Universal conductance of nanowires near the superconductor-metal quantum transition

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Talk online at http://sachdev.physics.harvard.edu
Why study quantum phase transitions?

- Theory for a quantum system with strong correlations: describe phases on either side of $g_c$ by expanding in deviation from the quantum critical point.

- Critical point is a novel state of matter without quasiparticle excitations

- Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Important property of ground state at $g=g_c$:
- temporal and spatial *scale* invariance;
- characteristic energy scale at other values of $g$: $\Delta \sim \vert g - g_c \vert^{\nu}$
Outline

I. Quantum Ising Chain
II. Landau-Ginzburg-Wilson theory
   Mean field theory and the evolution of the excitation spectrum.
III. Superfluid-insulator transition
    Boson Hubbard model at integer filling.
IV. Superconductor-metal transition in nanowires
    Universal conductance and sensitivity to leads
I. Quantum Ising Chain
I. Quantum Ising Chain

Degrees of freedom: \( j = 1 \ldots N \) qubits, \( N \) "large"

\[
|\uparrow\rangle_j, \; |\downarrow\rangle_j
\]

or

\[
|\rightarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j + |\downarrow\rangle_j \right), \quad |\leftarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j - |\downarrow\rangle_j \right)
\]

Hamiltonian of decoupled qubits:

\[
H_0 = -Jg \sum_j \sigma_j^x
\]
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z \]

Prefers neighboring qubits are either \(|\uparrow\rangle_j \uparrow\rangle_{j+1}\) or \(|\downarrow\rangle_j \downarrow\rangle_{j+1}\) (not entangled)

Full Hamiltonian

\[ H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right) \]

leads to entangled states at \(g\) of order unity
Experimental realization

LiHoF$_4$
Weakly-coupled qubits \((g \gg 1)\)

Ground state:
\[
|G\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \rangle
\]
\[
-\frac{1}{2g} |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \rightarrow \rightarrow \rightarrow \cdots \rangle - \cdots
\]

Lowest excited states:
\[
|\ell_j\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow_j \rightarrow \rightarrow \rightarrow \rightarrow \cdots \rangle + \cdots
\]

Coupling between qubits creates “flipped-spin” quasiparticle states at momentum \(p\)
\[
|p\rangle = \sum_j e^{ipx_j/h} |\ell_j\rangle
\]

Excitation energy \(\varepsilon(p) = \Delta + 4J \sin^2 \left(\frac{pa}{2\hbar}\right) + O\left(g^{-1}\right)\)

Excitation gap \(\Delta = 2gJ - 2J + O\left(g^{-1}\right)\)

Entire spectrum can be constructed out of multi-quasiparticle states
Dynamic Structure Factor $S(p, \omega)$: Weakly-coupled qubits ($g \gg 1$)

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)
while transferring energy $\hbar\omega$ and momentum $p$

$$S(p, \omega) \rightarrow Z\delta(\omega - \epsilon(p))$$

Quasiparticle pole

$$\sim 3\Delta$$

Three quasiparticle continuum

Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to
a Lorentzian of width $1/\tau_\phi$ where the phase coherence time $\tau_\phi$
is given by

$$\frac{1}{\tau_\phi} = \frac{2k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

Ground states:
\[
|G \uparrow\rangle = \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots
\]

\[\frac{-g}{2} |\cdots \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \cdots\rangle - \cdots\]

Second state \( |G \downarrow\rangle \) obtained by \( \uparrow \Leftrightarrow \downarrow \)

\( |G \downarrow\rangle \) and \( |G \uparrow\rangle \) mix only at order \( g^N \)

Lowest excited states: domain walls

\[
|d_j\rangle = |\cdots \uparrow \uparrow \uparrow \uparrow \downarrow_j \downarrow \downarrow \downarrow \downarrow \cdots \rangle + \cdots
\]

Coupling between qubits creates new “domain-wall” quasiparticle states at momentum \( p \)

\[
|p\rangle = \sum_j e^{ipx_j/h} |d_j\rangle
\]

Excitation energy \( \varepsilon(p) = \Delta + 4Jg \sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^2\right) \)

Excitation gap \( \Delta = 2J - 2gJ + O\left(g^2\right) \)
Dynamic Structure Factor $S(p, \omega)$: Strongly-coupled qubits ($g \ll 1$)

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)
while transferring energy $\hbar \omega$ and momentum $p$

Structure holds to all orders in $g$

At $T > 0$, motion of domain walls leads to a finite phase coherence time $\tau_\varphi$, and broadens coherent peak to a width $1/\tau_\varphi$ where

$$\frac{1}{\tau_\varphi} = \frac{2k_BT}{\pi \hbar} e^{-\Delta/k_BT}$$

Entangled states at $g$ of order unity

“Flipped-spin” Quasiparticle weight $Z$

$Z \sim (g - g_c)^{1/4}$


Ferromagnetic moment $N_0$

$N_0 \sim (g_c - g)^{1/8}$


Excitation energy gap $\Delta$

$\Delta \sim |g - g_c|$
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)

while transferring energy $\hbar \omega$ and momentum $p$

$S(p, \omega) \sim (\omega^2 - c^2 p^2)^{-7/8}$

No quasiparticles --- dissipative critical continuum
\[ H_I = -J \sum_i \left( g \sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \]

\[ \chi(\omega) = \frac{i}{\hbar} \sum_k^\infty \int_0^\infty dt \langle [\sigma_j^z(t), \sigma_k^z(0)] \rangle e^{i\omega t} \]

\[ = \frac{A}{T^{7/4}} \frac{1}{\left(1 - i\omega / \Gamma_R + \ldots\right)} \]

\[ \Gamma_R = \left( 2 \tan \frac{\pi}{16} \right) \frac{k_B T}{\hbar} \]

\[ \langle \sigma_j^z \sigma_k^z \rangle \sim \frac{1}{|j - k|^{1/4}} \]


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II. Landau-Ginzburg-Wilson theory

Mean field theory and the evolution of the excitation spectrum
• Identify order parameter \( \phi(x, \tau) \sim \sigma_j^z \)

• Symmetries:

  Spin inversion: \( \phi \rightarrow -\phi \)
  Time reversal \( \tau \rightarrow -\tau \)
  Spatial inversion \( x \rightarrow -x \)

• Write down most general Lagrangian consistent with symmetries

\[
\mathcal{Z} = \int D\phi(x, \tau) \exp \left( -\int d^d x \int d\tau \mathcal{L} [\phi] \right)
\]
\[
\mathcal{L} [\phi] = \frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla_x \phi)^2 + \frac{r}{2} \phi^2 + \frac{u}{4} \phi^4 + \ldots
\]

• Identify phases at \( r \gg 0 \) and \( r \ll 0 \) with the paramagnet and the ferromagnet respectively.
Quantum field theory formally resembles the classical statistical mechanics of an Ising model in $d+1$ dimensions. Theory of second-order classical phase transitions implies that at the critical point the susceptibility depends on the $d + 1$ dimensional momentum $k$ as

$$\chi(k) \sim \frac{1}{k^{2-\eta}}$$

After analytic continuation, and using the “Lorentz invariance” of the critical theory, the quantum critical point therefore has the following dynamic susceptibility at $T = 0$.

$$\chi(p, \omega) \sim \frac{1}{(c^2 p^2 - \omega^2)^{1-\eta/2}}$$

At $T > 0$, we have to consider a classical statistical mechanics problem in finite geometry with a ‘temporal’ direction of extent $L_\tau = \hbar/(k_B T)$. *Finite size scaling* now implies that the susceptibility at the critical point obeys

$$\chi(k) \sim L_\tau^{2-\eta} F(k L_\tau)$$

After analytic continuation, the quantum system has the dynamic response (note: can no longer use ”Lorentz invariance”)

$$\chi''(0, \omega) \sim \frac{1}{T^{2-\eta}} \Phi \left( \frac{\hbar \omega}{k_B T} \right)$$
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III. Superfluid-insulator transition

*Boson Hubbard model at integer filling*
Bosons at density $f = 1$

Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

LGW theory: continuous quantum transitions between these states

I. The Superfluid-Insulator transition

**Boson Hubbard model**

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, $j$, of a lattice, with short-range repulsive interactions.

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots$$

For small $U/t$, ground state is a superfluid BEC with superfluid density $\approx$ density of bosons

What is the ground state for large $U/t$?

Typically, the ground state remains a superfluid, but with superfluid density $\ll$ density of bosons.

The superfluid density evolves smoothly from large values at small $U/t$, to small values at large $U/t$, and there is no quantum phase transition at any intermediate value of $U/t$.

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions.)
What is the ground state for large $U/t$?

Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities.

Ground state has “density wave” order, which spontaneously breaks lattice symmetries.
Excitations of the insulator: infinitely long-lived, finite energy quasiparticles and quasiholes

Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m_{p,h}^*}$
Excitations of the insulator: infinitely long-lived, finite energy quasiparticles and quasiholes

Energy of quasi-particles/holes: $\epsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m_{p,h}^*}$
Excitations of the insulator: infinitely long-lived, finite energy quasiparticles and quasiholes

Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_p}$
LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x, \tau) \sim b_j^\dagger$

- Symmetries:
  
  \begin{align*}
  \text{Gauge invariance:} \quad & \Psi \rightarrow \Psi e^{i\theta} \\
  \text{Time reversal} \quad & \tau \rightarrow -\tau \ ; \ \Psi \rightarrow \Psi^* \\
  \text{Spatial inversion} \quad & x \rightarrow -x
  \end{align*}

- Write down most general Lagrangian consistent with symmetries

\begin{align*}
\mathcal{Z} &= \int \mathcal{D}\Psi(x, \tau) \exp \left( -\int d^d x \int d\tau \mathcal{L}[\Psi] \right) \\
\mathcal{L}[\Psi] &= K \Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r|\Psi|^2 + \frac{u}{2}|\Psi|^4 + \ldots
\end{align*}

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.

- For $K \neq 0$, the particle and hole excitations have different energies.
• Gauge-invariance of the underlying boson Hamiltonian shows that

\[ K = -\frac{\partial r}{\partial \mu} \]

• In mean-field theory, the ground state energy, \( E \), across the superfluid-insulator transition has the non-analytic term

\[ E = E_0 - \frac{r^2}{2u} \theta(-r) \]

(Beyond mean-field theory, the non-analytic term is \( E \sim r^{(d+z)\nu} \)).

• Because the density of bosons \( = -\partial E/\partial \mu \), this implies a change in the boson density across the transition \textit{unless} \( \partial r/\partial \mu = 0 \)

• A superfluid-insulator transition at fixed boson density must have

\[ K = 0 \]
Boson Green's function $G(p, \omega)$:

Cross-section to add a boson while transferring energy $\hbar \omega$ and momentum $p$

$G(p, \omega) = Z \delta(\omega - \varepsilon(p))$

Quasiparticle pole

Continuum of two quasiparticles + one quasihole

$\sim 3\Delta$

Similar result for quasi-hole excitations obtained by removing a boson
Entangled states at \( g \equiv t/U \) of order unity

Quasiparticle weight \( Z \)

\[ Z \sim (g_c - g)^{\eta \nu} \]


Excitation energy gap \( \Delta \)

\[ \Delta_{p,h} \sim (g_c - g)^{\nu} \text{ for } g < g_c \]
\[ \Delta_{p,h} = 0 \text{ for } g > g_c \]

Superfluid density \( \rho_s \)

\[ \rho_s \sim (g - g_c)^{(d+z-2)\nu} \]
Relaxational dynamics ("Bose molasses") with phase coherence/relaxation time $\tau_\phi$ given by

$$\frac{1}{\tau_\phi} = \left( \text{Universal number} \right) \frac{k_B T}{\hbar} \quad (1\mu K = 20.9\text{kHz})$$

Conductivity (in d=2) = $\frac{Q^2}{\hbar} \sum \left( \frac{\hbar\omega}{k_B T} \right) \sum \rightarrow \text{universal function}$

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$T=0$ Superconductor-metal transition

$T=0$ Superconductor-metal transition

\[ S = - \int d\tau \sum_{i,j} J_{ij} \psi_i^*(\tau) \psi_j(\tau) - \int d\tau d\tau' \sum_i \frac{\psi_i^*(\tau) \psi_i(\tau')}{(\tau - \tau')^2} \]
Continuum theory for quantum critical point

\[ S_{\text{bulk}} = \frac{A}{\hbar} \int_0^L dx \left[ \int_0^\beta d\tau \left( \delta |\partial_x \psi|^2 + a |\psi|^2 + \frac{b}{2} |\psi|^4 \right) \right. \\
\left. + \frac{\hbar \gamma}{\beta} \sum_{\omega_n} |\omega_n||\psi(x, \omega_n)|^2 \right], \]

Obeys strong hyperscaling properties in spatial dimensions \( d < 2 \). Critical properties can be determined by an expansion in \( \epsilon = 2 - d \) in a theory with \( n \)-component fields (\( n = 2 \) here).

\[
\begin{align*}
  z &= 2 - \eta \\
  \eta &= \frac{(n + 2)(12 - \pi^2)}{4(n + 8)^2} \epsilon^2 \\
  \nu &= \frac{1}{2} + \frac{(n + 2)}{4(n + 8)} \epsilon + \frac{(n + 2)(n^2 + (38 - 7\pi^2/6)n + 132 - 19\pi^2/3)}{8(n + 8)^3} \epsilon^2
\end{align*}
\]
Consequences of hyperscaling

**Diagram:**
- **Axes:** $T$ (vertical) and $R$ (horizontal)
- **Regions:**
  - **Normal** region
  - **Superconductor** region
- **Transition Points:**
  - **$R_c$**
  - **$T=0$**
- **Notes:**
  - No transition for $T>0$ in $d=1$
  - Sharp transition at $T=0$
Consequences of hyperscaling

Quantum Critical Region

No transition for $T>0$ in $d=1$

Sharp transition at $T=0$
Consequences of hyperscaling

Quantum Critical Region

The conductance $g$ obeys

$$g = \frac{4e^2}{h} \Phi \left( c_1 TL^z, \frac{\hbar \omega}{k_B T} \right)$$

where $\Phi$ is a universal function and only constant $c_1$ is non-universal.

For $L > (c_1 T)^{-1/z}$, we have hydrodynamic, “incoherent” transport and $g = \sigma/L$, where $\sigma$ is the conductivity which is independent of the leads and obeys

$$\sigma = \frac{4e^2}{h} \frac{1}{(c_1 T)^{1/z}} \Phi_1 \left( \frac{\hbar \omega}{k_B T} \right)$$
Consequences of hyperscaling

Quantum Critical Region

The conductance $g$ obeys

$$g = \frac{4e^2}{h} \Phi \left( c_1 TL^z, \frac{\hbar \omega}{k_B T} \right)$$

where $\Phi$ is a universal function and only constant $c_1$ is non-universal.

For $L < (c_1 T)^{-1/z}$, we have “coherent” transport, and the d.c. conductance is independent of $L$, but sensitive to the nature of the leads.

$$g = \frac{4e^2}{h} F (c_1 \omega L^z)$$
Effect of the leads

\[ S_{\text{lead}} = \int d\tau \left[ -H^* \psi(0, \tau) - H \psi^*(0, \tau) + C |\Psi(0, \tau)|^2 \right] \]

where $H \neq 0$ for a superconducting lead.

Both $H$ and $C$ scale to strong-coupling, and therefore we have Dirichlet boundary conditions ($\Psi = 0$) for a N lead, and Fixed boundary conditions for a S lead.

Conductance is independent of the specific bare values of $H$ and $C$. 
Large $n$ computation of conductance

\[ g = \frac{4e^2}{h} F_X (y) \quad ; \quad y = c_1 \omega L^z \]
Quantum Monte Carlo and large $n$ computation of d.c. conductance

\[ g = \frac{4e^2}{\hbar} C_{SN} \]
Conclusions

• Universal transport in wires near the superconductor-metal transition

• Theory includes contributions from thermal and quantum phase slips ---- reduces to the classical LAMH theory at high temperatures

• Sensitivity to leads should be a generic feature of the ``coherent” transport regime of quantum critical points.