Breakdown of the Landau-Ginzburg-Wilson paradigm at quantum phase transitions


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Talk online at http://sachdev.physics.harvard.edu
Outline

I. Statement of the problem
   A. Antiferromagnets
   B. Boson lattice models

II. Theory of defects: vortices near the superfluid-insulator transition
    Berry phases imply that vortices carry “flavor”

III. The cuprate superconductors
     Detection of vortex flavors?

IV. Defects in the antiferromagnet
    Hedgehog Berry phases and VBS order
I.A Quantum phase transitions of $S=1/2$ antiferromagnets
Ground state has long-range Néel order

Order parameter \( \tilde{\phi} = \eta_i \tilde{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\[ \langle \tilde{\phi} \rangle \neq 0 \]
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \quad ; \quad \vec{S}_i \Rightarrow \text{spin operator with } S=1/2 \]

Destroy Neel order by perturbations which preserve full square lattice symmetry e.g. second-neighbor or ring exchange.

What is the state with \( \langle \bar{\phi} \rangle = 0 \) ?
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What is the state with \( \langle \Phi \rangle = 0 \) ?
LGW theory for such a quantum transition

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\Phi$ by expanding in powers of $\Phi$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian.

$$S_{\phi} = \int d^2xd\tau \left[ \frac{1}{2} \left( (\nabla_x \Phi)^2 + \frac{1}{c^2} (\partial_c \Phi)^2 + r \Phi^2 \right) + \frac{u}{4!} (\Phi^2)^2 \right]$$

For $r > 0$ oscillations of $\Phi$ about $\Phi = 0$ lead to the following structure in the dynamic structure factor $S(p, \omega)$

$$\varepsilon(p) = \Delta + \frac{c^2 p^2}{2\Delta}; \quad \Delta = \sqrt{r/c}$$

Structure holds to all orders in $u$

Three particle continuum

Problem: there is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries
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“Liquid” of valence bonds has fractionalized $S=1/2$ excitations
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“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.
Another possible state, with $\langle \phi \rangle = 0$, is the valence bond solid (VBS)
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Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}} (i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
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Another possible state, with $\langle \tilde{\phi} \rangle = 0$, is the valence bond solid (VBS)

$\Psi = \sum_{ij} G_{ij} \psi_{\text{vbs}}(i)\psi_{\text{vbs}}(j)$

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The VBS state does have a stable $S=1$ quasiparticle excitation

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$$\langle \Psi_{\text{vbs}} \rangle \neq 0, \quad \langle \tilde{\phi} \rangle = 0$$
LGW theory of multiple order parameters

\[ F = F_{\text{vbs}} \left[ \Psi_{\text{vbs}} \right] + F_{\phi} \left[ \phi \right] + F_{\text{int}} \]

\[ F_{\text{vbs}} \left[ \Psi_{\text{vbs}} \right] = r_1 \left| \Psi_{\text{vbs}} \right|^2 + u_1 \left| \Psi_{\text{vbs}} \right|^4 + \cdots \]

\[ F_{\phi} \left[ \phi \right] = r_2 \left| \phi \right|^2 + u_2 \left| \phi \right|^4 + \cdots \]

\[ F_{\text{int}} = v \left| \Psi_{\text{vbs}} \right|^2 \left| \phi \right|^2 + \cdots \]

Distinct symmetries of order parameters permit couplings only between their energy densities.
LGW theory of multiple order parameters

First order transition

Neel order

Coexistence

"disordered"

VBS order

\[ \langle \Psi_{vbs} \rangle \]

\[ \langle \tilde{\phi} \rangle \]

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I.B Quantum phase transitions of boson lattice models
Bose condensation
Velocity distribution function of ultracold $^{87}\text{Rb}$ atoms

Apply a periodic potential (standing laser beams) to trapped ultracold bosons ($^{87}\text{Rb}$)

Momentum distribution function of bosons

Bragg reflections of condensate at reciprocal lattice vectors

Superfluid-insulator quantum phase transition at $T=0$

$V_0=0E_r$  

$V_0=3E_r$  

$V_0=7E_r$  

$V_0=10E_r$  

$V_0=13E_r$  

$V_0=14E_r$  

$V_0=16E_r$  

$V_0=20E_r$
Bosons at filling fraction $f = 1$

Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

Bosons at filling fraction $f = 1$

$\langle \Psi_{sf} \rangle \neq 0$

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Strong interactions: insulator
Bosons at filling fraction $f = 1/2$

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Strong interactions: insulator
Bosons at filling fraction \( f = 1/2 \)

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\langle \Psi_{sf} \rangle = 0
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Strong interactions: insulator
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Strong interactions: insulator

Insulator has “density wave” order
Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ.r}$

All insulators have $\langle \Psi_{sf} \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

**Insulating phases of bosons at filling fraction \( f = 1/2 \)**

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Insulating phases of bosons at filling fraction \( f = 1/2 \)

\[
\frac{1}{\sqrt{2}} \left( \Psi_r + \Psi_s \right) = \rho
\]

Charge density wave (CDW) order

Valence bond solid (VBS) order

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**Insulating phases of bosons at filling fraction \( f = 1/2 \)**

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Predictions of LGW theory

\[
\langle \Psi_{sf} \rangle \quad \text{Superfluid} \quad \langle \rho_0 \rangle \quad \text{Charge-ordered insulator}
\]

First order transition

Coexistence

\[
\langle \Psi_{sf} \rangle \quad \text{(Supersolid)} \quad \langle \rho_0 \rangle \quad \text{Charge-ordered insulator}
\]

"Disordered"

\[
\langle \Psi_{sf} \rangle = 0, \langle \rho_0 \rangle = 0 \quad \text{Charge-ordered insulator}
\]
Predictions of LGW theory

\[ \langle \Psi_{sf} \rangle \]
Superfluid

First order transition

\[ \langle \rho_Q \rangle \]
Charge-ordered insulator

\[ r_1 - r_2 \]

Coexistence
(Supersolid)

\[ \langle \Psi_{sf} \rangle \]
Superfluid

\[ \langle \rho_Q \rangle \]
Charge-ordered insulator

\[ r_1 - r_2 \]

"Disordered"
(\( \neq \) topologically ordered)

\[ \langle \Psi_{sf} \rangle = 0, \langle \rho_Q \rangle = 0 \]

\[ r_1 - r_2 \]

Charge-ordered insulator
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    *Berry phases imply that vortices carry “flavor”*

III. The cuprate superconductors
    *Detection of vortex flavors?*

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    *Hedgehog Berry phases and VBS order*
II. Theory of defects: vortices near the superfluid-insulator transition

*Berry phases imply that vortices carry “flavor”*
Excitations of the superfluid: Vortices

The circulation of a vortex is quantized:

$$\oint v_s \cdot dr = \frac{\hbar}{m} \oint \nabla \theta \cdot dr = n\frac{\hbar}{m}$$

where $n$ is an integer.
Observation of quantized vortices in rotating ultracold Na

Excitations of the superfluid: Vortices

Central question:
In two dimensions, we can view the vortices as point particle excitations of the superfluid. What is the quantum mechanics of these “particles”? 
In ordinary fluids, vortices experience the Magnus Force

\[ F_M = (\text{mass density of air}) \cdot (\text{velocity of ball}) \cdot (\text{circulation}) \]
For a vortex in a superfluid, this is

\[ F_M = (m\rho) \left( \left( v_s - \frac{dr_v}{dt} \right) \times \hat{z} \right) \left( \oint v_s \cdot dr \right) \]

\[ = n\hbar\rho \left( v_s - \frac{dr_v}{dt} \right) \times \hat{z} \]

where \( \rho = \) number density of bosons
\( v_s = \) local velocity of superfluid
\( r_v = \) position of vortex
For a vortex in a superfluid, this is

\[ F_M = (m\rho) \left( \left( v_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \oint v_s \cdot d\mathbf{r} \right) \]

\[ = nh\rho \left( v_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \]

\[ = n \left( \mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right) \]

where \( \mathbf{E} = \rho v_s \times \hat{z} \) and \( \mathbf{B} = -h\rho\hat{z} \)

**Dual picture:**

The vortex is a quantum particle with dual “electric” charge \( n \), moving in a dual “magnetic” field of strength \( = h \times (\text{number density of Bose particles}) \)
• The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual “magnetic” field of strength \( = \hbar \rho \), where \( \rho \) is the number density of bosons per unit cell.

• The vortex motion can be described by the effective Hofstadter Hamiltonian:

\[
\mathcal{H}_v = -t \sum_{\langle ij \rangle} (e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.})
\]

where \( \varphi_i \) is an operator which annihilates a vortex particle at site \( i \) of a square lattice.

\[
A_1 + A_2 + A_3 + A_4 = 2\pi f
\]

where \( f \) is the boson filling fraction.
Bosons at filling fraction $f = 1$

- At $f=1$, the “magnetic” flux per unit cell is $2\pi$, and the vortex does not pick up any phase from the boson density.

- The effective dual “magnetic” field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.
Bosons at rational filling fraction $f = p/q$

Quantum mechanics of the vortex “particle” in a periodic potential with $f$ flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

$T_x, T_y$ : Translations by a lattice spacing in the $x, y$ directions

$R$ : Rotation by 90 degrees.

Magnetic space group:

$T_x T_y = e^{2\pi i f} T_y T_x$ ;

$R^{-1} T_y R = T_x$ ; $R^{-1} T_x R = T_y^{-1}$ ; $R^4 = 1$

The low energy vortex states must form a representation of this algebra
Vortices in a superfluid near a Mott insulator at filling $f = p/q$

Hofstadter spectrum of the quantum vortex “particle” with field operator $\varphi$

At filling $f = p/q$, there are $q$ species of vortices, $\varphi_\ell$ (with $\ell = 1 \ldots q$), associated with $q$ degenerate minima in the vortex spectrum. These vortices realize the smallest, $q$-dimensional, representation of the magnetic algebra.

\[
T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i \ell f} \varphi_\ell \\
R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell mf}
\]
Boson-vortex duality

The $q \varphi_\ell$ vortices characterize both superconducting and density wave orders.

Superconductor/insulator: $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$
Boson-vortex duality

The \( q \varphi_\ell \) vortices characterize both superconducting and density wave orders

Density wave order:

Status of space group symmetry determined by density operators \( \rho_Q \) at wavevectors \( Q_{mn} = \frac{2\pi p}{q} (m,n) \)

\[
\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell mf}
\]

\( T_x : \rho_Q \rightarrow \rho_Q e^{i\mathbf{Q} \cdot \mathbf{\hat{x}}} \); \( T_y : \rho_Q \rightarrow \rho_Q e^{i\mathbf{Q} \cdot \mathbf{\hat{y}}} \)

\( R : \rho(Q) \rightarrow \rho(RQ) \)
Field theory with projective symmetry

Degrees of freedom:

- $q$ complex $\varphi_\ell$ vortex fields
- 1 non-compact U(1) gauge field $A_\mu$

$$S = \int d^2 x d\tau \left[ \sum_{\ell} \left\{ |(\partial_\mu - iA_\mu)\varphi_\ell|^2 + s|\varphi_\ell|^2 \right\} ight. 
\left. + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 + \sum_{\ell mn} \gamma_{mn} \varphi_\ell^* \varphi_{\ell+m} \varphi_{\ell+n} \varphi_{\ell+m-n} \right]$$

The projective symmetries constrain the couplings $\gamma_{mn}$ to obey

$$\gamma_{mn} = \gamma_{-m,-n} ; \quad \gamma_{mn} = \gamma_{m,m-n} ; \quad \gamma_{mn} = \gamma_{m-2n,-n}$$

$$\gamma_{\bar{m}\bar{n}} = \frac{1}{q} \sum_{mn} \gamma_{mn} e^{-2\pi i f[n(\bar{m}-\bar{n})+\bar{n}(m-n)]}$$
Field theory with projective symmetry

\[ \langle \Psi_{sf} \rangle \]

Superfluid

\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

Fluctuation-induced, weak, first order transition

\[ \langle \rho_Q \rangle \]

Charge-ordered insulator

\[ \langle \varphi_\ell \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]

\[ r_1 - r_2 \]
Field theory with projective symmetry

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\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

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\text{Charge-ordered insulator}
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Supersolid

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\text{Superfluid}
\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle \neq 0 \]

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Field theory with projective symmetry

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Superfluid
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Second order transition

\[ \langle \Psi_{sf} \rangle \]
Superfluid
\[ \langle \varphi_\ell \rangle = 0, \langle \rho_{mn} \rangle = 0 \]

\[ \langle \rho_Q \rangle \]
Charge-ordered insulator
\[ \langle \varphi_\ell \rangle \neq 0, \langle \rho_{mn} \rangle \neq 0 \]

\[ r_1 - r_2 \]
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of $q$ flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of density-wave order in its vicinity.
Mott insulators obtained by condensing vortices at $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Mott insulators obtained by condensing vortices at $f = 1/4, 3/4$

\[ a \times b \text{ unit cells; } q/a', q/b', ab/q', \text{ all integers} \]
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Outline

I. Statement of the problem
   A. Antiferromagnets
   B. Boson lattice models

II. Theory of defects: vortices near the superfluid-insulator transition
   *Berry phases imply that vortices carry “flavor”*

III. The cuprate superconductors
   *Detection of vortex flavors?*

IV. Defects in the antiferromagnet
   *Hedgehog Berry phases and VBS order*
III. The cuprate superconductors

*Detection of dual vortex wavefunction in STM experiments?*
La$_2$CuO$_4$
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Mott insulator: square lattice antiferromagnet

\[ H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j \]
\( \text{La}_{2-\delta}\text{Sr}_\delta\text{CuO}_4 \)

Superfluid: condensate of paired holes

\[ \langle \vec{S} \rangle = 0 \]
Many experiments on the cuprate superconductors show:

- Tendency to produce modulations in spin singlet observables at wavevectors \((2\pi/a)(1/4,0)\) and \((2\pi/a)(0,1/4)\).

- Proximity to a Mott insulator at hole density \(\delta = 1/8\) with long-range charge modulations at wavevectors \((2\pi/a)(1/4,0)\) and \((2\pi/a)(0,1/4)\).
The cuprate superconductor $\text{Ca}_{2-x}\text{Na}_x\text{CuO}_2\text{Cl}_2$

Possible structure of VBS order

This structure also explains spin-excitation spectra in neutron scattering experiments

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Do vortices in the superfluid “know” about these “density” modulations?
Consequences of our theory:

Information on VBS order is contained in the vortex flavor space

Density operators $\rho_q$ at wavevectors $Q_{mn} = \frac{2\pi p}{q} (m, n)$

$$\rho_{mn} = e^{i\pi m n f} \sum_{\ell=1}^{q} \phi_{\ell}^{*} \phi_{\ell+n} e^{2\pi i \ell m f}$$

Each pinned vortex in the superfluid has a halo of density wave order over a length scale $\approx$ the zero-point quantum motion of the vortex. This scale diverges upon approaching the insulator
Vortex-induced LDOS of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ integrated from 1meV to 12meV at 4K

Vortices have halos with LDOS modulations at a period $\approx$ 4 lattice spacings


Distinct experimental characteristics of underdoped cuprates at $T > T_c$

Measurements of Nernst effect are well explained by a model of a liquid of vortices and anti-vortices


STM measurements observe “density” modulations with a period of $\approx 4$ lattice spacings

LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ at 100 K.

Distinct experimental characteristics of underdoped cuprates at $T > T_c$

Our theory: modulations arise from pinned vortex-anti-vortex pairs – these thermally excited vortices are also responsible for the Nernst effect

LDOS of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ at 100 K.

Superfluids near Mott insulators

The Mott insulator has average Cooper pair density, $f = p/q$ per site, while the density of the superfluid is close (but need not be identical) to this value.

- Dual description using vortices with flux $\frac{h}{2e}$ which come in multiple (usually $q$) “flavors”
- The lattice space group acts in a projective representation on the vortex flavor space.
- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”
- Any pinned vortex must chose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.
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*Hedgehog Berry phases and VBS order*
Defects in the Neel state

Monopoles are points in spacetime, representing tunnelling events
• The Berry phases of the spins attach a non-trivial phase factor to each hedgehog tunnelling event


• Consequently, the hedgehog creation operator, $v^\dagger$, transforms non-trivially under square lattice space group operations (for spin $S = 1/2$):

$$T_x : v^\dagger \rightarrow -iv ; \quad T_y : v^\dagger \rightarrow iv ; \quad R : v^\dagger \rightarrow iv^\dagger$$

• These transformation properties allow the remarkable identification

$$v \sim e^{-i\pi/4} \Psi_{vbs}$$


So the defects in the Neel order are linked to the VBS order parameter.

• Condensation of hedgehogs induces long-range VBS order, and eliminates a purely “quantum disordered” state with vector spin excitations.