Insulators and superconductors with topological order

Ribhu Kaul (Harvard), Yong-Baek Kim (Toronto)
Alexei Kolezhuk (Hannover), Michael Levin (Harvard)
Subir Sachdev (Harvard), T. Senthil (MIT)
Outline

1. Quantum “disordering” magnetic order
   \textit{Collinear order and confinement}

2. $\mathbb{Z}_2$ spin liquids
   \textit{Noncollinear order and fractionalization}

3. Gapless U(1) spin liquids
   \textit{Deconfined criticality}

4. Doped spin liquids
   \textit{Superconductors with topological order}
Outline

1. Quantum “disordering” magnetic order
   Collinear order and confinement

2. $\mathbb{Z}_2$ spin liquids
   Noncollinear order and fractionalization

3. Gapless U(1) spin liquids
   Deconfined criticality

4. Doped spin liquids
   Superconductors with topological order
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state has long-range Néel order

Order parameter is a single vector field \( \vec{\varphi} = \eta_i \vec{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\( \langle \vec{\varphi} \rangle \neq 0 \) in Néel state.
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Destroy Neel order by perturbations which preserve full square lattice symmetry \textit{e.g.} second-neighbor or ring exchange.

What is the state with \( \langle \bar{\varphi} \rangle = 0 \) ?
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Destroy Neel order by perturbations which preserve full square lattice symmetry e.g. second-neighbor or ring exchange.

What is the state with \( \left\langle \vec{\varphi} \right\rangle = 0 \)?
**LGW theory for quantum criticality**

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\vec{\varphi}$ by expanding in powers of $\vec{\varphi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian

$$S_\varphi = \int d^2 x d\tau \left[ \frac{1}{2} \left( c^2 (\nabla_x \vec{\varphi})^2 + (\partial_\tau \vec{\varphi})^2 + s\vec{\varphi}^2 \right) + u (\vec{\varphi}^2)^2 \right]$$

LGW theory for quantum criticality

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\vec{\phi}$ by expanding in powers of $\vec{\phi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian.

$$S_\phi = \int d^2 x d\tau \left[ \frac{1}{2} \left( c^2 (\nabla_x \vec{\phi})^2 + (\partial_\tau \vec{\phi})^2 + s \vec{\phi}^2 \right) + u (\vec{\phi}^2)^2 \right]$$

State with no broken symmetries. Fluctuations of $\vec{\phi}$ about $\vec{\phi} = 0$ realize a stable $S = 1$ quasiparticle with energy $\varepsilon_k = \sqrt{s + c^2 k^2}$.

\[\langle \vec{\phi} \rangle \neq 0, \quad \text{Néel state}\]

\[\langle \vec{\phi} \rangle = 0, \quad S_C \leftrightarrow S\]
LGW theory for quantum criticality

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\vec{\phi}$ by expanding in powers of $\vec{\phi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian

$$S_{\phi} = \int d^2 x d\tau \left[ \frac{1}{2} \left( c^2 (\nabla_x \vec{\phi})^2 + (\partial_\tau \vec{\phi})^2 + s \vec{\phi}^2 \right) + u (\vec{\phi}^2)^2 \right]$$


However, $S = 1/2$ antiferromagnets on the square lattice have no such state.

\[ \langle \vec{\phi} \rangle \neq 0 \]

Néel state

\[ \langle \vec{\phi} \rangle = 0 \]
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries.

“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.

\[
\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\]
“Liquid” of valence bonds has fractionalized $S=1/2$ excitations

There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries.

“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.

\[ = \frac{1}{\sqrt{2}} (|↑↓⟩ - |↓↑⟩) \]
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries.

“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.

\[
\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\]
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries.

“Liquid” of valence bonds has fractionalized $S=1/2$ excitations

\[\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\]
There is no state with a gapped, stable $S=1$ quasiparticle and no broken symmetries.

“Liquid” of valence bonds has fractionalized $S=1/2$ excitations.

\[
\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\]
Decompose the Néel order parameter into spinors

\[ \vec{\varphi} = z^* \vec{\sigma}_{\alpha\beta} z_{\beta} \]

where \( \vec{\sigma} \) are Pauli matrices, and \( z_{\alpha} \) are complex spinors which carry spin \( S = 1/2 \).
Decompose the Néel order parameter into spinors

\[ \vec{\varphi} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta \]

where \( \vec{\sigma} \) are Pauli matrices, and \( z_\alpha \) are complex spinors which carry spin \( S = 1/2 \).

**Key question:** Can the \( z_\alpha \) become the needed \( S = 1/2 \) excitations of a fractionalized phase?
Possible theory for fractionalization and topological order

Decompose the Néel order parameter into spinors

\[ \vec{\varphi} = z^* \vec{\sigma}_{\alpha \beta} z_\beta \]

where \( \vec{\sigma} \) are Pauli matrices, and \( z_\alpha \) are complex spinors which carry spin \( S = 1/2 \).

**Key question:** Can the \( z_\alpha \) become the needed \( S = 1/2 \) excitations of a fractionalized phase?

Effective theory for spinons must be invariant under the U(1) gauge transformation

\[ z_\alpha \rightarrow e^{i\theta} z_\alpha \]
Possible theory for fractionalization and topological order

**Naive expectation:** Low energy spinon theory for “quantum disordering” a Néel state is

\[
S_z = \int d^2xd\tau \left[ c^2 |(\nabla_x - iA_x)z_\alpha|^2 + |(\partial_\tau - iA_\tau)z_\alpha|^2 + s |z_\alpha|^2 
+ u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]
\]
**Possible theory for fractionalization and topological order**

**Naive expectation:** Low energy spinon theory for “quantum dis-ordering” a Néel state is

\[
S_z = \int d^2xd\tau \left[ c^2 |(\nabla_x - iA_x)z_\alpha|^2 + |(\partial_\tau - iA_\tau)z_\alpha|^2 + s |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]
\]

Spin liquid state with stable \( S = 1/2 \) \( z_\alpha \) spinons, and a gapless U(1) photon \( A_\mu \) representing the topological order.

\[ \langle z_\alpha \rangle \neq 0 \]

Néel state

\[ \langle z_\alpha \rangle = 0 \]
Possible theory for fractionalization and topological order

**Naive expectation:** Low energy spinon theory for “quantum disordering” a Néel state is

\[
S_z = \int d^2xd\tau \left[ c^2 |(\nabla_x - iA_x)z_\alpha|^2 + |(\partial_\tau - iA_\tau)z_\alpha|^2 + s|z_\alpha|^2 
+ u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]
\]

However, **monopoles** in the \(A_\mu\) field will proliferate because of the gap to \(z_\alpha\) excitations, and lead to **confinement** of \(z_\alpha\).
Outline

1. Quantum “disordering” magnetic order
   Collinear order and confinement

2. $\mathbb{Z}_2$ spin liquids
   Noncollinear order and fractionalization

3. Gapless $U(1)$ spin liquids
   Deconfined criticality

4. Doped spin liquids
   Superconductors with topological order
Outline

1. Quantum “disordering” magnetic order
   Collinear order and confinement

2. $\mathbb{Z}_2$ spin liquids
   Noncollinear order and fractionalization

3. Gapless U(1) spin liquids
   Deconfined criticality

4. Doped spin liquids
   Superconductors with topological order
Discrete gauge theories do have deconfined phases in 2+1 dimensions.

What is $\Phi$ in the antiferromagnet? Its physical interpretation becomes clear from its allowed coupling to the spinons:

$$S_z, \Phi = \int d^2r d\tau \left[ \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + c.c. \right]$$

From this coupling it follows that the states with $\langle \Phi \rangle \neq 0$ have coplanar spin correlations.
**Z_2 gauge theory for fractionalization and topological order**

- *Discrete* gauge theories do have deconfined phases in 2+1 dimensions.

- Find a collective excitation Φ with the gauge transformation
  \[ \Phi \rightarrow e^{2i\theta} \Phi \]

- Higgs state with \( \langle \Phi \rangle \neq 0 \) is described by the fractionalized phase of a \( Z_2 \) gauge theory in the which the spinons \( z_\alpha \) carry \( Z_2 \) gauge charges (E. Fradkin and S. Shenker, Phys. Rev. D **19**, 3682 (1979)).

---

**Z$_2$ gauge theory for fractionalization and topological order**

- *Discrete* gauge theories do have deconfined phases in 2+1 dimensions.

- Find a collective excitation $\Phi$ with the gauge transformation

  $$\Phi \rightarrow e^{2i\theta} \Phi$$

- Higgs state with $\langle \Phi \rangle \neq 0$ is described by the fractionalized phase of a $Z_2$ gauge theory in the which the spinons $z_\alpha$ carry $Z_2$ gauge charges (E. Fradkin and S. Shenker, Phys. Rev. D **19**, 3682 (1979)).

- What is $\Phi$ in the antiferromagnet? Its physical interpretation becomes clear from its allowed coupling to the spinons:

  $$S_{z,\Phi} = \int d^2r d\tau [\lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.}]$$

  From this coupling it follows that the states with $\langle \Phi \rangle \neq 0$ have **coplanar spin correlations**.

Collinear magnetic order with $\langle \Phi \rangle = 0$.

A spin density wave:

$$\langle \tilde{S}_i \rangle \propto (\cos(\mathbf{K} \cdot \mathbf{r}_i), \sin(\mathbf{K} \cdot \mathbf{r}_i), 0)$$

$$\mathbf{K} = (\pi, \pi).$$
Coplanar magnetic order with $\langle \Phi \rangle \neq 0$.

A spin density wave:

$$\langle \tilde{S}_i \rangle \propto (\cos(K \cdot r_i), \sin(K \cdot r_i), 0)$$

with

$$K = (\pi + \langle \Phi \rangle, \pi + \langle \Phi \rangle).$$

*Experimental realization: CsCuCl$_3$*
Phase diagram of gauge theory of spinons

\[ S_z = \int d^2 x d\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right] \]

Phase diagram of gauge theory of spinons

\[ S_{z,\Phi} = \int d^2xd\tau \left[ |(\partial_\mu - i A_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \]

\[ \left. + |(\partial_\mu - 2i A_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.} \right] \]

\[ \langle z_\alpha \rangle \neq 0 , \langle \Phi \rangle = 0 \]

Néel state

\[ \langle z_\alpha \rangle \neq 0 , \langle \Phi \rangle \neq 0 \]

Spiral state

U(1) spin liquid unstable to confinement

\[ \langle z_\alpha \rangle = 0 , \langle \Phi \rangle = 0 \]

Z$_2$ spin liquid with bosonic spinons $z_\alpha$

\[ \langle z_\alpha \rangle = 0 , \langle \Phi \rangle \neq 0 \]

Phase diagram of gauge theory of spinons

\[ S_{z,\Phi} = \int d^2xd\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right. \\
+ |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.} \]

U(1) spin liquid unstable to confinement

\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle = 0 \]

\[ \langle z_\alpha \rangle \neq 0 \ , \ \langle \Phi \rangle = 0 \]

Néel state

\[ \langle z_\alpha \rangle \neq 0 \ , \ \langle \Phi \rangle \neq 0 \]

Spiral state

\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle \neq 0 \]
Characteristics of $Z_2$ spin liquid

- Two classes of gapped excitations:
  - Bosonic spinons $z_\alpha$ which carry $Z_2$ gauge charge
  - $Z_2$ vortex associated with $2\pi n$ winding in phase of $\Phi$. This vortex appears as a $\pi$ flux-tubes to spinons

Characteristics of $Z_2$ spin liquid

- Two classes of gapped excitations:
  - Bosonic spinons $z_\alpha$ which carry $Z_2$ gauge charge
  - $Z_2$ vortex associated with $2\pi n$ winding in phase of $\Phi$. This vortex appears as a $\pi$ flux-tubes to spinons

- Ground state degeneracy is sensitive to topology: a $\Phi$-vortex can be inserted without energy cost in each “hole”: 4-fold degeneracy on a torus.

Characteristics of $\mathbb{Z}_2$ spin liquid

- Two classes of gapped excitations:
  - Bosonic spinons $z_\alpha$ which carry $\mathbb{Z}_2$ gauge charge
  - $\mathbb{Z}_2$ vortex associated with $2\pi n$ winding in phase of $\Phi$. This vortex appears as a $\pi$ flux-tubes to spinons
- Ground state degeneracy is sensitive to topology: a $\Phi$-vortex can be inserted without energy cost in each “hole”: 4-fold degeneracy on a torus.

Characteristics of $Z_2$ spin liquid

- Two classes of gapped excitations:
  - Bosonic spinons $z_\alpha$ which carry $Z_2$ gauge charge
  - $Z_2$ vortex associated with $2\pi n$ winding in phase of $\Phi$. This vortex appears as a $\pi$ flux-tubes to spinons

- Ground state degeneracy is sensitive to topology: a $\Phi$-vortex can be inserted without energy cost in each “hole”: 4-fold degeneracy on a torus.


- Structure identical to that found later in exactly solvable model: the $Z_2$ toric code (A. Kitaev, quant-ph/9707021).

Characteristics of $Z_2$ spin liquid

- Two classes of gapped excitations:
  - Bosonic spinons $z_\alpha$ which carry $Z_2$ gauge charge
  - $Z_2$ vortex associated with $2\pi n$ winding in phase of $\Phi$. This vortex appears as a $\pi$ flux-tubes to spinons

- Ground state degeneracy is sensitive to topology: a $\Phi$-vortex can be inserted without energy cost in each “hole”: 4-fold degeneracy on a torus.


- Structure identical to that found later in exactly solvable model: the $Z_2$ toric code (A. Kitaev, quant-ph/9707021).

- Same states (without spinons) and $Z_2$ gauge theories found to describe liquid phases of quantum dimer models (R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001)).

Outline

1. Quantum “disordering” magnetic order
   *Collinear order and confinement*

2. \(Z_2\) spin liquids
   *Noncollinear order and fractionalization*

3. Gapless U(1) spin liquids
   *Deconfined criticality*

4. Doped spin liquids
   *Superconductors with topological order*
Outline

1. Quantum “disordering” magnetic order
   Collinear order and confinement

2. $\mathbb{Z}_2$ spin liquids
   Noncollinear order and fractionalization

3. Gapless U(1) spin liquids
   Deconfined criticality

4. Doped spin liquids
   Superconductors with topological order
Phase diagram of gauge theory of spinons

\[ S_{z,\Phi} = \int d^2xd\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \]

\[ + |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.} \]

U(1) spin liquid unstable to confinement

\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle = 0 \]

Néel state

\[ \langle z_\alpha \rangle \neq 0 \ , \ \langle \Phi \rangle = 0 \]

Z_2 spin liquid with bosonic spinons \( z_\alpha \)

\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle \neq 0 \]

Spiral state

\[ \langle z_\alpha \rangle \neq 0 \ , \ \langle \Phi \rangle \neq 0 \]


Phase diagram of gauge theory of spinons

\[ S_{z,\Phi} = \int d^2xd\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \\
\left. + |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.} \right] \\
\]

Néel state
\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle = 0 \]

Spiral state
\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle \neq 0 \]

U(1) spin liquid unstable to confinement
\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle = 0 \]

\[ Z_2 \text{ spin liquid with bosonic spinons } z_\alpha \]
\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle \neq 0 \]

Quantum theory for destruction of Neel order

Partition function on cubic lattice in spacetime

\[ Z = \prod_a \int d\vec{\varphi}_a \delta (\vec{\varphi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \vec{\varphi}_a \cdot \vec{\varphi}_{a+\mu} \right) \]

LGW theory: weights in partition function are those of a classical ferromagnet at a “temperature” \( g \)

Small \( g \) \( \Rightarrow \) ground state has Neel order with \( \langle \vec{\varphi} \rangle \neq 0 \)

Large \( g \) \( \Rightarrow \) paramagnetic ground state with \( \langle \vec{\varphi} \rangle = 0 \)
Missing ingredient: Spin Berry Phases

\[ e^{iA/2} \]
Quantum theory for destruction of Neel order

Partition function on cubic lattice in spacetime

\[ Z = \prod_a \int d\vec{\phi}_a \delta (\vec{\phi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \vec{\phi}_a \cdot \vec{\phi}_{a+\mu} \right) \]

LGW theory: weights in partition function are those of a classical ferromagnet at a “temperature” \( g \)

Small \( g \) ⇒ ground state has Neel order with \( \langle \vec{\phi} \rangle \neq 0 \)

Large \( g \) ⇒ paramagnetic ground state with \( \langle \vec{\phi} \rangle = 0 \)
Quantum theory for destruction of Neel order

Coherent state path integral on cubic lattice in spacetime

\[ Z = \prod_a \int d\tilde{\varphi}_a \delta \left( \tilde{\varphi}_a^2 - 1 \right) \exp \left( \frac{1}{g} \sum_{a, \mu} \tilde{\varphi}_a \cdot \tilde{\varphi}_{a+\mu} + iS_{\text{Berry}} \right) \]

Modulus of weights in partition function: those of a classical ferromagnet at a “temperature” \( g \)

Small \( g \) ⇒ ground state has Neel order with \( \langle \bar{\varphi} \rangle \neq 0 \)

Large \( g \) ⇒ paramagnetic ground state with \( \langle \bar{\varphi} \rangle = 0 \)

Berry phases lead to large cancellations between different time histories
Quantum theory for destruction of Neel order

Partition function on cubic lattice

\[ Z = \prod_a \int d\vec{\varphi}_a \delta (\vec{\varphi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a, \mu} \vec{\varphi}_a \cdot \vec{\varphi}_{a+\mu} + iS_{\text{Berry}} \right) \]

Rewrite partition function in terms of spinors \( z_{a\alpha} \), with \( \alpha = \uparrow, \downarrow \) and

\[ \vec{\varphi}_a = z_{a\alpha} \bar{\sigma}_{\alpha\beta} z_{a\beta} \]

Quantum theory for destruction of Neel order

Partition function on cubic lattice

\[ Z = \prod_a \int \! d\vec{\varphi}_a \delta \left( \vec{\varphi}_a^2 - 1 \right) \exp \left( \frac{1}{g} \sum_{a, \mu} \vec{\varphi}_a \cdot \vec{\varphi}_{a+\mu} + iS_{\text{Berry}} \right) \]

Partition function expressed as a gauge theory of spinor degrees of freedom

\[ Z = \prod_a \int \! d\varphi_{a\alpha} dA_{a\mu} \delta \left( \sum_{\alpha} |\varphi_{a\alpha}|^2 - 1 \right) \]

\[ \times \exp \left( \frac{1}{g} \sum_{a, \mu} \varphi_{a\alpha}^* e^{iA_{a\mu}} \varphi_{a+\mu, \alpha} + i \sum_a \eta_a A_{a\tau} \right) \]

Large $g$ effective action for the $A_{a\mu}$ after integrating $z_{a\mu}$

$$Z = \prod_{a,\mu} \int dA_{a\mu} \exp \left( \frac{1}{2e^2} \sum \cos (\Delta_\mu A_{a\nu} - \Delta_\nu A_{a\mu}) + i \sum \eta_a A_{a\tau} \right)$$

with $e^2 \sim g^2$

This is compact QED in 3 spacetime dimensions with static charges $\pm 1$ on two sublattices.

This theory can be reliably analyzed by a duality mapping.

The gauge theory is in a **confining** phase, and there is VBS order in the ground state. (Proliferation of monopoles in the presence of Berry phases).

Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$
\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_{ij} - r_i)}
$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
Characterization of VBS state with $\langle \vec{\phi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$
\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}
$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the **VBS order parameter**

$$
\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}
$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter.

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$
\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_je^{i \arctan(r_j - r_i)}
$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the \textit{VBS order parameter}

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \text{arctan}(r_j-r_i)}$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the VBS order parameter

$$\Psi_{\text{vbs}}(i) = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j - r_i)}$$
Characterization of VBS state with $\langle \vec{\varphi} \rangle = 0$

Such a state breaks the symmetry of rotations by $n\pi / 2$ about lattice sites, and has $\langle \Psi_{\text{vbs}} \rangle \neq 0$, where $\Psi_{\text{vbs}}$ is the \textit{VBS order parameter}

$$\Psi_{\text{vbs}}(\hat{i}) = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j e^{i \arctan(r_j-r_i)}$$
\[ Z = \prod_a \int d\bar{z}_{a\alpha} dA_{a\mu} \delta \left( \sum_\alpha |\bar{z}_{a\alpha}|^2 - 1 \right) \exp \left( \frac{1}{g} \sum_{a,\mu} \bar{z}_{a\alpha}^* e^{iA_{a\mu}} \bar{z}_{a+\mu,\alpha} + i \sum_a \eta_a A_{a\tau} \right) \]

Neel order
\[ \langle \bar{\varphi} \rangle \neq 0 \]

or

VBS order
\[ \langle \Psi_{\text{vbs}} \rangle \neq 0 \]
Not present in LGW theory of \( \bar{\varphi} \) order
Monopole fugacity; Arovas-Auerbach state
Phase diagram of $S=1/2$ square lattice antiferromagnet

Neel order

$\langle \vec{\varphi} \rangle \sim \langle z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta \rangle \neq 0$

or

VBS order $\langle \Psi_{\text{vbs}} \rangle \neq 0$

(at associated with condensation of monopoles in $A_\mu$),

$S = 1/2$ spinons $z_\alpha$ confined,

$S = 1$ triplon excitations

Second-order critical point described by

$$S_z = \int d^2x d\tau \left[ |(\partial_\mu - iA_\mu) z_\alpha|^2 + rz_\alpha|^2 + \frac{u}{2} (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \right]$$

at its critical point $r = r_c$ where $A_\mu$ is non-compact.

Phase diagram of gauge theory of spinons

\[ S_{z, \Phi} = \int d^2 x d\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \\
+ \left. |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_\alpha z_\beta + \text{c.c.} \right] \]

\( \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle = 0 \)

Néel state

\( \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle \neq 0 \)

Spiral state

U(1) spin liquid unstable to confinement

\( \langle z_\alpha \rangle = 0, \langle \Phi \rangle = 0 \)

\( \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle \neq 0 \)

Z_2 spin liquid with bosonic spinons \( z_\alpha \)
Phase diagram of gauge theory of spinons

\[ S_{z, \Phi} = \int d^2xd\tau \left[ \left| (\partial_\mu - iA_\mu)z_\alpha \right|^2 + s_1 |z_\alpha|^2 + u \left( |z_\alpha|^2 \right)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \\
\left. + |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + \text{c.c.} \right] + \text{monopoles} + S_{\text{Berry}} \]

U(1) spin liquid unstable to confinement
\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle = 0 \]

Néel state

Z_2 spin liquid with bosonic spinons \( z_\alpha \)
\[ \langle z_\alpha \rangle = 0 \ , \ \langle \Phi \rangle \neq 0 \]

Spiral state
Phase diagram of gauge theory of spinons

\[ S_{z,\Phi} = \int d^2x d\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s_1 |z_\alpha|^2 + u (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \\
\left. + |(\partial_\mu - 2iA_\mu)\Phi|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + c.c. \right] + \text{monopoles} + S_{\text{Berry}} \]

\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle = 0 \]
Néel state

\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle \neq 0 \]
Spiral state

\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle = 0 \]
U(1) spin liquid unstable to VBS order

\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle \neq 0 \]
Z_2 spin liquid with bosonic spinons \( z_\alpha \)
Large scale Quantum Monte Carlo studies

Easy-plane model:

\[ H_{XY} = 2J \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) - K \sum_{\langle i j k l \rangle} (S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+) \]


SU(2)-invariant model:

\[ H_{SU(2)} = J \sum_{\langle ij \rangle} S_i \cdot S_j - Q \sum_{\langle i j k l \rangle} (S_i \cdot S_j - \frac{1}{4}) (S_k \cdot S_l - \frac{1}{4}) \]

Easy-plane model

\[ \mathcal{H}_{XY} = 2J \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) - K \sum_{\langle ijkl \rangle} (S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+) \]
Easy-plane model

\[ \mathcal{H}_{XY} = 2J \sum_{<ij>} (S_i^x S_j^x + S_i^y S_j^y) - K \sum_{<ijkl>} (S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+) \]

Valence bond solid (VBS) order in expectation values of plaquette and exchange terms
SU(2) invariant model

\[ \mathcal{H}_{SU(2)} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - Q \sum_{\langle ijkl \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4}) (\mathbf{S}_k \cdot \mathbf{S}_l - \frac{1}{4}) \]

Strong evidence for a continuous “deconfined” quantum critical point


SU(2) invariant model

\[ H_{SU(2)} = J \sum_{\langle ij \rangle} S_i \cdot S_j - Q \sum_{\langle ijk \rangle} (S_i \cdot S_j - \frac{1}{4}) (S_k \cdot S_l - \frac{1}{4}) \]
SU(2) invariant model

\[ \mathcal{H}_{\text{SU}(2)} = J \sum_{\langle ij \rangle} S_i \cdot S_j - Q \sum_{\langle i j k l \rangle} (S_i \cdot S_j - \frac{1}{4}) (S_k \cdot S_l - \frac{1}{4}) \]

Probability distribution of VBS order \( \Psi \) at quantum critical point

Emergent circular symmetry is evidence for U(1) photon and topological order

Outline

1. Quantum “disordering” magnetic order
   *Collinear order and confinement*

2. $\mathbb{Z}_2$ spin liquids
   *Noncollinear order and fractionalization*

3. Gapless $\text{U}(1)$ spin liquids
   *Deconfined criticality*

4. Doped spin liquids
   *Superconductors with topological order*
Outline

1. Quantum “disordering” magnetic order
   
   Collinear order and confinement

2. $\mathbb{Z}_2$ spin liquids
   
   Noncollinear order and fractionalization

3. Gapless U(1) spin liquids
   
   Deconfined criticality

4. Doped spin liquids
   
   Superconductors with topological order
Hole dynamics in an antiferromagnet across a deconfined quantum critical point,
R. K. Kaul, A. Kolezhuk, M. Levin, S. Sachdev, and T. Senthil,

Algebraic charge liquids and the underdoped cuprates,
R. K. Kaul, Y. B. Kim, S. Sachdev, and T. Senthil,
arXiv:0706.2187
Phase diagram of gauge theory of spinons

\[ S_{z, \Phi} = \int d^2 x d\tau \left[ \left| (\partial\mu - iA_\mu)z_\alpha \right|^2 + s_1 |z_\alpha|^2 + u \left( |z_\alpha|^2 \right)^2 + \frac{1}{4e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \]

\[ + \left. \left| (\partial\mu - 2iA_\mu)\Phi \right|^2 + s_2 |\Phi|^2 + \tilde{u} |\Phi|^4 + \lambda \Phi^* \epsilon_{\alpha\beta} z_\alpha \partial_x z_\beta + c.c. \right] + \text{monopoles} + S_{\text{Berry}} \]

\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle = 0 \]
Néel state

\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle = 0 \]
U(1) spin liquid unstable to VBS order

\[ \langle z_\alpha \rangle \neq 0, \langle \Phi \rangle \neq 0 \]
Spiral state

\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle \neq 0 \]
Critical U(1) spin liquid

\[ \langle z_\alpha \rangle = 0, \langle \Phi \rangle \neq 0 \]
Z_2 spin liquid with bosonic spinons z_\alpha
Phase diagram of gauge theory of spinons and holons

Add a finite concentration of charge carriers

\[
\langle z_\alpha \rangle \neq 0 , \langle \Phi \rangle = 0 \\
\text{Néel state}
\]

\[
\langle z_\alpha \rangle = 0 , \langle \Phi \rangle = 0 \\
\text{U(1) spin liquid unstable to VBS order}
\]

\[
\langle z_\alpha \rangle \neq 0 , \langle \Phi \rangle \neq 0 \\
\text{Spiral state}
\]

\[
\langle z_\alpha \rangle = 0 , \langle \Phi \rangle \neq 0 \\
Z_2 \text{ spin liquid with bosonic spinons } z_\alpha
\]
Phase diagram of doped antiferromagnets

$La_2CuO_4$

VBS order

Neél order

Hole density $x$

$s_1$
• Begin with the representation of the quantum antiferromagnet as the lattice $\mathbb{C}P^1$ model:

$$S_z = -\frac{1}{g} \sum_{a,\mu} \bar{z}_{a\alpha}^* e^{iA_{a\mu}} z_{a+\mu,\alpha} + i \sum_a \eta_a A_{a\tau}$$

• Write the electron operator at site $r$, $c_\alpha(r)$ in terms of fermionic holon operators $f_\pm$

$$c_\alpha(r) = \begin{cases} f_+^\dagger(r) z_{r\alpha} & \text{for } r \text{ on sublattice } A \\ \varepsilon_{\alpha\beta} f_-^\dagger(r) z_{r\beta}^* & \text{for } r \text{ on sublattice } B \end{cases}$$

Note that the holons $f_s$ have charge $s$ under the U(1) gauge field $A_\mu$. 
• Choose the dispersion, $\epsilon(\vec{k})$ of the $f_{\pm}$ in momentum space so that its minima are at $(\pm \pi/2, \pm \pi/2)$. To avoid double-counting, these dispersions must be restricted to be within the diamond Brillouin zone.

\[ S_f = \int d\tau \sum_{s=\pm} \int \frac{d^2k}{4\pi^2} f_s^\dagger(\vec{k}) \left( \partial_\tau - isA_\tau + \epsilon(\vec{k} - s\vec{A}) \right) f_s(\vec{k}) \]

• Include the hopping between opposite sublattices (Shraiman-Siggia term):

\[ S_t = -t \sum_{\langle rr' \rangle} c_\alpha^\dagger(r)c_\alpha(r') + \text{h.c.} \]

\[ = -t \sum_{\langle rr' \rangle} (f_+^\dagger(r)z_{r\alpha})^\dagger \varepsilon_{\alpha\beta} f_-^\dagger(r')z_{r\beta}^* \]

• Complete theory for doped antiferromagnet:

\[ S = S_z + S_f + S_t \]
Phase diagram of lightly doped antiferromagnet

\[ A = (2\pi)^2 x / 8 \]

\[ A = (2\pi)^2 x / 4 \]
Pictorial explanation of factor of 2:

- In the Néel phase, sublattice index is identical to spin index. So for each valley and momentum, degeneracy of the hole state is 2.

- In the VBS state, the sublattice index and the spin index are distinct. So for each valley and momentum, degeneracy of the hole state is 4.
Phase diagram of lightly doped antiferromagnet

\[ A = (2\pi)^2 x/8 \]

\[ A = (2\pi)^2 x/4 \]
Phase diagram of lightly doped antiferromagnet

\[ A = (2\pi)^2 x / 8 \]

\[ A = (2\pi)^2 x / 4 \]
A new non-Fermi liquid phase:  
The holon metal  
An algebraic charge liquid.

- Ignore compactness in $A_\mu$ and Berry phase term.
- Neutral spinons $z_\alpha$ are gapped.
- Charge $e$ fermions $f_s$ form Fermi surfaces and carry charges $s = \pm 1$ under the U(1) gauge field $A_\mu$.
- Quasi-long range order in a variety of VBS and pairing correlations.

Area of each Fermi pocket,

$$A = (2\pi)^2 x/4.$$  

The Fermi pocket will show sharp magnetoresistance oscillations, but it is invisible to photoemission.
Quantum oscillations and the Fermi surface in an underdoped high $T_c$ superconductor (ortho-II ordered YBa$_2$Cu$_3$O$_{6.5}$).

Holon pairing leading to $d$-wave superconductivity

First consider holon pairing in the Neel state, where holon=hole.

This was studied in V. V. Flambaum, M. Yu. Kuchiev, and O. P. Sushkov, *Physica C* 227, 267 (1994); V. I. Belincher et al., *Phys. Rev. B* 51, 6076 (1995). They found $p$-wave pairing of holons, induced by spin-wave exchange from the sublattice mixing term $S_t$. This corresponds to $d$-wave pairing of physical electrons.
Holon pairing leading to $d$-wave superconductivity
Holon pairing leading to $d$-wave superconductivity

Gap nodes
Holon pairing leading to \(d\)-wave superconductivity

We assume the same pairing holds across a transition involving loss of long-range Néel order. The resulting phase is another algebraic charge liquid - the \textit{holon superconductor}. This superconductor has gapped spinons with no electrical charge, and spinless, nodal Bogoliubov-Dirac quasiparticles. The superconductivity does \textbf{not} gap the U(1) gauge field \(A_\mu\), because the Cooper pairs are gauge neutral.
Low energy theory of holon superconductor

4 two-component Dirac quasiparticles coupled to a U(1) gauge field

\[ S_{\text{holon superconductor}} = \int d\tau d^2r \left[ \frac{1}{2e_0^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right. \]

\[ + \sum_{i=1}^{4} \left. \psi_i^\dagger (D_\tau - i v_F (\partial_x - i A_x) \tau^x - i v_F (\partial_y - i A_y) \tau^y) \psi_i \right. \]
Low energy theory of holon superconductor

External vector potential $\vec{A}$ couples as

$$\mathcal{H}_A = \vec{j} \cdot \vec{A}$$

where

$$j_x = v_F \left( \psi_3^\dagger \psi_3 - \psi_1^\dagger \psi_1 \right) , \quad j_x = v_F \left( \psi_4^\dagger \psi_4 - \psi_2^\dagger \psi_2 \right)$$

are conserve charges of $S_{\text{holon superconductor}}$.

**Fundamental property:** The superfluid density, $\rho_s$, has the following $x$ and $T$ dependence:

$$\rho_s(x, T) = cx - R k_B T$$

where $c$ is a non-universal constant and $R$ is a universal constant obtained in a $1/N$ expansion ($N = 4$ is the number of Dirac fermions):

$$R = 0.4412 + \frac{0.074}{N} + \ldots$$
AF Mott Insulator

AF Metal

Holon-hole metal

Algebraic charge liquid

Superconducting algebraic charge liquid

Holon-hole superconductor

AF+ dSC
Conclusions

1. Theory for $\mathbb{Z}_2$ and U(1) spin liquids in quantum antiferromagnets, and evidence for their realization in model spin systems.

2. *Algebraic charge liquids* appear naturally upon adding fermionic carriers to spin liquids with bosonic spinons. These are conducting states with topological order.

3. The holon metal/superconductor, obtained by doping a Neel-ordered insulator, matches several observed characteristics of the underdoped cuprates.