Quantum Criticality


Talk online: sachdev.physics.harvard.edu
What is a quantum phase transition?

Non-analyticity in ground state properties as a function of some control parameter $g$
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True level crossing: Usually a first-order transition
What is a quantum phase transition?

Non-analyticity in ground state properties as a function of some control parameter $g$

True level crossing: Usually a \textit{first}-order transition

Avoided level crossing which becomes sharp in the infinite volume limit: \textit{second}-order transition
Many levels are important near a second-order quantum phase transition
Why study quantum phase transitions?

- The ground states at large and small $g$ have wavefunctions which can usually be written as products of wavefunctions of local degrees of freedom i.e. they have negligible quantum entanglement. The quantum critical state at $g_c$ often has long-range quantum entanglement: the “spooky” non-local quantum correlations pointed out by Einstein, Podolsky, and Rosen survive in a macrosopic system at the longest distances.
Why study quantum phase transitions?

- We are often able to describe the quantum state at $g_c$ by methods drawn from quantum field theory; expansion in $g-g_c$ then allows for a controlled theory in an intermediate coupling regime important for many experimental systems.
Why study quantum phase transitions?

- The quantum critical point controls properties over a wide regime of “quantum criticality” at non-zero temperatures. I will argue that this regime is the key to understanding the physical properties of a variety of modern electronic materials.
Outline

1. The quantum Ising chain
   A. The magnetic insulator CoNb$_2$O$_6$
   B. Ultracold Rb atoms in an optical lattice

2. Nonzero temperatures and quantum criticality
   Antiferromagnetic insulators

3. Higher temperature superconductors and “strange metals”
   Quantum criticality of fermions and Fermi surfaces
1. The quantum Ising chain
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Degrees of freedom: $j = 1 \ldots N$ qubits, $N$ "large"

$|\uparrow\rangle_j, |\downarrow\rangle_j$

or $|\rightarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j + |\downarrow\rangle_j \right)$, $|\leftarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j - |\downarrow\rangle_j \right)$
Degrees of freedom: \( j = 1 \ldots N \) qubits, \( N \) "large"

\[
\begin{align*}
|\uparrow\rangle_j , & \quad |\downarrow\rangle_j \\
\text{or} \quad |\rightarrow\rangle_j = \frac{1}{\sqrt{2}} (|\uparrow\rangle_j + |\downarrow\rangle_j) , \quad & \quad |\leftarrow\rangle_j = \frac{1}{\sqrt{2}} (|\uparrow\rangle_j - |\downarrow\rangle_j)
\end{align*}
\]

Hamiltonian of decoupled qubits:

\[
H_0 = -Jg \sum_j \sigma_j^x
\]
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z \]
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma^z_j \sigma^z_{j+1} \]

\[
\left( \langle \rightarrow \rangle_j \langle \leftrightarrow | + | \leftrightarrow \rangle_j \langle \rightarrow | \right) \left( \langle \rightarrow \rangle_{j+1} \langle \leftrightarrow | + | \leftrightarrow \rangle_{j+1} \langle \rightarrow | \right)
\]
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z \]

Prefers neighboring qubits are either \( \uparrow \rangle_j \uparrow \rangle_{j+1} \) or \( \downarrow \rangle_j \downarrow \rangle_{j+1} \) (not entangled)
Full Hamiltonian

\[ H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right) \]
**Full Hamiltonian**

\[
H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)
\]

Product state for large \( g \)
Full Hamiltonian

\[
H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)
\]

Product state for small \( g \)

or

\[
\begin{align*}
|\uparrow\rangle_1 & |\uparrow\rangle_2 \cdots |\uparrow\rangle_j \cdots |\uparrow\rangle_{N-1} & |\uparrow\rangle_N \\
|\downarrow\rangle_1 & |\downarrow\rangle_2 \cdots |\downarrow\rangle_j \cdots |\downarrow\rangle_{N-1} & |\downarrow\rangle_N
\end{align*}
\]
Full Hamiltonian

$$H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)$$

Entangled state at quantum critical point, involving complicated superposition of $2^N$ qubit configurations.
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   Antiferromagnetic insulators

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Quasi-1D Ising ferromagnet CoNb$_2$O$_6$

Co$^{2+}$ spin chain along c-axis

Ferromagnetic superexchange
\(~ 90^\circ \) bond Co-O-Co
\(~ 20K \sim 2\text{meV} \)

Strong easy-axis (Ising)

30 meV

Single crystal of CoNb$_2$O$_6$
(Oxford image furnace)

Quasi-1D Ising ferromagnet \( \text{CoNb}_2\text{O}_6 \)

Co\(^{2+}\) spin chain along c-axis

Magnetic long-range order Bragg peak

Transverse field \( J_g \) (Tesla)


\(\text{Thursday, May 5, 2011}\)
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Superfluid-insulator transition of $^{87}\text{Rb}$ atoms in a magnetic trap and an optical lattice potential

Mott insulator of $^{87}$Rb atoms in a magnetic trap and an optical lattice potential

Applying an “electric” field to the Mott insulator
Why is there a peak (and not a threshold) when $E = U$?
\[ H = -t \sum_{\langle ij \rangle} (b_i^+ b_j + b_j^+ b_i) + \frac{U}{2} \sum_i n_i (n_i - 1) - \sum_i E \cdot r_i n_i \]

\[ n_i = b_i^+ b_i \]

\[ |U - E|, t \ll E, U \]
Resonant transition when $E \approx U$
Virtual state
Virtual state
Resonant transition when $E \approx U$
Resonant transition when $E \approx U$
Phase diagram

Phase diagram

Ising quantum phase transition

Effective Hamiltonian can be written as spin model

\[ \uparrow = \text{up} \]
\[ \downarrow = \text{down} \]
\[ \leftrightarrow = \text{x-y plane} \]
Effective Hamiltonian can be written as spin model

\[ \begin{align*}
\text{up} & = \uparrow \\
\text{down} & = \downarrow \\
x-y \text{ plane} & = \rightarrow
\end{align*} \]
Hamiltonian of resonant subspace

Effective Hamiltonian can be written as spin model

\[ = \uparrow \text{ up} \]
\[ = \downarrow \text{ down} \]
\[ = \rightarrow \text{ x-y plane} \]
Effective Hamiltonian can be written as spin model

\[ \text{up} = \uparrow \]

\[ \text{down} = \downarrow \]

\[ \text{x-y plane} = \rightarrow \]
Effective Hamiltonian can be written as spin model:

- \( \uparrow \) = up
- \( \downarrow \) = down
- \( \rightarrow \) = x-y plane
Effective Hamiltonian can be written as spin model

Constraint:

forbidden!
Include a term \( (J/4) \sum_i (\sigma_i^z - 1) (\sigma_{i+1}^z - 1) \) and send \( J \to \infty \). Infinite exchange interaction!
Effective Hamiltonian can be written as spin model

\[ \Delta = E - U < 0 \]

Paramagnetic state
Effective Hamiltonian can be written as a spin model

Antiferromagnetic state, two fold degenerate
Hamiltonian of resonant subspace

Antiferromagnetic state, two fold degenerate
\[ H = \sum_i \left[ \frac{J}{4} \sigma_i^z \sigma_{i+1}^z + 1 \right. \\
\left. - \frac{h_z}{2} \sigma_i^z - \frac{h_x}{2} \sigma_i^x \right] \]

\[ J \to \infty, \]

\[ h_z = J + (U - E), \]

\[ h_x = 2\sqrt{2t} \]

hx = 0: classical first order phase transition
Finite hx: quantum phase transition, second order
Quantum gas microscope

In-situ imaging of antiferromagnetic chains

In-situ imaging of antiferromagnetic chains

In-situ imaging of antiferromagnetic chains

Antiferromagnetic order

Normalized Neel order

Gradient (E/U) = \frac{1}{g}

Hanbury-Brown-Twiss noise correlations measure Fourier transform of boson density

Hanbury-Brown-Twiss noise correlations measure Fourier transform of boson density

Peaks from antiferromagnetic order

Outline

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The cuprate superconductors
Square lattice antiferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Ground state has long-range Néel order

Order parameter is a single vector field $\vec{\varphi} = \eta_i \vec{S}_i$

$\eta_i = \pm 1$ on two sublattices

$\langle \vec{\varphi} \rangle \neq 0$ in Néel state.
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Weaken some bonds to induce spin entanglement in a new quantum phase.
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state is a “quantum paramagnet” with spins locked in valence bond singlets

\[ = \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow \rangle - |\downarrow \uparrow \rangle \right) \]
\[ \lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
\[ \lambda = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]

An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
TlCuCl$_3$

Quantum paramagnet at ambient pressure
TlCuCl$_3$

Neel order under pressure

\[ \lambda \quad = \quad \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
Excitation spectrum in the paramagnetic phase

Spin $S = 1$
"triplon"
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

“triplon”
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

“triplon”
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

"triplon"
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

"triplon"
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

Spin waves
Triplon in the quantum paramagnet. Spin waves and a new “Higgs” particle in the Néel phase: the latter represents longitudinal oscillations in the magnitude of the Néel order.

Triplon in the quantum paramagnet.

\[ \Delta \sim (\lambda - \lambda_c)^{z\nu} \]

Triplon in the quantum paramagnet.

Spin waves and a new “Higgs” particle in the Néel phase: the latter represents longitudinal oscillations in the magnitude of the Néel order.

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Triplon in the quantum paramagnet.
Spin waves and a new “Higgs” particle in the Néel phase: the latter represents longitudinal oscillations in the magnitude of the Néel order.

Quantum critical point with non-local entanglement in spin wavefunction

\[ \lambda = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Classical spin waves

Dilute triplon gas

Quantum critical


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Triplon energy gap
\[ \Delta \sim (\lambda - \lambda_c)^{z\nu}; \]
crossover at
\[ \Delta \sim k_B T. \]

Pressure in TlCuCl$_3$

Classical Boltzmann theory of triplon particles: Leads to relaxation and thermal equilibration times of order \((\hbar/k_B T)e^{\Delta/k_B T}\)
Neutron scattering measurements of the collisions between triplons in $\text{Y}_2\text{BaNiO}_5$

Neutron scattering measurements of the collisions between triplons in Y$_2$BaNiO$_5$


Theoretical prediction
\[ \hbar \Gamma = 1.20 k_B T e^{\Delta / k_B T} \]

Classical Boltzmann theory of triplon particles:
Leads to relaxation and thermal equilibration times of order $(\hbar/k_B T) e^{\Delta/k_B T}$

Classical spin waves, also with relaxational and equilibration times $\gg \hbar / k_B T$

Classical spin waves

Quantum critical

Dilute triplon gas

Neel order

Pressure in TlCuCl$_3$


Classical spin waves

Quantum critical

Dilute triplon gas

Neel order

Pressure in TlCuCl$_3$
Strongly coupled dynamics and transport with no particle/wave interpretation, and relaxation thermal equilibration times are universally proportional to $\frac{\hbar}{k_B T}$.
Quantum critical transport

Quantum “nearly perfect fluid” with shortest possible relaxation time, $\tau_R$

$$\tau_R = C \frac{\hbar}{k_B T}$$

where $C$ is a universal constant

Non-zero temperature crossovers for the quantum Ising chain

Color density plot of $d\xi^{-1}/dT$, where $\xi$ is the spin correlation length. This quantity measures the strength of the interactions between the thermal excitations.
Neutron scattering measurements on La$_{1.86}$ Sr$_{0.14}$ Cu O$_4$, showing scaling of the dynamic spin susceptibility at an incommensurate wavevector:

$$\chi''(\omega, T) = \frac{A}{T^{2-\eta}} \Phi \left( \frac{\hbar \omega}{k_B T} \right)$$

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The cuprate superconductors

Na-CCOC

- Cu
- Ca/Na
- O
- Cl
Square lattice antiferromagnet

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Ground state has long-range Néel order

Order parameter is a single vector field \( \vec{\varphi} = \eta_i \vec{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\( \langle \vec{\varphi} \rangle \neq 0 \) in Néel state.
Electron-doped cuprate superconductors

La$_{2-x}$Sr$_x$CuO$_4$  Hole-doped

RE$_{2-x}$Ce$_x$CuO$_4$  Electron-doped

T

$T^*$

$T_N$

$\sim 300K$

$\sim 30K$

Hole doping / Sr content (x)

Electron doping / Ce content (x)
Electron-doped cuprate superconductors

Figure prepared by K. Jin and R. L. Greene based on N. P. Fournier, P. Armitage, and R. L. Greene, Rev. Mod. Phys. 82, 2421 (2010).

Resistivity
\[ \rho \approx \rho_0 + AT^n \]
Electron-doped cuprate superconductors

Figure prepared by K. Jin and R. L. Greene based on N. P. Fournier, P. Armitage, and R. L. Greene, Rev. Mod. Phys. 82, 2421 (2010).

Resistivity
\[ \rho \sim \rho_0 + AT^n \]
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Resistivity
\[ \rho \sim \rho_0 + AT^n \]
Iron pnictides: 
a new class of high temperature superconductors

LaOFeAs

BaFe$_2$As$_2$

FeSe

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Iron pnictides:
a new class of high temperature superconductors

Temperature-doping phase diagram of the iron pnictides:

SDW = spin density wave = antiferromagnetism in a metal


Temperature-doping phase diagram of the iron pnictides:

$\text{Resistivity } \sim \rho_0 + AT^\alpha$

$\text{SDW} = \text{spin density wave} = \text{antiferromagnetism in a metal}$


The temperature-doping phase diagram of the iron pnictides:

SDW = spin density wave = antiferromagnetism in a metal

Organic Bechgaard compounds

Resistivity $\sim \rho_0 + AT^\alpha$

Doiron-Leyraud et al., PRB 80, 214531 (2009)
Organic Bechgaard compounds

\[ \rho \sim \rho_0 + AT^\alpha \]

Doiron-Leyraud et al., PRB 80, 214531 (2009)
Organic Bechgaard compounds

Strange Metal

Resistivity $\sim \rho_0 + AT^\alpha$

Doiron-Leyraud et al., PRB 80, 214531 (2009)
Lower $T_c$ superconductivity in the heavy fermion compounds

Fermi surface

Metal with “large” Fermi surface

Momenta with electronic states empty

Momenta with electronic states occupied
The electron spin polarization obeys

\[ \langle \vec{S}(r, \tau) \rangle = \overline{\varphi}(r, \tau)e^{iK \cdot r} \]

where \( K \) is the ordering wavevector.
Metal with “large” Fermi surface
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$. 
Electron and hole pockets in antiferromagnetic phase with $\langle \varphi \rangle \neq 0$
Fermi surface + antiferromagnetism

\[ \langle \varphi \rangle \neq 0 \]

Metal with electron and hole pockets

Increasing interaction

\[ \langle \varphi \rangle = 0 \]

Metal with “large” Fermi surface


Nd$_{2-x}$Ce$_x$CuO$_4$

Quantum oscillations

Quantum oscillations

\[ \text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4 \]

T. Helm, M.V. Kartsovnik, M. Bartkowiak, N. Bittner, M. Lambacher, A. Erb, J. Wosnitza, and R. Gross, 
Fermi surface + antiferromagnetism

Metal with electron and hole pockets

Metal with “large” Fermi surface

Increasing interaction

\[ \langle \varphi \rangle \neq 0 \]

\[ \langle \varphi \rangle = 0 \]


Quantum criticality of antiferromagnetism and Fermi surfaces

Fluctuating Fermi pockets

Quantum Critical

Large Fermi surface

Spin density wave (SDW)

$T^*$

$T$

$\chi_m$
Quantum criticality of antiferromagnetism and Fermi surfaces

Fluctuating Fermi pockets

Quantum Critical

Spin density wave (SDW)

T

T*

Large Fermi surface

Increasing SDW order

Quantum Criticality of antiferromagnetism and Fermi surfaces
Quantum criticality of antiferromagnetism and Fermi surfaces

- Fluctuating Fermi pockets
- Strange Metal
- Large Fermi surface
- Spin density wave (SDW)

\[ T^* \]

\[ x_m \]
Conclusions

Paradigm of quantum phase transitions: the quantum Ising chain, realized in the ferromagnetic insulator CoNb$_2$O$_6$, and ultracold atoms in “tilted” optical lattices

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Conclusions

Described excitations spectrum of the coupled-dimer antiferromagnet TICuCl$_3$. It has regimes of classical dynamics of triplon particles, and of non-linear spin waves, and a regime of quantum-criticality with characteristic equilibration time $\hbar/k_B T$. 

$\hbar/k_B T$
Conclusions

Quantum criticality of antiferromagnetism and Fermi surface reconstruction controls “strange metal” regime of quasi-two dimensional higher temperature superconductors.

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