Topological order in quantum matter

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1. Classical $XY$ model in 2 and 3 dimensions

2. Topological order in the classical $XY$ model in 3 dimensions

3. Topological order in the quantum $XY$ model in 2+1 dimensions

4. Topological order in the Hubbard model
1. Classical XY model in 2 and 3 dimensions

2. Topological order in the classical XY model in 3 dimensions

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4. Topological order in the Hubbard model
\[ Z_{XY} = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\frac{H}{T} \right) \]

\[ H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \]

Describes non-zero \( T \) phase transitions of superfluids, magnets with `easy-plane' spins, …..
In spatial dimension $d = 3$, in the low $T$ phase, the symmetry $\theta_i \to \theta_i + c$ is “spontaneously broken”. There is (off-diagonal) long-range order (LRO) characterized by $|\Psi_i| = e^{i \theta_i}$

$$\lim_{|r_i - r_j| \to \infty} \langle \Psi_i \Psi_j^* \rangle = |\Psi_0|^2 \neq 0.$$

We break the symmetry by choosing an overall phase so that

$$\langle \Psi_i \rangle = \Psi_0 \neq 0$$

**Figure 1:** Schematic picture of ferro- and antiferromagnets. The chequerboard pattern in the antiferromagnet is called a Néel state.

The role of symmetry in physics. Using new experimental techniques, hidden patterns of symmetry were discovered. For example, there are magnetic materials where the moments form a chequerboard pattern where the neighbouring moments are anti-parallel, see Fig. 1. In spite of not having any net magnetization, such antiferromagnets are nevertheless ordered states, and the pattern of microscopic spins can be revealed by neutron scattering. The antiferromagnetic order can again be understood in terms of the associated symmetry breaking.

In a mathematical description of ferromagnetism, the important variable is the magnetization, $\mathbf{m}_i = \mu \mathbf{S}_i$, where $\mu$ is the magnetic moment and $\mathbf{S}_i$ the spin on site $i$. In an ordered phase, the average value of the spin site is different from zero, $\langle \mathbf{m}_i \rangle \neq 0$. The magnetization is an example of an order parameter, which is a quantity that has a non-zero average in the ordered phase. In a crystal it is natural to think of the sites as just the atomic positions, but more generally one can define “block spins” which are averages of spins on many neighbouring atoms. The “renormalization group” techniques used to understand the theory of such aggregate spins are crucial for understanding phase transitions, and resulted in a Nobel Prize for Ken Wilson in 1982.

It is instructive to consider a simple model, introduced by Heisenberg, that describes both ferro- and antiferromagnets. The Hamiltonian is

$$H = J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \mu \sum_i \mathbf{B} \cdot \mathbf{S}_i.$$

**Figure 2:** (Left) Long-range order (LRO). (Right) Short-range order (SRO).
In spatial dimension $d = 3$, in the low $T$ phase, the symmetry $\theta_i \to \theta_i + c$ is “spontaneously broken”. There is (off-diagonal) long-range order (LRO) characterized by ($\Psi_i \equiv e^{i\theta_i}$)

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Wilson-Fisher critical theory (Nobel Prize, 1982)
In spatial dimension $d = 2$, the symmetry $\theta_i \rightarrow \theta_i + c$ is preserved at all non-zero $T$. There is no LRO, and

$$\langle \Psi_i \rangle = 0 \text{ for all } T > 0.$$ 

Nevertheless, there is a phase transition at $T = T_{KT}$, where the nature of the correlations changes

$$\lim_{|r_i - r_j| \rightarrow \infty} \langle \Psi_i \Psi_j^* \rangle \sim \begin{cases} 
|r_i - r_j|^{-\alpha}, & \text{for } T < T_{KT}, \text{(QLRO)} \\
\exp(-|r_i - r_j|/\xi), & \text{for } T > T_{KT}, \text{(SRO)} 
\end{cases}$$

The low $T$ phase also has topological order associated with the suppression of vortices.
Kosterlitz-Thouless theory in $d=2$

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3. Topological order in the quantum XY model in 2+1 dimensions

4. Topological order in the Hubbard model
Can we modify the XY model Hamiltonian to obtain a phase with “topological order” in d=3?

\[ \langle \Psi_i \rangle = 0 \]

SRO

LRO

\[ \langle \Psi_i \rangle \neq 0 \]

J

J_c

\[ \Psi_0 \neq 0 \]
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It is instructive to consider a simple model, introduced by Heisenberg, that describes both ferro- and antiferromagnets. The Hamiltonian is:

\[
H_F = J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \mu K \sum_i \mathbf{B} \cdot \mathbf{S}_i
\]

\[ \langle \Psi_i \rangle = 0 \]

SRO

Topological order

\[ \langle \Psi_i \rangle = 0 \]

SRO

No topological order

\[ \langle \Psi_i \rangle = \Psi_0 \neq 0 \]

LRO

Figure 1: Schematic picture of ferro- and antiferromagnets. The chequerboard pattern in the antiferromagnet is called a Néel state.

The role of symmetry in physics. Using new experimental techniques, hidden patterns of symmetry were discovered. For example, there are magnetic materials where the moments form a chequerboard pattern where the neighbouring moments are anti-parallel, see Fig. 1. In spite of not having any net magnetization, such antiferromagnets are nevertheless ordered states, and the pattern of microscopic spins can be revealed by neutron scattering. The antiferromagnetic order can again be understood in terms of the associated symmetry breaking.
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Figure 1:

Schematic picture of ferro- and antiferromagnets. The chequerboard pattern in the antiferromagnet is called a Néel state.
\[
\tilde{Z}_{XY} = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\tilde{H}/T \right)
\]
\[
\tilde{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + \ldots .
\]

Add terms which suppress single but not double vortices…. 
\[
\tilde{Z}_{XY} = \sum_{\{\sigma_{ij}\} = \pm 1} \prod_{i} \int_{0}^{2\pi} \frac{d\theta_i}{2\pi} \exp \left(-\tilde{H}/T\right)
\]

\[
\tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left[ \left( \theta_i - \theta_j \right)/2 \right] - K \sum_{\Box (ij) \in \Box} \prod \sigma_{ij}
\]
\[
\tilde{Z}_{\text{XY}} = \sum_{\{\sigma_{ij}\} = \pm 1} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\tilde{H} / T \right)
\]

\[
\tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left[ (\theta_i - \theta_j) / 2 \right] - K \sum_{\square} \prod_{(ij) \in \square} \sigma_{ij}
\]

- With \( K = 0 \), we can explicitly sum over \( \sigma_{ij} \), and the theory reduces to the ordinary XY model.
\[ \tilde{Z}_{XY} = \sum_{\{\sigma_{ij}\} = \pm 1} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\frac{\tilde{H}}{T} \right) \]

\[ \tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left[ \left( \theta_i - \theta_j \right)/2 \right] - K \sum_{\square} \prod_{(ij) \in \square} \sigma_{ij} \]

• The theory has a \( \mathbb{Z}_2 \) gauge invariance: we can change

\[ \theta_i \rightarrow \theta_i + \pi(1 - \eta_i) \]

\[ \sigma_{ij} \rightarrow \eta_i \sigma_{ij} \eta_j , \]

with \( \eta_i = \pm 1 \), and the energy remains unchanged.

• The XY order parameter \( \Psi_i = \epsilon^{i\theta_i} \) is gauge invariant, as are all physical observables. So this is an XY model with a modified Hamiltonian, and no additional degrees of freedom have been introduced.
\[ \tilde{Z}_{XY} = \sum_{\{\sigma_{ij}\} = \pm 1} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\tilde{H}/T \right) \]

\[ \tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left[ (\theta_i - \theta_j)/2 \right] - K \sum_{\square} \prod_{(ij) \in \square} \sigma_{ij} \]

- A single (odd) $2\pi$ vortex in $\theta_i$ has
  \[ \prod_{(ij) \in \square} \cos \left[ (\theta_i - \theta_j)/2 \right] < 0. \]

- So for $J > 0$, such a vortex will prefer $\prod_{(ij) \in \square} \sigma_{ij} = -1$, i.e. a $2\pi$ vortex has $\mathbb{Z}_2$ flux $= -1$ in its core.

- So a large $K > 0$ will suppress (odd) $2\pi$ vortices.

- There is no analogous suppression of (even) $4\pi$ vortices.
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\[
\widetilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left( \frac{(\theta_i - \theta_j)}{2} \right) - K \sum_{\square} \prod_{(ij) \in \square} \sigma_{ij}
\]

The diagram illustrates the phase diagram for this model, with the order parameter \( \langle \Psi_i \rangle \) and the interaction parameters \( J \) and \( K \). The phase transition occurs at \( J_c \), where the order parameter changes from zero to a non-zero value, indicating a loss of long-range order (LRO) to short-range order (SRO) or vice versa. The topological order is also indicated, with different phases depending on the values of \( K \) and \( J \).
\begin{align*}
\tilde{Z}_{XY} &= \sum_{\{\sigma_{ij}\} = \pm 1} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left(-\tilde{H}/T\right) \\
\tilde{H} &= -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left[(\theta_i - \theta_j)/2\right] - K \sum_{\square} \prod_{(ij) \in \square} \sigma_{ij}
\end{align*}

Deconfined phase of $Z_2$ gauge theory. $Z_2$ flux is expelled

Higgs phase of $Z_2$ gauge theory

Confined phase of $Z_2$ gauge theory. $Z_2$ flux fluctuates

$\langle \Psi_i \rangle = 0$

$\langle \Psi_i \rangle = \Psi_0 \neq 0$
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A quantum Hamiltonian in 2+1 dimensions

\[ \tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij}^z \cos [(\theta_i - \theta_j)/2] - K \prod_{(ij) \in \square} \sigma_{ij}^z \]

\[ -U \sum_i \frac{\partial^2}{\partial \theta_i^2} - g \sum_{\langle ij \rangle} \sigma_{ij}^x \]
A quantum Hamiltonian in 2+1 dimensions

\[ \tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij}^z \cos \left[ \left( \theta_i - \theta_j \right)/2 \right] - K \sum_{\square (ij) \in \square} \prod \sigma_{ij}^z \]

\[ -U \sum_i \frac{\partial^2}{\partial \theta_i^2} - g \sum_{\langle ij \rangle} \sigma_{ij}^x \]

- In the topological phase, the suppressed \( Z_2 \) fluxes of -1 become well-defined gapped quasiparticle excitations (‘visons’) above the ground state.
A quantum Hamiltonian in 2+1 dimensions

\[ \tilde{H} = -J \sum_{\langle ij \rangle} \sigma^z_{ij} \cos \left[ \frac{(\theta_i - \theta_j)}{2} \right] - K \sum_{\Box} \prod_{(ij) \in \Box} \sigma^z_{ij} \]

\[ -U \sum_i \frac{\partial^2}{\partial \theta_i^2} - g \sum_{\langle ij \rangle} \sigma^x_{ij} \]

- In the topological phase, on a torus, inserting the $\mathbb{Z}_2$ flux of -1 into one of the cycles of the torus leads to an orthogonal state whose energy cost vanishes exponentially in the size of the torus: there are 4 degenerate ground states on a large torus.
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Figure: K. Fujita and J. C. Seamus Davis

$\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$
Figure: K. Fujita and J. C. Seamus Davis

\[ \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \]

d-wave superconductor
Figure: K. Fujita and J. C. Seamus Davis
“Undoped” insulating anti-ferromagnet
Anti-ferromagnet with $p$ mobile holes per square
But relative to the band insulator, there are $1 + p$ holes per square.
In a conventional metal (a Fermi liquid), with no broken symmetry, the area enclosed by the Fermi surface must be $1+p$.

Anti-ferromagnet with $p$ mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square.
A conventional metal: the Fermi liquid with Fermi surface of size \( 1 + p \).
Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size $p$ and not $1+p$. 

Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size $p$ and not $1+p$. If present at $T=0$, a metal with a size $p$ Fermi surface (and translational symmetry preserved) must have topological order.
The Hubbard Model

$$H = - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

$t_{ij}$ → “hopping”. $U$ → local repulsion, $\mu$ → chemical potential

Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha}$$

$$c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta}^\dagger c_{i\alpha} = \delta_{ij}\delta_{\alpha\beta}$$

$$c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0$$

Will study on the square lattice
Fermi surface + antiferromagnetism

Electron spin

\[ \langle \mathbf{S}_i \rangle = \eta_i \mathbf{\Phi}_i \]

where \( \eta_i = \pm 1 \)

\[ \langle \mathbf{\Phi} \rangle \neq 0 \]

Insulator with AF order

\[ \langle \mathbf{\Phi} \rangle = 0 \]

Metal with “large” Fermi surface of size 1
We can (exactly) transform the Hubbard model to the “spin-fermion” model: electrons $c_{i\alpha}$ on the square lattice with dispersion

$$\mathcal{H}_c = -\sum_{i,\rho} t_\rho \left( c_{i,\alpha}^\dagger c_{i+\mathbf{v}_\rho,\alpha} + c_{i+\mathbf{v}_\rho,\alpha}^\dagger c_{i,\alpha} \right) - \mu \sum_i c_{i,\alpha}^\dagger c_{i,\alpha} + \mathcal{H}_{\text{int}}$$

are coupled to an antiferromagnetic order parameter $\Phi^\ell(i)$, $\ell = x, y, z$

$$\mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i \Phi^\ell(i) c_{i,\alpha}^\dagger \sigma^\ell_{\alpha\beta} c_{i,\beta} + V_\Phi$$

where $\eta_i = \pm 1$ on the two sublattices. (For suitable $V_\Phi$, integrating out the $\Phi$ yields back the Hubbard model).
We can (exactly) transform the Hubbard model to the “spin-fermion” model: **electrons** \( c_{i\alpha} \) on the square lattice with dispersion

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are coupled to an **antiferromagnetic order parameter** \( \Phi^\ell(i) \), \( \ell = x, y, z \)

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\mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i \Phi^\ell(i) c_{i,\alpha}^\dagger \sigma^\ell_{\alpha\beta} c_{i,\beta} + V_\Phi
\]

where \( \eta_i = \pm 1 \) on the two sublattices. (For suitable \( V_\Phi \), integrating out the \( \Phi \) yields back the Hubbard model).

When \( \Phi^\ell(i) = \) (non-zero constant) independent of \( i \), we have long-range AF order, which gaps out the fermions into an insulating state.
For fluctuating antiferromagnetism, we transform to a rotating reference frame using the SU(2) rotation $R_i$

$$
\begin{pmatrix}
c_{i\uparrow} \\
c_{i\downarrow}
\end{pmatrix} = R_i \begin{pmatrix}
\psi_{i,+} \\
\psi_{i,-}
\end{pmatrix},
$$

in terms of fermionic “chargons” $\psi_s$ and a **Higgs field** $H^a(i)$

$$
\sigma^\ell \Phi^\ell(i) = R_i \sigma^a H^a(i) R^\dagger_i
$$

The Higgs field is the AFM order in the rotating reference frame.
For fluctuating antiferromagnetism, we transform to a rotating reference frame using the SU(2) rotation $R_i$

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$$
\sigma^\ell \Phi^\ell(i) = R_i \sigma^a H^a(i) R_i^\dagger
$$

The Higgs field is the AFM order in the rotating reference frame. Note that this representation is ambiguous up to a SU(2) gauge transformation, $V_i$

$$
\begin{pmatrix}
  \psi_{i,+} \\
  \psi_{i,-}
\end{pmatrix}
\rightarrow V_i \begin{pmatrix}
  \psi_{i,+} \\
  \psi_{i,-}
\end{pmatrix}
\quad R_i \rightarrow R_i V_i^\dagger
\quad \sigma^a H^a(i) \rightarrow V_i \sigma^b H^b(i) V_i^\dagger.
$$
The simplest effective Hamiltonian for the fermionic chargons is the same as that for the electrons, with the AFM order replaced by the Higgs field.

\[ \mathcal{H}_\psi = - \sum_{i, \rho} t_\rho \left( \psi_{i,s}^{\dagger} \psi_{i+\nu_\rho,s} + \psi_{i+\nu_\rho,s}^{\dagger} \psi_{i,s} \right) - \mu \sum_i \psi_{i,s}^{\dagger} \psi_{i,s} + \mathcal{H}_{\text{int}} \]

\[ \mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i H^a(i) \psi_{i,s}^{\dagger} \sigma^a_{ss'} \psi_{i,s'} + V_H \]
The simplest effective Hamiltonian for the fermionic chargons is the same as that for the electrons, with the AFM order replaced by the Higgs field.

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\]

\[
\mathcal{H}_{\text{int}} = -\lambda \sum_i \eta_i H^a(i) \psi_{i,s}^\dagger \sigma_{ss'}^a \psi_{i,s'} + V_H
\]

**IF** we can transform to a rotating reference frame in which \( H^a(i) = \) a constant independent of \( i \) and time, **THEN** the \( \psi \) fermions in the presence of fluctuating AFM will inherit the insulating nature of the electrons in the presence of static AFM.
Fluctuating antiferromagnetism

We cannot always find a single-valued SU(2) rotation $R_i$ to make the Higgs field $H^a(i)$ a constant!

Fluctuating antiferromagnetism

We cannot always find a single-valued SU(2) rotation $R_i$ to make the Higgs field $H^a(i)$ a constant!

$n$-fold vortex in AFM order

A.V. Chubukov, T. Senthil and S. Sachdev, PRL 72, 2089 (1994);
S. Sachdev, E. Berg, S. Chatterjee, and Y. Schattner, PRB 94, 115147 (2016)
Topological order

We cannot always find a single-valued SU(2) rotation $R_i$ to make the Higgs field $H^a(i)$ a constant!

Vortices with $n$ odd must be suppressed: such a metal with “fluctuating antiferromagnetism” has BULK $\mathbb{Z}_2$ TOPOLOGICAL ORDER and fermions which inherit the insulating behavior of the antiferromagnetic metal.
Fermi surface + antiferromagnetism

Electron spin
\[
\langle \vec{S}_i \rangle = \eta_i \vec{\Phi}_i
\]
where \( \eta_i = \pm 1 \)

\[ p=0 \]

\[ \langle \vec{\Phi} \rangle \neq 0 \]
Insulator with AF order

Metal with “large” Fermi surface of size 1
Fermi surface + antiferromagnetism + topological order

Insulator with $Z_2$ topological order
Higgs phase of a SU(2) gauge theory
(SU(2) is broken down to $Z_2$)

$\langle \Phi \rangle = 0$

$p=0$

Insulator with AF order

$\langle \Phi \rangle \neq 0$

Metal with “large” Fermi surface of size 1

$U/t$
Using new experimental techniques, hidden patterns of symmetry were discovered. For example, there are magnetic materials where the moments form a chequerboard pattern where the neighbouring moments are anti-parallel, see Fig. 1. In spite of not having any net magnetization, such antiferromagnets are nevertheless ordered states, and the pattern of microscopic spins can be revealed by neutron scattering. The antiferromagnetic order can again be understood in terms of the associated symmetry breaking.

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The Hamiltonian is given by:

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\tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left( \frac{\theta_i - \theta_j}{2} \right) - K \sum_{(ij) \in \square} \prod_{\square} \sigma_{ij}
\]

This Hamiltonian describes both ferro- and antiferromagnets. It is instructive to consider a simple model, introduced by Heisenberg, that describes both ferro- and antiferromagnets. The Hamiltonian is given by:

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\tilde{H}_{XY} = \sum_{\{\sigma_{ij}\} = \pm 1} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp \left( -\frac{\tilde{H}}{T} \right)
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\tilde{H} = -J \sum_{\langle ij \rangle} \sigma_{ij} \cos \left( \frac{\theta_i - \theta_j}{2} \right) - K \sum_{(ij) \in \square} \prod_{\square} \sigma_{ij}
\]

Deconfined phase of \( Z_2 \) gauge theory. \( Z_2 \) flux is expelled.

Higgs phase of \( Z_2 \) gauge theory

\[ \langle \Psi_i \rangle = 0 \]

\[ \langle \Psi_i \rangle = \Psi_0 \neq 0 \]

Confined phase of \( Z_2 \) gauge theory. \( Z_2 \) flux fluctuates.

\[ \langle \Psi_i \rangle = 0 \]
Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size \( p \) and not \( 1 + p \).

If present at \( T=0 \), a metal with a size \( p \) Fermi surface (and translational symmetry preserved) must have topological order.
Topological order in the pseudogap metal

Mathias S. Scheurer,1 Shubhayu Chatterjee,1 Wei Wu,2,3 Michel Ferrero,2,3 Antoine Georges,2,4,3,5 and Subir Sachdev1,6,7

We compute the electronic Green’s function of the topologically ordered Higgs phase of a SU(2) gauge theory of fluctuating antiferromagnetism on the square lattice. The results are compared with cluster extensions of dynamical mean field theory, and quantum Monte Carlo calculations, on the pseudogap phase of the strongly interacting hole-doped Hubbard model. Good agreement is found in the momentum, frequency, hopping, and doping dependencies of the spectral function and electronic self-energy. We show that lines of (approximate) zeros of the zero-frequency electronic Green’s function are signs of the underlying topological order of the gauge theory, and describe how these lines of zeros appear in our theory of the Hubbard model. We also derive a modified, non-perturbative version of the Luttinger theorem that holds in the Higgs phase.

to appear.....
Kagome lattice antiferromagnet

Candidate for an insulator with $\mathbb{Z}_2$ topological order which has charge neutral, spin $S = 1/2$, “spinon” excitations.


See talk tomorrow by H. Changlani

Herbertsmithite: $\text{ZnCu}_3 \,(\text{OH})_6 \,\text{Cl}_2$
Evidence for a gapped spin-liquid ground state in a kagome Heisenberg antiferromagnet

Mingxuan Fu, Takashi Imai, Tian-Heng Han, Young S. Lee

The kagome Heisenberg antiferromagnet is a leading candidate in the search for a spin system with a quantum spin-liquid ground state. The nature of its ground state remains a matter of active debate. We conducted oxygen-17 single-crystal nuclear magnetic resonance (NMR) measurements of the spin-1/2 kagome lattice in herbertsmithite [ZnCu$_3$(OH)$_6$Cl$_2$], which is known to exhibit a spinon continuum in the spin excitation spectrum. We demonstrated that the intrinsic spin susceptibility $\chi_{\text{kagome}}$ deduced from the oxygen-17 NMR frequency shift, asymptotes to zero below temperatures of 0.03|J|, where J = 200 Kelvin is the copper-copper superexchange interaction. Combined with the magnetic field dependence of $\chi_{\text{kagome}}$, that we observed at low temperatures, these results imply that the kagome Heisenberg antiferromagnet has a spin-liquid ground state with a finite gap.

Gap to topological $S = 1/2$ spinon excitations: $\approx J/20$.

Science 350, 655 (2015)

Neutron scattering observations


Theory by Punk, Chowdhury, Sachdev

Nature Physics, 2013
Gapped Spin-1/2 Spinon Excitations in a New Kagome Quantum Spin Liquid Compound Cu₃Zn(OH)₆FBr

Zili Feng (冯子力)¹, Zheng Li 李政)¹,², Xin Meng 孟鑫)¹, Wei Yi 衣玮)¹, Yuan Wei 魏源)¹, Jun Zhang (张骏)³, Yan-Cheng Wang 王艳成)¹, Wei Jiang (蒋伟)⁴, Zheng Liu (刘峥)⁵, Shiyan Li (李世燕)³,⁶

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Gap to topological $S = 1/2$ spinon excitations: $\approx J/20$.  

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**TABLE I.** Structure parameters of Cu echo as a function of the RF frequency at constant NMR spectra of specific heat) measurements were carried out on the polycrystalline samples by replacing the interkagome intensities. The vertical lines indicate peak positions.

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**FIG. 1.** (a) Schematic crystal structure of Cu. Gapped Spin-1/2 Spinon Excitations in a New Kagome Quantum Spin Liquid Compound Cu₃Zn(OH)₆FBr - IOPscience
New classes of quantum states with topological order

- (a) defect suppression in states with fluctuating order
- (b) Higgs phases of emergent gauge fields

A metal with bulk topological order (i.e. long-range quantum entanglement) can explain existing experiments in cuprates, and agrees well with cluster-DMFT. Direct evidence for topological order in kagome antiferromagnets.
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