Confinement-deconfinement transitions in $\mathbb{Z}_2$ gauge theories, and deconfined criticality

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1. $\mathbb{Z}_2$ lattice gauge theory and topological order

2. The Ising* confinement transition

3. “Odd” $\mathbb{Z}_2$ lattice gauge theory and deconfined criticality with an emergent $U(1)$ gauge field

4. $\mathbb{Z}_2$ lattice gauge theory with fermions at half-filling, and deconfined criticality with an emergent $SU(2)$ gauge field
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$\mathbb{Z}_2$ lattice gauge theory

(Wegner, 1971)

\[
H = - \sum \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x
\]

\[
G_i = \begin{array}{c|ccc}
\tau^x & \tau^x & \tau^x \\
\hline
\tau^x & \tau^x & \tau^x \\
\tau^x & \tau^x & \tau^x \\
\end{array}
\]

Gauss’s Law: $[H, G_i] = 0$, $G_i = 1$
\[ W_C = \prod_c \tau^z \]

Deconfined phase
\[ W_C \sim \text{Perimeter Law} \]

Confined phase
\[ W_C \sim \text{Area Law} \]
Topological order

\[ V_x = \prod_{\overline{C}_x} \tau^x, \quad V_y = \prod_{\overline{C}_y} \tau^x \]

\[ W_x = \prod_{C_x} \tau^z, \quad W_y = \prod_{C_y} \tau^z \]

\[ V_x W_y = -W_y V_x, \quad V_y W_x = -W_x V_y \]

and all other pairs commute.

On a torus, there are two additional independent operators, \( V_x \) and \( V_y \) which commute with the Hamiltonian:

\[ [H, V_x] = [H, V_y] = 0 \]

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\[ [H, V_x] = [H, V_y] = 0 \]

Deconfined phase.
4-fold degenerate ground state: \( V_x = \pm 1, V_y = \pm 1 \).
Can take linear combinations to make eigenstates with \( W_x = \pm 1, W_y = \pm 1 \).

Confined phase.
Unique ground state has \( V_x = 1, V_y = 1 \).
No topological order
Topological order

On a torus, there are two additional independent operators, $V_x$ and $V_y$ which commute with the Hamiltonian:

$$[H, V_x] = [H, V_y] = 0$$

and all other pairs commute.

Topological phase has degenerate states with $Z_2$ flux $W = \pm 1$ through the holes of the torus

(N. Read and S.S., 1991)
**Topological order**

Topological order can distinguish the phases when dynamical (or even gapless) matter fields are present.

Deconfined phase.
4-fold degenerate ground state: $V_x = \pm 1$, $V_y = \pm 1$.
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No topological order.

$V_x = \prod_{\overline{C}_x} \tau^x$, $V_y = \prod_{\overline{C}_y} \tau^x$

$W_x = \prod_{\overline{C}_x} \tau^z$, $W_y = \prod_{\overline{C}_y} \tau^z$

$V_x W_y = -W_y V_x$, $V_y W_x = -W_x V_y$

and all other pairs commute.

On a torus, there are two additional independent operators, $V_x$ and $V_y$ which commute with the Hamiltonian:

$[H, V_x] = [H, V_y] = 0$
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4-fold degenerate ground state: $V_x = \pm 1, V_y = \pm 1$.
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Topological order

This criterion can distinguish the phases when dynamical (or even gapless) matter fields are present.

\[
H = -\sum_i \tau_z \tau_z \tau_z \tau_z - g \sum_i \tau_x
\]

G_i = 1

Kramers-Wannier-Wegner duality, Ising* criticality
Embed in a compact $U(1)$ gauge theory

Define

$$\tau \sim e^{iA}$$

and impose $A = 0, \pi$ by adding a potential $\sim -\cos(2A)$. Then make a gauge transformation $A_\mu \rightarrow A_\mu - \Delta_\mu \theta/2$, and make $\theta$ dynamical to make the Hamiltonian gauge invariant. In this manner the $\mathbb{Z}_2$ gauge theory becomes a compact $U(1)$ gauge theory with a charge 2 Higgs field:

$$\mathcal{L} = \sum_i \left[ \frac{1}{2g} \left( \frac{\partial \tilde{A}_i}{\partial \tau} \right)^2 + \frac{1}{2U} \left( \frac{\partial \theta_i}{\partial \tau} \right)^2 \right] - \sum_i \cos(\tilde{A} \times \tilde{A}) - J \sum_i \cos(\tilde{\theta}_i - 2\tilde{A}_i)$$

E. Fradkin and S. Shenker, Phys Rev D 19, 3682 (1979)

Now we take the naive continuum limit with the Higgs field $\Phi \sim e^{i\theta}$, and obtain a theory of complex scalar $\Phi$ coupled to a $U(1)$ gauge field

$$\mathcal{L} = |(\partial_\mu - 2iA_\mu)\Phi|^2 + s|\Phi|^2 + u|\Phi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2$$

However, this turns continuum theory out to be incorrect: we cannot ignore the monopoles is the compact $U(1)$ gauge field, $A_\mu$ near the confinement transition.
Particle-vortex duality
But we proceed anyway, and perform a Dasgupta-Halperin-Peskin particle-vortex duality on $\mathcal{L}$. This requires a complex scalar $\phi$ which creates a vortex with flux $\pi$, and the dual theory is

$$\widetilde{\mathcal{L}} = |\partial_\mu \phi|^2 + \tilde{s}|\phi|^2 + \tilde{u}|\phi|^4.$$ 

Now we have to impose the requirement that vortices with flux $\pi$ and flux $-\pi$ are the same i.e. allow $2\pi$ monopoles to be created from the vacuum. This modifies the Lagrangian to

$$\widetilde{\mathcal{L}} = |\partial_\mu \phi|^2 + \tilde{s}|\phi|^2 + \tilde{u}|\phi|^4 - \lambda(\phi^2 + (\phi^*|^2).$$

Finally, we write $\phi = \sigma + iv\vartheta$. The field $\sigma$ has a smaller mass then $\vartheta$, and so we can integrated out $\vartheta$ to obtain the final correct dual theory

$$\widetilde{\mathcal{L}} = (\partial_\mu \sigma)^2 + \tilde{s}\sigma^2 + \tilde{u}\sigma^4.$$ 

This is the promised dual Ising* theory of the confinement-deconfinement transition. The $\sigma$ field creates the ‘vison’ particle. The * refers to the fact that a single vison cannot be created locally, and this changes some topological properties on compact spaces.
Deconfined phase.
4-fold degenerate ground state: $V_x = \pm 1$, $V_y = \pm 1$.
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This criterion can distinguish the phases when dynamical (or even gapless) matter fields are present

$$H = -\sum \tau^z \tau^z \tau^z \tau^z - g \sum \tau^x$$
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Symmetry-enriched topological (SET) order

\[ H = - \sum \tau^z \tau^z \tau^z \tau^z - g \sum_a \tau^x, \quad G_i = -1 \]
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Topological order

Confined phase.
Broken symmetry and valence bond solid (VBS) order

Symmetry-enriched topological (SET) order and deconfined criticality

\[
H = - \sum \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x , \quad G_i = -1
\]

Deconfined quantum criticality with a U(1) gauge theory and a charge 2 complex scalar

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Topological order

Berry phases suppress monopoles at the critical point

Embedding the \( \mathbb{Z}_2 \) gauge theory in a compact U(1) gauge theory as before, the \( G_i = -1 \) background charges lead to a source term for \( A_\tau \) (a Polyakov loop)

\[
\mathcal{L} = \sum_i \left[ \frac{1}{2g} \left( \frac{\partial \vec{A}_i}{\partial \tau} \right)^2 + \frac{1}{2U} \left( \frac{\partial \theta_i}{\partial \tau} \right)^2 \right] - \sum_i \cos(\vec{\Delta} \times \vec{A}) - J \sum_i \cos(\vec{\Delta} \theta_i - 2\vec{A}_i) \\
+ i \sum_i (-1)^{i_x+i_y} A_{i\tau}
\]

Performing the Dasgupta-Halperin duality transform directly on this lattice model with the source term, we now find a dual vortex theory in which only *quadrupled* monopoles are permitted.

\[
\widetilde{\mathcal{L}} = |\partial_\mu \phi|^2 + \widetilde{s}|\phi|^2 + \widetilde{u}|\phi|^4 - \lambda_4 (\phi^8 + (\phi^*)^8).
\]

The \( \lambda_4 \) coupling is known to be irrelevant at the (Wilson-Fisher) critical point, and so monopoles can be ignored in the critical theory! Undualizing back to the original theory, this means that it is now valid to take the naive continuum limit of \( \mathcal{L} \) to obtain the deconfined critical theory with a U(1) gauge field

\[
\mathcal{L} = |(\partial_\mu - 2i A_\mu) \Phi|^2 + s|\Phi|^2 + u|\Phi|^4 + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2.
\]
\[ \tilde{\mathcal{L}} = |\partial_\mu \phi|^2 + \tilde{s}|\phi|^2 + \tilde{u}|\phi|^4 - \lambda_4 (\phi^8 + (\phi^*)^8). \]
Symmetry-enriched topological (SET) order and deconfined criticality

\[ H = - \sum \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x, \quad G_i = -1 \]

Deconfined quantum criticality with a U(1) gauge theory and a charge 2 complex scalar

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Fermionic matter at half filling

\[ H = - \sum \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x - t \sum_{\langle ij \rangle} \psi_{i\alpha}^\dagger \tau_i^z \psi_{j\alpha} \]

**Deconfined phase.**
Massless Dirac fermions
Topological order

**Confined phase.**
Fermion pairing and superconductivity

\[ S. \text{Gazit, M. Randeria, and A. Vishwanath, } \textit{Nature Physics} \ 13, \ 484 \ (2017) \]
Fermionic matter at half filling

\[ H = - \sum_\square \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x - t \sum_{\langle ij \rangle} \psi^\dagger_{i\alpha} \tau^z_{ij} \psi_{j\alpha} \]

Deconfined quantum criticality with a SU(2) gauge theory and a critical SO(3) Higgs scalar

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S. Gazit, F. F. Assaad, Chong Wang, S. Sachdev, and A. Vishwanath, to appear
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