Modern quantum materials realize a remarkably rich set of electronic phases. This school will explore the many new concepts and methods which have been developed in recent years, moving beyond the traditional paradigms of Fermi liquid theory and spontaneous symmetry breaking. In particular, long-range quantum entanglement appears in topological and quantum-critical states, and the school will discuss new techniques required to describe their observable properties.

For more details:
www.as.huji.ac.il/horizons-in-quantum
The SYK model of non-Fermi liquids and black holes

Applications of Gauge-Gravity Duality 2016
Chalmers University of Technology
Gothenburg, Sweden, October 4, 2016

Subir Sachdev

Talk online: sachdev.physics.harvard.edu
Quantum matter without quasiparticles

- Quasiparticles are long-lived excitations which can be combined to yield the complete low-energy many-body spectrum.
Quantum matter without quasiparticles

- Quasiparticles are long-lived excitations which can be combined to yield the complete low-energy many-body spectrum.

- Quasiparticles need not be electrons: they can be emergent excitations which involve non-local changes in the wave function of the underlying electrons e.g. Laughlin quasiparticles, visons . . .

\[ \tau \approx \frac{1}{T^2} \text{ in gapped systems,} \]
\[ \tau \approx e^T \text{ in Fermi liquids,} \]

Systems without quasiparticles saturate a (conjectured) lower bound on the local-equilibration/de-phasing/transition-to-quantum-chaos time \( \tau_C \sim k_B T \) where \( C \) is a \( T \)-independent constant.
Quantum matter without quasiparticles

- Quasiparticles are long-lived excitations which can be combined to yield the complete low-energy many-body spectrum.

- Quasiparticles need not be electrons: they can be emergent excitations which involve non-local changes in the wave function of the underlying electrons e.g. Laughlin quasiparticles, visons . . .

- How do we rule out quasiparticle excitations? Examine the time it takes to reach local thermal equilibrium. Equilibration takes a long time while quasiparticles collide (in Fermi liquids, $\tau \sim 1/T^2$; in gapped systems, $\tau \sim e^{\Delta/T}$). Systems without quasiparticles saturate a (conjectured) lower bound on the local-equilibration/de-phasing/transition-to-quantum-chaos time

\[ \tau_\varphi \geq C \frac{\hbar}{k_B T} \]

where $C$ is a $T$-independent constant.
Quantum matter without quasiparticles

- Shortest possible local-equilibration/de-phasing/transition-to-quantum-chaos with

\[
\tau_\phi \geq C \frac{\hbar}{k_BT} \\
\eta \geq \frac{\hbar}{4\pi k_B} \\
D \geq \tilde{C} \frac{\hbar}{k_BT} \\
\tau_L \geq \frac{1}{2\pi} \frac{\hbar}{k_BT}
\]

S. Sachdev, Quantum Phase Transitions (1999)
K. Damle and Sachdev, PRB 56, 8714 (1997)


Saturation requires fixed point with disorder and interactions
M. Blake, PRL 117, 091601 (2016)

J. Maldacena, S. H. Shenker and D. Stanford, JHEP 08 (2016) 106

In Fermi liquids, \( \tau \sim 1/T^2 \);
in gapped systems, \( \tau \sim e^{\Delta/T} \).
\[ \frac{1}{\tau} = \alpha \frac{k_B T}{\hbar} \]

Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
- Coupled SYK models: diffusive metals without quasiparticles
- Holographic Einstein-Maxwell-axion theory with momentum dissipation
Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model

- Coupled SYK models: diffusive metals without quasiparticles

- Holographic Einstein-Maxwell-axion theory with momentum dissipation
To obtain a non-Fermi liquid, we set $t_{ij} = 0:

$$H_{\text{SYK}} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij,k\ell} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$Q = \frac{1}{N} \sum_i c_i^\dagger c_i$$

A fermion can move only by entangling with another fermion: the Hamiltonian has “nothing but entanglement”.

Cold atom realization:

**SYK model**

Feynman graph expansion in $J_{ij...}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} , \quad \Sigma(\tau) = -J^2 G^2(\tau)G(-\tau)
\]

\[
G(\tau = 0^-) = Q.
\]

Feynman graph expansion in $J_{ij}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
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\]

\[
G(\tau = 0^-) = Q.
\]

Low frequency analysis shows that the solutions must be gapless and obey

\[
\Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \ldots, \quad G(z) = \frac{A}{\sqrt{z}}
\]

for some complex $A$. The ground state is a non-Fermi liquid, with a continuously variable density $Q$. 

SYK model

- $T = 0$ Green’s function $G \sim 1/\sqrt{\tau}$

SYK model

- $T = 0$ Green’s function $G \sim 1/\sqrt{\tau}$

- $T > 0$ Green’s function implies conformal invariance
  $G \sim 1/(\sin(\pi T \tau))^{1/2}$

  A. Georges and O. Parcollet PRB 59, 5341 (1999)
SYK model

• \(T = 0\) Green’s function \(G \sim 1/\sqrt{\tau}\)

• \(T > 0\) Green’s function implies conformal invariance
  \(G \sim 1/(\sin(\pi T \tau))^{1/2}\)

• Non-zero entropy as \(T \to 0\), \(S(T \to 0) = N S_0 + \ldots\)

A better understanding of the above facts can be reached from the perspective of symmetry-protected topological (SPT) phases. As shown recently in Ref. 14, the complex SYK model can be thought of as the boundary of a 1D SPT system in the symmetry class AIII. The periodicity of 4 in $N$ arises from the fact that we need to put 4 chains to gap out the boundary degeneracy without breaking the particle-hole symmetry. In the Majorana SYK case, the symmetric Hamiltonian can be constructed as a symmetric matrix in the Clifford algebra $\mathbb{Cl}_{0, N}$, and the Bott periodicity in the real representation of the Clifford algebra gives rise to a $\mathbb{Z}_8$ classification. Here, for the complex SYK case, we can similarly construct the Clifford algebra by dividing one complex fermion into two Majorana fermions, and then we will have a periodicity of 4.

From the above definition of retarded Green's function, we can relate them to the imaginary time Green's function as defined in Eq. (16), $G_R(\tau) = G(\tau + i\eta_n + i\eta_P)$. In Fig. 3, we show a comparison between the imaginary part of the Green's function from large $N$ and from the exact diagonalization computation. The spectral function from ED is particle-hole symmetric for all $N$, whereas for the fermionic case, we have $G_F(z) \sim 1/pz$; this inverse square-root behavior also holds in the bosonic case without spin glass order. Fig. 10 is our result from ED, with a comparison between $G_B$ with $G_F$. It is evident that the behavior of $G_B$ is qualitatively different from $G_F$, and so an inverse square-root behavior is ruled out. Instead, we can clearly see that, as system size gets larger, $G_B$'s peak value increases much faster than the $G_F$'s peak value. This supports the presence of spin glass order.

Large $N$ solution of equations for $G$ and $\Sigma$ agree well with exact diagonalization of the finite $N$ Hamiltonian $\Rightarrow$ no spin-glass order

However, exact diagonalization of the same model with hard-core bosons indicates the presence of spin-glass order in the ground state.

W. Fu and S. Sachdev, PRB 94, 035135 (2016)
SYK model

After integrating the fermions, the partition function can be written as a path integral with an action $S$ analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det [\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)]$$

$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) [G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)]$$
After integrating the fermions, the partition function can be written as a path integral with an action $S$ analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det [\delta(\tau_1 - \tau_2) (\partial_\tau + \mathbf{\Sigma}) - \Sigma(\tau_1, \tau_2)]$$

$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left[ G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2) \right]$$

At frequencies $\ll J$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = \left[ f'(\sigma_1)f'(\sigma_2) \right]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = \left[ f'(\sigma_1)f'(\sigma_2) \right]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
Let us write the large $N$ saddle point solutions of $S$ as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2}, \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$ 

These are not invariant under the reparametrization symmetry but are invariant only under a SL(2,R) subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$ 

So the (approximate) reparametrization symmetry is spontaneously broken.

J. Maldacena and D. Stanford, arXiv:1604.07818

See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768
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So the (approximate) reparametrization symmetry is spontaneously broken.

Reparametrization zero mode
Expand about the saddle point by writing $G(\tau_1, \tau_2) = \left[f_0(\tau_1) f_0(\tau_2)\right]^{1/4}G_s(f(\tau_1) f(\tau_2))$ (and similarly for $\Sigma$) and obtain an effective action for $f(\tau)$. This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

Connections of SYK to gravity and AdS$_2$ horizons

- Reparameterization and gauge invariance are the ‘symmetries’ of the Einstein-Maxwell theory of gravity and electromagnetism
- SL(2,R) is the isometry group of AdS$_2$.

J. Maldacena and D. Stanford, arXiv:1604.07818
See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768
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So the (approximate) reparametrization symmetry is spontaneously broken.

**Reparametrization zero mode**

Expand about the saddle point by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))$$

(and similarly for $\Sigma$) and obtain an effective action for $f(\tau)$. This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

J. Maldacena and D. Stanford, arXiv:1604.07818

See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768
SYK model

With \( g(\tau) = e^{-i\phi(\tau)} \), the action for \( \phi(\tau) \) and \( f(\tau) = \frac{1}{\pi T} \tan(\pi T (\tau + \epsilon(\tau)) \) fluctuations is

\[
S_{\phi,f} = \frac{K}{2} \int_0^{1/T} d\tau (\partial_{\tau} \phi + i(2\pi E T) \partial_{\tau} \epsilon)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ f, \tau \},
\]

where \( \{ f, \tau \} \) is the Schwarzian:

\[
\{ f, \tau \} \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.
\]

The couplings are given by thermodynamics (\( \Omega \) is the grand potential)

\[
K = - \left( \frac{\partial^2 \Omega}{\partial \mu^2} \right)_T, \quad \gamma + 4\pi^2 E^2 K = - \left( \frac{\partial^2 \Omega}{\partial T^2} \right)_\mu
\]

\[
2\pi E = \frac{\partial S_0}{\partial Q}
\]
SYK model

With $g(\tau) = e^{-i\phi(\tau)}$, the action for $\phi(\tau)$ and $f(\tau) = \frac{1}{\pi T} \tan(\pi T(\tau + \epsilon(\tau)))$ fluctuations is

$$S_{\phi,f} = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi + i (2\pi \mathcal{E} T) \partial_\tau \epsilon)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{f, \tau\},$$

where $\{f, \tau\}$ is the Schwarzian:

$$\{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

The correlators of the density fluctuations, $Q(\tau)$, and the energy fluctuations $\delta E - \mu \delta Q(\tau)$ are time independent and given by

$$\begin{pmatrix}
\langle \delta Q(\tau) \delta Q(0) \rangle \\
\langle (\delta E(\tau) - \mu \delta Q(\tau)) \delta Q(0) \rangle / T \\
\langle (\delta E(\tau) - \mu \delta Q(\tau))(\delta E(0) - \mu \delta Q(0)) \rangle / T
\end{pmatrix} = T \chi_s$$

where $\chi_s$ is the static susceptibility matrix given by

$$\chi_s \equiv \begin{pmatrix}
-(\partial^2 \Omega / \partial \mu^2)_T & -\partial^2 \Omega / (\partial T \partial \mu) \\
-T \partial^2 \Omega / (\partial T \partial \mu) & -T (\partial^2 \Omega / \partial T^2)_{\mu}
\end{pmatrix}.$$
Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
- Coupled SYK models: diffusive metals without quasiparticles
- Holographic Einstein-Maxwell-axion theory with momentum dissipation
One can also derive the thermodynamic properties from the large-$N$ saddle point free energy:

$$F_N = \frac{1}{2} \log \mathcal{P} f (\tau) + \frac{1}{2} \int d\tau_1 d\tau_2 \mathcal{G} (\tau_1, \tau_2) \mathcal{J}^2 \mathcal{G} (\tau_1, \tau_2)^2 \mathcal{C} \quad (8)$$

In the second line we write the free energy in a low temperature expansion, where $U \propto 0.0406 \mathcal{J}$ is the ground state energy, $S_0 \propto 0.234$ is the zero temperature entropy \cite{32, 4}, and $T = \frac{\epsilon}{v} = \frac{\tau}{16} p \mathcal{J} \propto 0.396 \mathcal{J}$ is the specific heat \cite{11}. The entropy term can be derived by inserting the conformal saddle point solution (2) in the effective action. The specific heat can be derived from knowledge of the leading ($1 / \mathcal{J}$) correction to the conformal saddle, but the energy requires the exact (numerical) finite $\mathcal{J}$ solution.

3 The generalized SYK model

In this section, we will present a simple way to generalize the SYK model to higher dimensions while keeping the solvable properties of the model in the large-$N$ limit. For concreteness of the presentation, in this section we focus on a (1 + 1)-dimensional example, which describes a one-dimensional array of SYK models with coupling between neighboring sites. It should be clear how to generalize, and we will discuss more details of the generalization to arbitrary dimensions and generic graphs in section 6.

3.1 Definition of the chain model

Figure 1: A chain of coupled SYK sites: each site contains $N \gg 1$ fermion with SYK interaction. The coupling between nearest neighbor sites are four fermion interaction with two from each site.

Yingfei Gu, Xiao-Liang Qi, and D. Stanford, arXiv:1609.07832
**SYK model**

The correlators of the density fluctuations, $Q(\tau)$, and the energy fluctuations $\delta E - \mu \delta Q(\tau)$ are time independent and given by

\[
\begin{pmatrix}
\langle \delta Q(\tau) \delta Q(0) \rangle \\
\langle (\delta E(\tau) - \mu \delta Q(\tau)) \delta Q(0) \rangle / T
\end{pmatrix}
= T \chi_s
\]

where $\chi_s$ is the static susceptibility matrix given by

\[
\chi_s \equiv \begin{pmatrix}
-\left(\frac{\partial^2 \Omega}{\partial \mu^2}\right)_T & \frac{\partial^2 \Omega}{\partial T \partial \mu} \\
\frac{T \partial^2 \Omega}{\partial T \partial \mu} & -T \left(\frac{\partial^2 \Omega}{\partial T^2}\right)_\mu
\end{pmatrix}.
\]

**Coupled SYK models**

\[
\begin{pmatrix}
\langle Q; Q \rangle_{k,\omega} \\
\langle E - \mu Q; Q \rangle_{k,\omega} / T
\end{pmatrix}
= [i\omega (-i\omega + D k^2)^{-1} + 1] \chi_s
\]

where the diffusivities are related to the thermoelectric conductivities by the Einstein relations

\[
D = \begin{pmatrix}
\sigma & \alpha \\
\alpha T & \frac{\alpha}{\kappa}
\end{pmatrix} \chi_s^{-1}.
\]
**Coupled SYK models**

\[
\begin{pmatrix}
    \langle Q; Q \rangle_{k,\omega} & \langle E - \mu Q; Q \rangle_{k,\omega} / T \\
    \langle E - \mu Q; Q \rangle_{k,\omega} & \langle E - \mu Q; E - \mu Q \rangle_{k,\omega} / T
\end{pmatrix}
\]

where the diffusivities are related to the thermoelectric conductivities by the Einstein relations

\[
D = \begin{pmatrix}
    \sigma & \alpha \\
    \alpha T & \bar{\kappa}
\end{pmatrix} \chi_s^{-1}.
\]

The coupled SYK models realize a diffusive, metal with no quasiparticle excitations. (a “strange metal”)
Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
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- Holographic Einstein-Maxwell-axion theory with momentum dissipation
SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{T}$
- $T > 0$ Green's function implies conformal invariance
  $G \sim 1/(\sin(\pi T \tau))^{1/2}$
- Non-zero entropy as $T \to 0$, $S(T \rightarrow 0) = N S_0 + \ldots$
SYK model

• $T = 0$ Green’s function $G \sim 1/\sqrt{T}$

• $T > 0$ Green’s function implies conformal invariance $G \sim 1/\left(\sin(\pi T \tau)\right)^{1/2}$

• Non-zero entropy as $T \to 0$, $S(T \to 0) = NS_0 + \ldots$

• These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with AdS$_2$ near-horizon geometry. The Bekenstein-Hawking entropy is $NS_0$.

S. Sachdev, PRL 105, 151602 (2010)
**SYK and AdS$_2$**

\[
\text{AdS}_2 \times T^2 \\
\text{charge density } Q \\
ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + dx^2 \\
\text{Gauge field: } A = (E/\zeta)dt
\]

---

**PHYSICAL REVIEW LETTERS** 105, 151602 (2010)

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Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a “small” Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, AdS$_2 \times \mathbb{R}^2$ physics of Reissner-Nordström black holes.
SYK model

- $T = 0$ Green’s function $G \sim 1/\sqrt{T}$

- $T > 0$ Green’s function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$

- Non-zero entropy as $T \to 0$, $S(T \to 0) = NS_0 + \ldots$

- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with AdS$_2$ near-horizon geometry. The Bekenstein-Hawking entropy is $NS_0$.

- There is a scalar zero mode associated with the breaking of reparameterization invariance down to SL(2,R). The same pattern of symmetries is present in gravity theories on AdS$_2$.

A. Kitaev, KITP talk, 2015
SYK model

- The dependence of $S_0$ on the density $Q$ matches the behavior of the Wald-Bekenstein-Hawking entropy of AdS$_2$ horizons in a large class of gravity theories.

S. Sachdev PRX 5, 041025 (2015)
**SYK model**

- The dependence of $S_0$ on the density $Q$ matches the behavior of the Wald-Bekenstein-Hawking entropy of AdS$_2$ horizons in a large class of gravity theories.

- The scalar zero mode leads to a linear-in-$T$ specific heat

\[ S(T \to 0) = NS_0 + N\gamma T + \ldots \]

An identical scalar zero model is also present in the low energy limit of theories of quantum gravity on AdS$_2$.

J. Maldacena and D. Stanford, arXiv:1604.07818
**SYK model**

- The dependence of $S_0$ on the density $Q$ matches the behavior of the Wald-Bekenstein-Hawking entropy of AdS$_2$ horizons in a large class of gravity theories.

- The scalar zero mode leads to a linear-in-$T$ specific heat

\[ S(T \to 0) = NS_0 + N\gamma T + \ldots. \]

An identical scalar zero model is also present in the low energy limit of theories of quantum gravity on AdS$_2$.

- The Lyapunov time to quantum chaos saturates the lower bound both in the SYK model and in quantum gravity.

\[ \tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T} \]

A. Kitaev, KITP talk, 2015

J. Maldacena and D. Stanford, arXiv:1604.07818
It would be nice to have a solvable model of holography.

<table>
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<th>theory</th>
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<td>$O(1)$</td>
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<td>yes</td>
<td>yes</td>
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Slide by D. Stanford at Strings 2016, Beijing
Einstein-Maxwell-axion theory

\[ S = \int d^4x \sqrt{-\hat{g}} \left( \hat{R} + 6/L^2 - \frac{1}{2} \sum_{i=1}^{2} (\partial \hat{\varphi}_i)^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right), \]

- For \( \hat{\varphi}_i = 0 \), we obtain the Reissner-Nördstrom-AdS charged black hole, with a near-horizon AdS\(_2 \times T^2\) near-horizon geometry.

- For \( \hat{\varphi}_i = kx_i \), we obtain a similar solution but with momentum dissipation (a bulk massive graviton).

Y. Bardoux, M. M. Caldarelli, and C. Charmousis, JHEP 05 (2012) 054
M. Blake and D. Tong, PRD 88 (2013), 106004.

M. Blake and D. Tong, PRD 88 (2013), 106004.
Einstein-Maxwell-axion theory

\[ S = \int d^4 x \sqrt{-\hat{g}} \left( \hat{R} + 6/L^2 - \frac{1}{2} \sum_{i=1}^{2} (\partial \hat{\phi}_i)^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right), \]

In the small torus limit, \( T \ll 1/R \), where \( R \) is the size of the torus, the theory dimensionally reduces to an Einstein-Maxwell-dilaton theory in two dimensions

\[ S = \int d^2 x \sqrt{-g} \left( e^{\phi} R + e^{\phi/2} (6/L^2) - m^2 e^{-\phi/2} - \frac{1}{4} e^{3\phi/2} F_{ab} F^{ab} \right), \]

Y. Bardoux, M. M. Caldarelli, and C. Charmousis, JHEP 05 (2012) 054
M. Blake and D. Tong, PRD 88 (2013), 106004.

The Einstein-Maxwell-dilaton theory of the small torus limit, $T \ll 1/R$, is equivalent on its boundary to the Schwarzian theory discussed earlier for the SYK model

$$S_{\phi,f} = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ f, \tau \} + i \frac{L}{4\pi^2} \int_0^{1/T} d\tau (\partial_\tau \phi)\{ f, \tau \}.$$ 

So the correlators of the density fluctuations, $Q(\tau)$, and the energy fluctuations $\delta E - \mu \delta Q(\tau)$ are time independent and given by

$$\begin{pmatrix} \langle \delta Q(\tau) \delta Q(0) \rangle \\ \langle (\delta E(\tau) - \mu \delta Q(\tau)) \delta Q(0) \rangle \\ \langle (\delta E(\tau) - \mu \delta Q(\tau))(\delta E(0) - \mu \delta Q(0)) \rangle / T \end{pmatrix} = T \chi_s$$

where $\chi_s$ is the static susceptibility matrix given by

$$\chi_s = \begin{pmatrix} -(\partial^2 \Omega/\partial \mu^2)_T & -\partial^2 \Omega/(\partial T \partial \mu) \\ -T \partial^2 \Omega/(\partial T \partial \mu) & -T(\partial^2 \Omega/\partial T^2)_\mu \end{pmatrix}.$$ 

Einstein-Maxwell-axion theory

Finally, in the large torus limit, $T \gg 1/R$, we have the behavior of the diffusive metal without quasiparticles found in the coupled SYK models

\[
\begin{pmatrix}
\langle Q; Q \rangle_{k, \omega} & \langle E - \mu Q; Q \rangle_{k, \omega} / T \\
\langle E - \mu Q; Q \rangle_{k, \omega} & \langle E - \mu Q; E - \mu Q \rangle_{k, \omega} / T
\end{pmatrix}
= \left[ i \omega (-i \omega + D k^2)^{-1} + 1 \right] \chi_s
\]

where the diffusivities are related to the thermoelectric conductivities by the Einstein relations

\[
D = \left( \begin{array}{cc}
\sigma & \alpha \\
\alpha T & \bar{K}
\end{array} \right) \chi_s^{-1}.
\]

Y. Bardoux, M. M. Caldarelli, and C. Charmousis, JHEP 05 (2012) 054
M. Blake and D. Tong, PRD 88 (2013), 106004.
Non-Fermi liquids

- Shortest possible “phase coherence” time, fastest possible local equilibration time, or fastest possible Lyapunov time towards quantum chaos, all of order $\frac{\hbar}{k_B T}$.

- Realization in solvable SYK model, which saturates the lower bound on the Lyapunov time.

- Coupled SYK models realize diffusive metal without quasi-particles.

- Remarkable holographic match to Einstein-Maxwell-axion theories with momentum dissipation via the Schwarzian effective action.