Quantum entanglement and the phases of matter

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1. Conformal quantum matter
   Entanglement, emergent dimensions and string theory

2. Compressible quantum matter
   Holography of strange metals
Outline

1. Conformal quantum matter
   Entanglement, emergent dimensions and string theory

2. Compressible quantum matter
   Holography of strange metals
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Examine ground state as a function of \( \lambda \)

S=1/2 spins

Examine ground state as a function of \( \lambda \)
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

At large \( \lambda \) ground state is a “quantum paramagnet” with spins locked in valence bond singlets
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Nearest-neighor spins are “entangled" with each other.
Can be separated into an Einstein-Podolsky-Rosen (EPR) pair.
Square lattice antiferromagnet

\[
H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

For \( \lambda \approx 1 \), the ground state has antiferromagnetic ("Néel") order, and the spins align in a checkerboard pattern.
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No EPR pairs
\[ = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
\[
\lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
\]

TlCuCl$_3$

An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
TlCuCl$_3$

Quantum paramagnet at ambient pressure
TlCuCl$_3$

Neel order under pressure

A. Oosawa, K. Kakurai, T. Osakabe, M. Nakamura, M. Takeda, and H. Tanaka, 
Excitations of TlCuCl$_3$ with varying pressure

Excitations of TlCuCl$_3$ with varying pressure

Broken valence bond excitations of the quantum paramagnet

Excitations of TlCuCl$_3$ with varying pressure

Spin waves above the Néel state

Excitations of $\text{TlCuCl}_3$ with varying pressure

“Higgs” particle appears at theoretically predicted energy

S. Sachdev, arXiv:0901.4103

Quantum critical point with non-local entanglement in spin wavefunction

\[ \lambda = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Tensor network representation of entanglement at quantum critical point

D-dimensional space

depth of entanglement
Long-range entanglement
• Long-range entanglement

• Long distance and low energy correlations near the quantum critical point are described by a quantum field theory which is relativistically invariant (where the spin-wave velocity plays the role of the velocity of “light”).
• Long-range entanglement

• Long distance and low energy correlations near the quantum critical point are described by a quantum field theory which is relativistically invariant (where the spin-wave velocity plays the role of the velocity of “light”).

• The quantum field theory is invariant under scale and conformal transformations at the quantum critical point: a CFT3
• Allows unification of the standard model of particle physics with gravity.

• Low-lying string modes correspond to gauge fields, gravitons, quarks ...
A $D$-brane is a $D$-dimensional surface on which strings can end.

The low-energy theory on a $D$-brane is an ordinary quantum field theory with no gravity.
- A $D$-brane is a $D$-dimensional surface on which strings can end.
- The low-energy theory on a $D$-brane is an ordinary quantum field theory with no gravity.
- In $D = 2$, we obtain strongly-interacting CFTs. These are "dual" to string theory on anti-de Sitter space: $\text{AdS}_4$. 
Tensor network representation of entanglement at quantum critical point

D-dimensional space

String theory near a D-brane

Emergent direction of AdS4
Tensor network representation of entanglement at quantum critical point

Emergent direction of AdS4

Brain Swingle, arXiv:0905.1317

Monday, January 23, 2012
\[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]

\[ \text{Entanglement entropy } S_{EE} = -\text{Tr} (\rho_A \ln \rho_A) \]
Entanglement entropy

D-dimensional space

depth of entanglement
Entanglement entropy

D-dimensional space

Draw a surface which intersects the minimal number of links

Emergent direction of AdS4

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The entanglement entropy of a region $A$ on the boundary equals the minimal area of a surface in the higher-dimensional space whose boundary co-incides with that of $A$.

This can be seen both the string and tensor-network pictures

Brian Swingle, arXiv:0905.1317
AdS$_{d+2}$

Emergent holographic direction

CFT$_{d+1}$

Quantum matter with long-range entanglement
AdS_{d+2} \rightarrow R^{d,1}_{\text{Minkowski}}

Emergent holographic direction

CFT_{d+1}

Quantum matter with long-range entanglement
AdS \_{d+2} \quad \mathbb{R}^{d,1} \\
\text{Minkowski} \\
\text{Emergent holographic direction}

CFT \_{d+1} \\
\text{Quantum matter with long-range entanglement}

Area measures entanglement entropy

Emergent holographic direction
Why $\text{AdS}_{d+2}$?
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation

\[ x_i \rightarrow \zeta x_i, \quad t \rightarrow \zeta t, \quad ds^2 \rightarrow ds^2 \]

This gives the unique metric

\[ ds^2 = \frac{1}{r^2} dt^2 + dr^2 + dx_i^2 \]

Reparametrization invariance in $r$ has been used to the prefactor of $dx_i^2$ equal to $1/r^2$. This fixes $r \rightarrow \zeta r$ under the scale transformation. This is the metric of the space $\text{AdS}_{d+2}$.

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For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation ($i = 1 \ldots d$)

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**Why AdS$_{d+2}$?**
Quantum critical point with non-local entanglement in spin wavefunction

\[ \lambda = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Classical spin waves

Dilute triplon gas

Quantum critical

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Pressure in TlCuCl$_3$

Classical spin waves

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Classical spin waves

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Quantum critical

Neel order

Pressure in TlCuCl$_3$

AdS/CFT correspondence at non-zero temperatures

$\text{AdS}_4$-Schwarzschild black-brane

A 2+1 dimensional system at its quantum critical point
AdS/CFT correspondence at non-zero temperatures

AdS$_4$-Schwarzschild black-brane

Black-brane at temperature of 2+1 dimensional quantum critical system

A 2+1 dimensional system at its quantum critical point
AdS/CFT correspondence at non-zero temperatures

AdS$_4$-Schwarzschild black-brane

- Black-brane at temperature of 2+1 dimensional quantum critical system
- Friction of quantum criticality = waves falling into black brane
- A 2+1 dimensional system at its quantum critical point
A 2+1 dimensional system at its quantum critical point

AdS/CFT correspondence at non-zero temperatures

AdS$_4$-Schwarzschild black-brane

Black-brane at temperature of 2+1 dimensional quantum critical system

Produces successful description of many properties of quantum critical points at non-zero temperatures
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   *Holography of strange metals*
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2. Compressible quantum matter

*Holography of strange metals*
Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved $U(1)$ charge $Q$ (the "electron density") in spatial dimension $d > 1$.

Compressible systems must be gapless.

Conformal quantum matter is compressible in $d = 1$, but not for $d > 1$. 

Compressible quantum matter
Compressible quantum matter

- Consider an infinite, continuum, translationally-invariant quantum system with a globally conserved U(1) charge $Q$ (the "electron density") in spatial dimension $d > 1$.

- Describe zero temperature phases where $d\langle Q \rangle / d\mu \neq 0$, where $\mu$ (the "chemical potential") which changes the Hamiltonian, $H$, to $H - \mu Q$. 

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The only compressible phase of traditional condensed matter physics which does not break the translational or $U(1)$ symmetries is the Landau Fermi liquid.
Conformal quantum matter

Graphene
Compressible quantum matter

Fermi Liquid with a Fermi surface
The only low energy excitations are long-lived quasiparticles near the Fermi surface.

- The *only* low energy excitations are long-lived quasiparticles near the Fermi surface.
• **Luttinger relation:** The total “volume (area)” \( A \) enclosed by the Fermi surface is equal to \( \langle Q \rangle \).
The cuprate superconductors
Electron-doped cuprate superconductors
Electron-doped cuprate superconductors

Superconductor
Bose condensate of pairs of electrons
Short-range entanglement
Electron-doped cuprate superconductors

Figure prepared by K. Jin and R. L. Greene based on N. P. Fournier, P. Armitage, and R. L. Greene, Rev. Mod. Phys. 82, 2421 (2010).

Resistivity \( \sim \rho_0 + AT^n \)
Electron-doped cuprate superconductors

Figure prepared by K. Jin and R. L. Greene based on N. P. Fournier, P. Armitage, and R. L. Greene, Rev. Mod. Phys. 82, 2421 (2010).

Ordinary metal (Fermi liquid)

Resistivity
\[ \sim \rho_0 + AT^n \]
Electron-doped cuprate superconductors

\( \rho_0 + AT^n \)
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Strange Metal

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Resistivity
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Excitations of a ground state with long-range entanglement

Strange Metal

Electron-doped

Resistivity
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Figure prepared by K. Jin and R. L. Greene based on N. P. Fournier, P. Armitage, and R. L. Greene, Rev. Mod. Phys. 82, 2421 (2010).
Temperature-doping phase diagram of the iron pnictides:

\[ \text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2 \]

\[ T \sim \rho_0 + AT^\alpha \]


Temperature-doping phase diagram of the iron pnictides:

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Temperature-doping phase diagram of the iron pnictides:

Temperature-doping phase diagram of the iron pnictides:

\[ T_{SDW} \]

\[ T_{c} \]

\[ \nabla T_{0} \]

\[ \text{Resistivity} \sim \rho_{0} + AT^{\alpha} \]


The simplest example of an exotic compressible phase (a “strange metal”) is realized by fermions with a Fermi surface coupled to an Abelian or non-Abelian gauge field.
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The theory of this strange metal is strongly coupled in two spatial dimensions, and the traditional field-theoretic expansion methods break down.

Study the large N limit of a SU(N) gauge field coupled to adjoint (matrix) fermions at a non-zero chemical potential.
Consider the following (most) general metric for the holographic theory

\[
ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r 2d(z-1)/(d-\theta)} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)\]

This metric transforms under rescaling as 

\[x_i \rightarrow \zeta x_i, \quad t \rightarrow \zeta t, \quad ds \rightarrow \zeta^{\theta/d} ds.\]

This identifies \(z\) as the dynamic critical exponent (\(z = 1\) for "relativistic" quantum critical points).

What is \(\theta\)? (\(\theta = 0\) for "relativistic" quantum critical points).
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This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

What is \( \theta \)? (\( \theta = 0 \) for “relativistic” quantum critical points).
At $T > 0$, there is a “black-brane” at $r = r_h$. 

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$. 

The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$.
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Under rescaling $r \to \zeta^{(d-\theta)/d} r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z}$$

So $\theta$ is the “violation of hyperscaling” exponent.
A non-Fermi liquid has gapless fermionic excitations on the Fermi surface, which disperse in the single transverse direction with dynamic critical exponent $z$. So we expect compressible quantum states to have an effective dimension $d - \theta$ with

$$\theta = d - 1$$

L. Huijse, S. Sachdev, B. Swingle, arXiv:1112.0573
Logarithmic violation of “area law”: \( S_{EE} = \frac{1}{12} (k_F P) \ln(k_F P) \)

for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape.

Logarithmic violation of “area law”: 

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for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape.

Non-Fermi liquids have, at most, the “1/12” prefactor modified.

Holography of non-Fermi liquids

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + r^2\theta/(d-\theta) dr^2 + dx_i^2 \right) \]

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- The entanglement entropy exhibits logarithmic violation of the area law only for this value of \( \theta \)!
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L. Huijse, S. Sachdev, B. Swingle, arXiv:1112.0573
Holography of non-Fermi liquids

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2 d(z-1)/(d-\theta)} + \frac{r^{2\theta}/(d-\theta)}{d} dr^2 + dx_i^2 \right) \]

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- The entanglement entropy exhibits logarithmic violation of the area law only for this value of \( \theta \).
- The co-efficient of the logarithmic term is consistent with the Luttinger relation.
- Many other features of the holographic theory are consistent with a boundary theory which has “hidden” Fermi surfaces of gauge-charged fermions.

L. Huijse, S. Sachdev, B. Swingle, arXiv:1112.0573
Conclusions

Phases of matter with long-range quantum entanglement are prominent in numerous modern materials.
Simplest examples of long-range entanglement are at quantum-critical points of insulating antiferromagnets.
Conclusions

More complex examples in metallic states are experimentally ubiquitous, but pose difficult strong-coupling problems to conventional methods of field theory
Conclusions

String theory and holography offer a remarkable new approach to describing states with long-range quantum entanglement.
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Much recent progress offers hope of a holographic description of “strange metals”