Quantum phase transitions: from antiferromagnets and superconductors to black holes

Talk online: sachdev.physics.harvard.edu
Outline

1. Introduction to quantum phase transitions: quantum spin systems and relativistic field theories

2. Quantum phase transitions in $d$-wave superconductors and metals

3. The AdS/CFT correspondence: quantum criticality at strong coupling

4. The cuprate high temperature superconductors: competing orders and quantum criticality
1. Introduction to quantum phase transitions: quantum spin systems and relativistic field theories

2. Quantum phase transitions in $d$-wave superconductors and metals

3. The AdS/CFT correspondence: quantum criticality at strong coupling

4. The cuprate high temperature superconductors: competing orders and quantum criticality
The cuprate superconductors
Ground state has long-range Néel order

Order parameter is a single vector field $\vec{\varphi} = \eta_i \vec{S}_i$

$\eta_i = \pm 1$ on two sublattices

$\langle \vec{\varphi} \rangle \neq 0$ in Néel state.
Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface

K.M. Shen et al., Science 2005
M. Platé et al., PRL 2005

Smaller hole Fermi-pockets

Large hole Fermi surface
Antiferromagnetism

Fermi surface

d-wave superconductivity
Outline

2. Quantum phase transitions in \textit{d}-wave superconductors and metals

A. \textit{d}-wave superconductivity

B. Discrete symmetry breaking in a \textit{d}-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal
2. Quantum phase transitions in \( d \)-wave superconductors and metals

A. \( d \)-wave superconductivity

B. Discrete symmetry breaking in a \( d \)-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal
Antiferromagnetism

Fermi surface

d-wave superconductivity
d-wave superconductivity in cuprates

\[ \mathbf{H}_0 = - \sum_{i<j} t_{ij} c^\dagger_{i\alpha} c_{i\alpha} \equiv \sum_{k} \varepsilon_k c^\dagger_{k\alpha} c_{k\alpha} \]

- Begin with free electrons.
**d-wave superconductivity in cuprates**

\[
H = \sum_{k} \left( \varepsilon_{k} c_{k\alpha}^\dagger c_{k\alpha} + \Delta_{k} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.} \right)
\]

- Begin with free electrons.

- Add *d*-wave pairing interaction
  \[\Delta_{k} \sim \cos k_{x} - \cos k_{y}\] which vanishes along diagonals
d-wave superconductivity in cuprates

\[ H = \sum_k \left( \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} + \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.} \right) \]

- Begin with free electrons.
- Add d-wave pairing interaction \( \Delta_k \) which vanishes along diagonals.
- Obtain Bogoliubov quasiparticles with dispersion \( \sqrt{\varepsilon_k^2 + \Delta_k^2} \).
d-wave superconductivity in cuprates

\[ S_{\Psi} = \int \frac{d^2 k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{1a}^\dagger (-i\omega_n + v_F k_x \tau^z + v_{\Delta} k_y \tau^x) \Psi_{1a} \]

\[ + \int \frac{d^2 k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{2a}^\dagger (-i\omega_n + v_F k_y \tau^z + v_{\Delta} k_x \tau^x) \Psi_{2a}. \]

4 two-component Dirac fermions
2. Quantum phase transitions in $d$-wave superconductors and metals

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C. Nematic ordering in a metal
Nematic order in YBCO

Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D. A. Bonn, W. N. Hardy, and Louis Taillefer
arXiv: 0909.4430

d-wave superconductivity in cuprates

Now consider a discrete spontaneous symmetry breaking, with Ising symmetry, described by a real scalar field $\phi$. Two cases of experimental interest are:

- Break 4-fold lattice rotation symmetry to 2-fold lattice rotations: leads to a superconductor with nematic order: equivalent to $d_{x^2-y^2} + s$ pairing.

$$H = H_\phi + \sum_k \left( \varepsilon_k c_{k\alpha}^{\dagger} c_{k\alpha} + \Delta_k c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} + \text{c.c.} \right)$$

$$H_\phi = \phi \sum_k c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} + \text{c.c.}$$
**d-wave superconductivity in cuprates**

Now consider a discrete spontaneous symmetry breaking, with Ising symmetry, described by a real scalar field $\phi$.

Two cases of experimental interest are:

- Break 4-fold lattice rotation symmetry to 2-fold lattice rotations: leads to a superconductor with **nematic** order: equivalent to $d_{x^2-y^2} + s$ pairing.

- Time-reversal symmetry breaking: leads to a $d_{x^2-y^2} + id_{xy}$ superconductor, in which the Dirac fermions are massive

\[
H = H_\phi + \sum_k \left( \varepsilon_k c_{k\alpha}^\dagger c_{k\alpha} + \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.} \right)
\]

\[
H_\phi = i\phi \sum_k \sin k_x \sin k_y c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \text{c.c.}
\]
Lattice rotation symmetry breaking

\[ d_{x^2 - y^2} \text{ superconductor} + \text{nematic order} \]
\[ \langle \phi \rangle \neq 0 \]

\[ \langle \phi \rangle = 0 \]

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Time-reversal symmetry breaking

\[ d_{x^2 - y^2} \pm id_{xy} \]

superconductor

\[ \langle \phi \rangle \neq 0 \]

\[ d_{x^2 - y^2} \] superconductor

\[ \langle \phi \rangle = 0 \]
Discrete symmetry breaking in d-wave superconductors

Field theory for transition with Ising order described by a real scalar field $\phi$:

$$\mathcal{S} = \mathcal{S}_\Psi + \mathcal{S}_\phi + \mathcal{S}_{\Psi\phi}$$

4 two-component Dirac fermions

$$\mathcal{S}_\Psi = \int \frac{d^2 k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{1a}^\dagger \left( -i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x \right) \Psi_{1a}$$

$$+ \int \frac{d^2 k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{2a}^\dagger \left( -i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x \right) \Psi_{2a}.$$
Discrete symmetry breaking in d-wave superconductors

Field theory for transition with Ising order described by a real scalar field $\phi$:

$$S = S_\Psi + S_\phi + S_{\Psi\phi}$$

4 two-component Dirac fermions

$$S_\Psi = \int \frac{d^2 k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{1a}^\dagger \left(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x \right) \Psi_{1a}$$

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Ising field theory

$$S_\phi = \int d^2 x d\tau \left[ \frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2 + \frac{u_0}{24} \phi^4 \right];$$
Ising order and Dirac fermions couple via a “Yukawa” term.

\[ S_{\Psi \phi} = \int d^2 x d\tau \left[ \lambda_0 \phi \left( \Psi_1^\dagger \tau^x \Psi_1 + \Psi_2^\dagger \tau^x \Psi_2 \right) \right], \]

Nematic ordering

\[ S_{\Psi \phi} = \int d^2 x d\tau \left[ \lambda_0 \phi \left( \Psi_1^\dagger \tau^y \Psi_1 + \Psi_2^\dagger \tau^y \Psi_2 \right) \right], \]

Time reversal symmetry breaking

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\[ S_{\Psi \phi} = \int d^2x d\tau \left[ \lambda_0 \phi \left( \Psi_{1a}^\dagger \tau^y \Psi_{1a} + \Psi_{2a}^\dagger \tau^y \Psi_{2a} \right) \right], \]

Time reversal symmetry breaking

For the latter case only, with \( v_F = v_{\Delta} = c \), theory reduces to relativistic Gross-Neveu model

Integrating out the fermions yields an effective action for the scalar order parameter

\[
S_\phi = \frac{N_f}{v_\Delta v_F} \Gamma \left[ \lambda_0 \phi(x, \tau); \frac{v_\Delta}{v_F} \right] + \frac{N_f}{2} \int d^2xd\tau \left( r \phi^2(x, \tau) \right) + \text{irrelevant terms}
\]

where \( \Gamma \) is a non-local and non-analytic functional of \( \phi \).

The theory has only 2 couplings constants: \( r \) and \( v_\Delta/v_F \).

Integrating out the fermions yields an effective action for the nematic order parameter

\[ S_\phi = \frac{N_f}{2} \int_{k,\omega} |\phi(k, \omega)|^2 r \]

\[ + \frac{\lambda_0^2}{8 v_F v_\Delta} \left( \frac{\omega^2 + v_F^2 k_x^2}{\sqrt{\omega^2 + v_F^2 k_x^2 + v_\Delta^2 k_y^2}} + (x \leftrightarrow y) \right) \]

+higher order terms which cannot be neglected

Integrating out the fermions yields an effective action for the $T$-breaking order parameter.

\[ S_\phi = \frac{N_f}{2} \int_{k, \omega} |\phi(k, \omega)|^2 \left[ r + \frac{\lambda_0^2}{8v_Fv_\Delta} \left( \sqrt{\omega^2 + v_F^2 k_x^2 + v_\Delta^2 k_y^2} + (x \leftrightarrow y) \right) \right] + \text{higher order terms which cannot be neglected} \]

Integrating out the fermions yields an effective action for the scalar order parameter 

\[ S_\phi = \frac{N_f}{v_\Delta v_F} \Gamma \left[ \lambda_0 \phi(x, \tau); \frac{v_\Delta}{v_F} \right] \]

\[ + \frac{N_f}{2} \int d^2x d\tau \left( r \phi^2(x, \tau) \right) \]

\[ + \text{irrelevant terms} \]

where \( \Gamma \) is a non-local and non-analytic functional of \( \phi \).

The theory has only 2 couplings constants: \( r \) and \( v_\Delta / v_F \).

Integrating out the fermions yields an effective action for the scalar order parameter

\[
S_\phi = \frac{N_f}{v_\Delta v_F} \Gamma \left[ \lambda_0 \phi(x, \tau); \frac{v_\Delta}{v_F} \right] \\
+ \frac{N_f}{2} \int d^2 x d\tau \left( r \phi^2(x, \tau) \right) \\
+ \text{irrelevant terms}
\]

where \( \Gamma \) is a non-local and non-analytic functional of \( \phi \).

There is a systematic expansion in powers of \( 1/N_f \) for renormalization group equations and all critical properties.

Outline

2. Quantum phase transitions in \textit{d}-wave superconductors and metals

\begin{enumerate}
  \item \textit{d}-wave superconductivity
  \item \textit{Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal}
  \item Nematic ordering in a metal
\end{enumerate}
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2. Quantum phase transitions in $d$-wave superconductors and metals

   A. $d$-wave superconductivity

   B. Discrete symmetry breaking in a $d$-wave superconductor: reflection (nematic ordering) or time-reversal

   C. Nematic ordering in a metal
Quantum criticality of Pomeranchuk instability

Fermi surface with full square lattice symmetry
Electron Green’s function in Fermi liquid (T=0)

\[ G(k, \omega) = \frac{Z}{\omega - v_F(k - k_F) - i\omega^2 \mathcal{F}\left(\frac{k-k_F}{\omega}\right)} + \ldots \]
Electron Green’s function in Fermi liquid (T=0)

\[ G(k, \omega) = \frac{Z}{\omega - v_F(k - k_F) - i\omega^2 \mathcal{F} \left( \frac{k - k_F}{\omega} \right)} + \ldots \]

Green’s function has a pole in the LHP at

\[ \omega = v_F(k - k_F) - i\alpha(k - k_F)^2 + \ldots \]

Pole is at \( \omega = 0 \) precisely at \( k = k_F \) i.e. on a sphere of radius \( k_F \) in momentum space. This is the Fermi surface.
Quantum criticality of Pomeranchuk instability

Fermi surface with full square lattice symmetry
Quantum criticality of Pomeranchuk instability

Spontaneous elongation along $x$ direction:
Ising order parameter $\phi > 0$. 

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Quantum criticality of Pomeranchuk instability

Spontaneous elongation along $y$ direction:
Ising order parameter $\phi < 0$. 
Quantum criticality of Pomeranchuk instability

Pomeranchuk instability as a function of coupling $\lambda$

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$
Quantum criticality of Pomeranchuk instability

Phase diagram as a function of $T$ and $\lambda$

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$T_c$

$\lambda_c$

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Phase diagram as a function of $T$ and $\lambda$

Classical $d=2$ Ising criticality

Quantum critical

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$T_c$

$\lambda_c$
Quantum criticality of Pomeranchuk instability

Phase diagram as a function of $T$ and $\lambda$

$\langle \phi \rangle = 0$

$\langle \phi \rangle \neq 0$

$D=2+1$ Ising criticality?
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

\[
S_\phi = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]
\]

Effective action for electrons:

\[
S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right]
\]
\[
\equiv \sum_{\alpha=1}^{N_f} \sum_\mathbf{k} \int d\tau c_{\mathbf{k}\alpha}^\dagger (\partial_\tau + \varepsilon_\mathbf{k}) c_{\mathbf{k}\alpha}
\]
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

\[ S_{\phi c} = -\gamma \int d\tau \phi \sum_{\alpha=1}^{N_f} \sum_{k} (\cos k_x - \cos k_y) c_{k\alpha}^\dagger c_{k\alpha} \]

for spatially independent \( \phi \)

\[ \langle \phi \rangle > 0 \]

\[ \langle \phi \rangle < 0 \]
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^{\dagger} c_{k-q/2,\alpha} \]

for spatially dependent \( \phi \)

\[ \langle \phi \rangle > 0 \]

\[ \langle \phi \rangle < 0 \]
Quantum criticality of Pomeranchuk instability

\[ S_\phi = \int d^2 r d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k^\alpha) c_{k\alpha} \]

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,\mathbf{q}} \phi_{\mathbf{q}} (\cos k_x - \cos k_y) c_{k+\mathbf{q}/2,\alpha}^\dagger c_{k-\mathbf{q}/2,\alpha} \]

Quantum critical field theory

\[ \mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}c_{i\alpha} \exp (-S_\phi - S_c - S_{\phi c}) \]

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Integrate out $c_\alpha$ fermions and obtain non-local corrections to $\phi$ action

$$
\delta S_\phi \sim N_f \gamma^2 \int \frac{d^2 q}{4\pi^2} \int \frac{d\omega}{2\pi} |\phi(q, \omega)|^2 \left[ \frac{\omega}{q} + q^2 \right] + \ldots
$$

This leads to a critical point with dynamic critical exponent $z = 3$ and quantum criticality controlled by the Gaussian fixed point.
Quantum criticality of Pomeranchuk instability

Hertz theory

Self energy of $c_\alpha$ fermions to order $1/N_f$

$$\Sigma_c(k, \omega) \sim \frac{i}{N_f} \omega^{2/3}$$

This leads to the Green's function

$$G(k, \omega) \approx \frac{1}{\omega - v_F(k - k_F) - \frac{i}{N_f} \omega^{2/3}}$$

Note that the order $1/N_f$ term is more singular in the infrared than the bare term; this leads to problems in the bare $1/N_f$ expansion in terms that are dominated by low frequency fermions.
Quantum criticality of Pomeranchuk instability

The infrared singularities of fermion particle-hole pairs are most severe on planar graphs: these all contribute at leading order in $1/N_f$.

Sung-Sik Lee, Physical Review B 80, 165102 (2009)
Quantum criticality of Pomeranchuk instability

A string theory for the Fermi surface?