}^\dagger |v\rangle & \quad \hat{b}_{\alpha}^\dagger |v\rangle \\
\end{align*}
\]

\[
\begin{align*}
c_{\alpha} &= b_{\alpha} f^\dagger \\
\vec{S} &= \frac{1}{2} b_{\alpha}^\dagger \sigma_{\alpha\beta} b_{\beta} \\
b_{\alpha}^\dagger b_{\alpha} + f^\dagger f &= 1 \\
U(1) \text{ gauge invariance,} & \quad f \rightarrow f e^{i\phi}, \quad b_{\alpha} \rightarrow b_{\alpha} e^{i\phi}
\end{align*}
\]

The physical electron (\( c_{\alpha} \)) and spin (\( \vec{S} \)) operators are rotations in this SU(2|1) superspin space.
Random $t$-$J$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Each site has 3 states which we map to the ‘superspin’ space of a fermion $f$ (the holon) and a boson $b_{\alpha}$ (the spinon):

$$f^{\dagger} |v\rangle \quad b_{\alpha}^{\dagger} |v\rangle \quad b_{\alpha}^{\dagger} |v\rangle$$

$$c_{\alpha} = b_{\alpha} f^{\dagger}$$

$$\vec{S} = \frac{1}{2} b_{\alpha}^{\dagger} \sigma_{\alpha\beta} b_{\beta}$$

$$b_{\alpha}^{\dagger} b_{\alpha} + f^{\dagger} f = 1$$

U(1) gauge invariance,

$$f \rightarrow f e^{i\phi}, \quad b_{\alpha} \rightarrow b_{\alpha} e^{i\phi}$$

The physical electron ($c_{\alpha}$) and spin ($\vec{S}$) operators are rotations in this SU(2|1) superspin space.
Both the $t$ and the $J$ terms involve four single-particle operators. Consequently, in a large $M$ limit, the saddle-point equations are very similar to the $q = 4$ SYK equations. These equations realize a critical phase with SYK criticality, provided none of the bosons condense.

Assuming the bosons are not condensed, we obtain SYK-like equations for the boson and fermion Green's functions:

$$G_b(i\omega_n) = \frac{1}{i\omega_n + \mu_b - \Sigma_b(i\omega_n)}$$
$$\Sigma_b(\tau) = -t^2 G_f(\tau) G_f(-\tau) G_b(\tau)$$
$$G_f(i\omega_n) = \frac{1}{i\omega_n + \mu_f - \Sigma_f(i\omega_n)}$$
$$\Sigma_f(\tau) = -J^2 G_f^2(\tau) G_f(-\tau) + k t^2 G_f(\tau) G_b(\tau) G_b(-\tau)$$

Here $\mu_f$ and $\mu_b$ are chemical potentials chosen to satisfy

$$\langle f^\dagger f \rangle = \frac{1}{2} - k\delta \, , \, \langle b^\dagger b \rangle = \delta.$$
SYK criticality of fractionalized excitations

\[
\langle \vec{S}_i(\tau) \cdot \vec{S}_i(0) \rangle \sim \frac{1}{|\tau|}
\]

\[
\langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \rangle \sim \frac{1}{\tau}
\]
Random $t$-$J$ model: large $M$ limit

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Zeroth order, $p_c = 1/3$

$p_c \quad \rho$

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Random $t$-$J$ model: large $M$ limit

SYK criticality of fractionalized excitations

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Disordered Fermi liquid.
Condense holon $b$, $f_\alpha$ carrier density $1 + p$

$$f_\uparrow^\dagger |v\rangle \quad f_\downarrow^\dagger |v\rangle$$

$$b^\dagger |v\rangle$$

$$\langle \vec{S}_i(\tau) \cdot \vec{S}_i(0) \rangle \sim \frac{1}{\tau^2}$$

$$\langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \rangle \sim \frac{1}{\tau}$$

Zeroth order, $p_c = 1/3$

$p_c$

$p$

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Random $t$-$J$ model: large $M$ limit

**SU(2|1) theory**

Metallic spin glass. Condense spinon $b_\alpha$, $f$ carrier density $p$

$$f^\dagger |v\rangle$$

$$b_\uparrow^\dagger |v\rangle, b_\downarrow^\dagger |v\rangle$$

$$\langle \vec{S}_i(\tau) \cdot \vec{S}_i(0) \rangle \sim \text{constant}$$

$$\langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \rangle \sim \frac{1}{\tau}$$

**SU(1|2) theory**

Disordered Fermi liquid. Condense holon $b$, $f_\alpha$ carrier density $1 + p$

$$f_\uparrow^\dagger |v\rangle, f_\downarrow^\dagger |v\rangle$$

$$b_\uparrow^\dagger |v\rangle$$

$$\langle \vec{S}_i(\tau) \cdot \vec{S}_i(0) \rangle \sim \frac{1}{\tau^2}$$

$$\langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \rangle \sim \frac{1}{\tau}$$

Zeroth order, $p_c = 1/3$

\[ p \]

D. Joshi, Chenyuan Li, G. Tarnopolsky, A. Georges, S. Sachdev, PRX 10, 021033 (2020)
The random $t$-$J$ model has

- Spin glass order for $p < p_c$.
- Fermi liquid with Luttinger volume Fermi surface for $p > p_c$.
- Maxima in entropy, specific heat, and entanglement entropy near $p = p_c$.
- SYK-Planckian criticality near $p_c$.
- ‘Marginal’ spin susceptibility near criticality, with boundary graviton correction ‘observed’ in SU(2) model.
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Partial understanding of these properties is achieved by a theory of SYK criticality of bosonic and fermionic partons.