Quantum phase transitions of correlated electrons and atoms

Subir Sachdev
Harvard University

See also: Quantum phase transitions of correlated electrons in two dimensions, cond-mat/0109419.

Quantum Phase Transitions
Cambridge University Press
What is a quantum phase transition?

Non-analyticity in ground state properties as a function of some control parameter $g$

True level crossing:
Usually a *first*-order transition

Avoided level crossing which becomes sharp in the infinite volume limit:
*second*-order transition
Why study quantum phase transitions?

- Theory for a quantum system with strong correlations: describe phases on either side of $g_c$ by expanding in deviation from the quantum critical point.
- Critical point is a novel state of matter without quasiparticle excitations.
- Critical excitations control dynamics in the wide quantum-critical region at non-zero temperatures.

Important property of ground state at $g = g_c$: temporal and spatial scale invariance; characteristic energy scale at other values of $g$: $\Delta \sim |g - g_c|^{\nu}$.
I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
   *Mean field theory and the evolution of the excitation spectrum.*

III. Superfluid-insulator transition
   *Boson Hubbard model at integer filling.*

IV. Bosons at fractional filling
   *Beyond the Landau-Ginzburg-Wilson paradigm.*

V. Quantum phase transitions and the Luttinger theorem
   *Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell*
I. Quantum Ising Chain
I. Quantum Ising Chain

Degrees of freedom: \( j = 1 \ldots N \) qubits, \( N \) "large"

\[ |\uparrow\rangle_j, |\downarrow\rangle_j \]

or \( |\rightarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j + |\downarrow\rangle_j \right) \), \( |\leftarrow\rangle_j = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_j - |\downarrow\rangle_j \right) \)

Hamiltonian of decoupled qubits:

\[ H_0 = -Jg \sum_j \sigma_j^x \]
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z \]

Prefers neighboring qubits are either \( \uparrow_j \uparrow_{j+1} \) or \( \downarrow_j \downarrow_{j+1} \) (not entangled)

Full Hamiltonian

\[ H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right) \]

leads to entangled states at \( g \) of order unity
Experimental realization

\[ \text{LiHoF}_4 \]
Weakly-coupled qubits \((g \gg 1)\)

Ground state:
\[
|G\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \rangle
- \frac{1}{2g} |\cdots \rightarrow \rightarrow \rightarrow \leftrightarrow \leftrightarrow \rightarrow \rightarrow \rightarrow \cdots \rangle - \cdots
\]

Lowest excited states:
\[
|\ell_j\rangle = |\cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftrightarrow_j \rightarrow \rightarrow \rightarrow \rightarrow \cdots \rangle + \cdots
\]

Coupling between qubits creates “flipped-spin” quasiparticle states at momentum \(p\)
\[
|p\rangle = \sum_j e^{ipx_j/\hbar} |\ell_j\rangle
\]

Excitation energy \(\varepsilon(p) = \Delta + 4J \sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^{-1}\right)\)

Excitation gap \(\Delta = 2gJ - 2J + O\left(g^{-1}\right)\)

Entire spectrum can be constructed out of multi-quasiparticle states
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)
while transferring energy $\hbar \omega$ and momentum $p$

$Z \delta(\omega - \varepsilon(p))$

Quasiparticle pole

Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to
a Lorentzian of width $1/\tau_\phi$, where the phase coherence time $\tau_\phi$
is given by

$$\frac{1}{\tau_\phi} = \frac{2k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

Ground states:

\[ |G \uparrow\rangle = \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots \]

\[ -\frac{g}{2} \left| \cdots \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \cdots \right\rangle - \cdots \]

Second state \( |G \downarrow\rangle \) obtained by \( \uparrow \Leftrightarrow \downarrow \)

\( |G \downarrow\rangle \) and \( |G \uparrow\rangle \) mix only at order \( g^N \)

Lowest excited states: domain walls

\[ |d_j\rangle = |\cdots \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \cdots \rangle + \cdots \]

Coupling between qubits creates new “domain-wall” \emph{quasiparticle} states at momentum \( p \)

\[ |p\rangle = \sum_j e^{ipx_j/\hbar} |d_j\rangle \]

Excitation energy \( \epsilon(p) = \Delta + 4Jg \sin^2 \left( \frac{pa}{2\hbar} \right) + O(g^2) \)

Excitation gap \( \Delta = 2J - 2gJ + O(g^2) \)
Dynamic Structure Factor $S(p, \omega)$: Strongly-coupled qubits ($g \ll 1$)

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum $p$

Structure holds to all orders in $g$

At $T > 0$, motion of domain walls leads to a finite phase coherence time $\tau_\varphi$, and broadens coherent peak to a width $1/\tau_\varphi$ where

$$\frac{1}{\tau_\varphi} = \frac{2k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

Entangled states at $g$ of order unity

"Flipped-spin" Quasiparticle weight $Z$

$Z \sim (g - g_c)^{1/4}$


Ferromagnetic moment $N_0$

$N_0 \sim (g_c - g)^{1/8}$


Excitation energy gap $\Delta$

$\Delta \sim |g - g_c|$
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $\left\langle - \right\rangle$ to a $\left\langle + \right\rangle$ (or vice versa)

while transferring energy $\hbar \omega$ and momentum $p$

Critical coupling ($g = g_c$)

$$S(p, \omega) \sim \left( \omega^2 - c^2 p^2 \right)^{-7/8}$$

No quasiparticles --- dissipative critical continuum
\[ H_I = -J \sum_i \left( g \sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \]

\[ \chi(\omega) = \frac{i}{\hbar} \sum_{k} \int_{0}^{\infty} dt \left\langle \left[ \sigma_j^z(t), \sigma_k^z(0) \right] \right\rangle e^{i\omega t} \]

\[ \Gamma_R = \left( 2 \tan \frac{\pi}{16} \right) \frac{k_B T}{\hbar} \]

\[ \langle \sigma_j^z \sigma_k^z \rangle \sim \frac{1}{|j-k|^{1/4}} \]


Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
   Mean field theory and the evolution of the excitation spectrum.

III. Superfluid-insulator transition
   Boson Hubbard model at integer filling.

IV. Bosons at fractional filling
   Beyond the Landau-Ginzburg-Wilson paradigm.

V. Quantum phase transitions and the Luttinger theorem
   Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell
II. Landau-Ginzburg-Wilson theory

Mean field theory and the evolution of the excitation spectrum
• Identify order parameter $\phi(x, \tau) \sim \sigma_j^z$

• Symmetries:

  Spin inversion: $\phi \rightarrow -\phi$

  Time reversal $\tau \rightarrow -\tau$

  Spatial inversion $x \rightarrow -x$

• Write down most general Lagrangian consistent with symmetries

$$Z = \int \mathcal{D}\phi(x, \tau) \exp \left(-\int d^d x \int d\tau \mathcal{L}[\phi]\right)$$

$$\mathcal{L}[\phi] = \frac{1}{2} (\partial_\tau \phi)^2 + \frac{c^2}{2} (\nabla_x \phi)^2 + \frac{r}{2} \phi^2 + \frac{u}{4} \phi^4 + \ldots$$

• Identify phases at $r \gg 0$ and $r \ll 0$ with the paramagnet and the ferromagnet respectively.
Quantum field theory formally resembles the classical statistical mechanics of an Ising model in $d + 1$ dimensions. Theory of second-order classical phase transitions implies that at the critical point the susceptibility depends on the $d + 1$ dimensional momentum $k$ as

$$\chi(k) \sim \frac{1}{k^{2-\eta}}$$

After analytic continuation, and using the “Lorentz invariance” of the critical theory, the quantum critical point therefore has the following dynamic susceptibility at $T = 0$.

$$\chi(p, \omega) \sim \frac{1}{(c^2 p^2 - \omega^2)^{1-\eta/2}}$$

At $T > 0$, we have to consider a classical statistical mechanics problem in finite geometry with a ‘temporal’ direction of extent $L_\tau = \hbar/(k_B T)$. Finite size scaling now implies that the susceptibility at the critical point obeys

$$\chi(k) \sim L^{2-\eta}_\tau F(k; L_\tau)$$

After analytic continuation, the quantum system has the dynamic response (note: can no longer use ”Lorentz invariance”)

$$\chi''(0, \omega) \sim \frac{1}{T^2} \frac{\Phi \left( \frac{\hbar \omega}{k_B T} \right)}{\eta}$$
Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
   *Mean field theory and the evolution of the excitation spectrum.*

III. Superfluid-insulator transition
   *Boson Hubbard model at integer filling.*

IV. Bosons at fractional filling
   *Beyond the Landau-Ginzburg-Wilson paradigm.*

V. Quantum phase transitions and the Luttinger theorem
   *Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell*
III. Superfluid-insulator transition

Boson Hubbard model at integer filling
Bose condensation
Velocity distribution function of ultracold $^{87}$Rb atoms

Apply a periodic potential (standing laser beams) to trapped ultracold bosons (\(^{87}\text{Rb}\))
Momentum distribution function of bosons

Bragg reflections of condensate at reciprocal lattice vectors

Superfluid-insulator quantum phase transition at $T=0$

$V_0=0E_r$  $V_0=3E_r$  $V_0=7E_r$  $V_0=10E_r$

$V_0=13E_r$  $V_0=14E_r$  $V_0=16E_r$  $V_0=20E_r$
Bosons at filling fraction $f = 1$

Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

Bosons at filling fraction $f = 1$

Weak interactions: superfluidity
Bosons at filling fraction $f = 1$

$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity
Bosons at filling fraction $f = 1$

Weak interactions: superfluidity

$\langle \Psi \rangle \neq 0$
Bosons at filling fraction $f = 1$

$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity
Bosons at filling fraction $f = 1$

$$\langle \Psi \rangle = 0$$

Strong interactions: insulator
The **Superfluid-Insulator transition**

**Boson Hubbard model**

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, $j$, of a lattice, with short-range repulsive interactions.

\[
H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots
\]

\[
n_j \equiv b_j^\dagger b_j
\]


For small $U/t$, ground state is a superfluid BEC with

superfluid density $\approx$ density of bosons
What is the ground state for large $U/t$?

Typically, the ground state remains a superfluid, but with superfluid density $\ll$ density of bosons

The superfluid density evolves smoothly from large values at small $U/t$, to small values at large $U/t$, and there is no quantum phase transition at any intermediate value of $U/t$.

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions)
What is the ground state for large $U/t$?

Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities

\[ \langle n_j \rangle = 3 \quad -\frac{t}{U} \]

\[ \langle n_j \rangle = 7/2 \]

Ground state has “density wave” order, which spontaneously breaks lattice symmetries
Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*

Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_p}$
Excitations of the insulator: infinitely long-lived, finite energy quasiparticles and quasiholes

Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_p}$
Excitations of the insulator: infinitely long-lived, finite energy quasiparticles and quasiholes

Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_p,h}$
LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x, \tau) \sim b_j^\dagger$

- Symmetries:
  
  Gauge invariance: $\Psi \to \Psi e^{i\theta}$
  
  Time reversal $\tau \to -\tau$ ; $\Psi \to \Psi^*$
  
  Spatial inversion $x \to -x$

- Write down most general Lagrangian consistent with symmetries

  \[ Z = \int D\Psi(x, \tau) \exp \left( -\int d^d x \int d\tau \mathcal{L} [\Psi] \right) \]

  \[ \mathcal{L} [\Psi] = K \Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r |\Psi|^2 + \frac{u}{2} |\Psi|^4 + \ldots \]

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.

- For $K \neq 0$, the particle and hole excitations have different energies.
• Gauge-invariance of the underlying boson Hamiltonian shows that

\[ K = -\frac{\partial r}{\partial \mu} \]

• In mean-field theory, the ground state energy, \( E \), across the superfluid-insulator transition has the non-analytic term

\[ E = E_0 - \frac{r^2}{2u} \theta(-r) \]

(Beyond mean-field theory, the non-analytic term is \( E \sim r^{(d+z)\nu} \)).

• Because the density of bosons = \(-\partial E/\partial \mu\), this implies a change in the boson density across the transition \textit{unless} \( \partial r/\partial \mu = 0 \)

• A superfluid-insulator transition at fixed boson density must have

\[ K = 0 \]
Boson Green's function $G(p, \omega)$:

Cross-section to add a boson while transferring energy $\hbar \omega$ and momentum $p$

Insulating ground state

Continuum of two quasiparticles + one quasihole

Quasiparticle pole

$Z \delta(\omega - \varepsilon(p))$

Similar result for quasi-hole excitations obtained by removing a boson
Entangled states at $g \equiv t/U$ of order unity

Quasiparticle weight $Z$

$Z \sim (g_c - g)^{\eta \nu}$


Excitation energy gap $\Delta$

$\Delta_{p,h} \sim (g_c - g)^{\nu}$ for $g < g_c$

$\Delta_{p,h} = 0$ for $g > g_c$

Superfluid density $\rho_s$

$\rho_s \sim (g - g_c)^{(d+z-2)\nu}$
Crossovers at nonzero temperature

Relaxational dynamics ("Bose molasses") with phase coherence/relaxation time $\tau_\phi$ given by

$$\frac{1}{\tau_\phi} = \left( \text{Universal number} \right) \frac{k_B T}{\hbar} \quad (1\mu K = 20.9\text{kHz})$$

Conductivity (in $d=2$) = $\frac{Q^2}{\hbar} \sum \left( \frac{\hbar \omega}{k_B T} \right) \quad \Sigma \rightarrow \text{universal function}$


Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
   *Mean field theory and the evolution of the excitation spectrum.*

III. Superfluid-insulator transition
   *Boson Hubbard model at integer filling.*

IV. Bosons at fractional filling
   *Beyond the Landau-Ginzburg-Wilson paradigm.*

V. Quantum phase transitions and the Luttinger theorem
   *Depleting the Bose-Einstein condensate of trapped ultracold atoms – see talk by Stephen Powell*
IV. Bosons at fractional filling

*Beyond the Landau-Ginzburg-Wilson paradigm*

Bosons at filling fraction $f = 1/2$

\[ \langle \Psi \rangle \neq 0 \]

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

\[ \langle \Psi \rangle \neq 0 \]

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

$\langle \Psi \rangle \neq 0$

Weak interactions: superfluidity

Bosons at filling fraction $f = 1/2$

$\langle \Psi \rangle = 0$

Strong interactions: insulator

Bosons at filling fraction $f = 1/2$

Strong interactions: insulator

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ.r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Insulating phases of bosons at filling fraction $f = 1/2$

Charge density wave (CDW) order

Valence bond solid (VBS) order

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

**Insulating phases of bosons at filling fraction \( f = 1/2 \)**

Can define a common CDW/VBS order using a generalized "density" \( \rho(r) = \sum_Q \rho_Q e^{iQ \cdot r} \)

All insulators have \( \langle \Psi \rangle = 0 \) and \( \langle \rho_Q \rangle \neq 0 \) for certain \( Q \)

**Insulating phases of bosons at filling fraction \( f = 1/2 \)**

Can define a common CDW/VBS order using a generalized "density" \( \rho(r) = \sum_Q \rho_Q e^{iQ \cdot r} \)

All insulators have \( \langle \Psi \rangle = 0 \) and \( \langle \rho_Q \rangle \neq 0 \) for certain \( Q \)

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Insulating phases of bosons at filling fraction $f = 1/2$

\[ \frac{1}{\sqrt{2}}(\psi + \psi^*) \]

Charge density wave (CDW) order  
Valence bond solid (VBS) order  
Valence bond solid (VBS) order

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

**Insulating phases of bosons at filling fraction \( f = \frac{1}{2} \)**

Can define a common CDW/VBS order using a generalized "density" \( \rho(r) = \sum Q \rho_Q e^{iQ \cdot r} \)

All insulators have \( \langle \Psi \rangle = 0 \) and \( \langle \rho_Q \rangle \neq 0 \) for certain \( Q \)

**Insulating phases of bosons at filling fraction \( f = 1/2 \)**

Can define a common CDW/VBS order using a generalized "density" \( \rho(r) = \sum_{\mathbf{Q}} \rho_{\mathbf{Q}} e^{i\mathbf{Q} \cdot r} \)

All insulators have \( \langle \Psi \rangle = 0 \) and \( \langle \rho_{\mathbf{Q}} \rangle \neq 0 \) for certain \( \mathbf{Q} \)

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum Q \rho_Q e^{iQ.r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Insulating phases of bosons at filling fraction $f = 1/2$

Can define a common CDW/VBS order using a generalized "density" $\rho(r) = \sum_Q \rho_Q e^{iQ \cdot r}$

All insulators have $\langle \Psi \rangle = 0$ and $\langle \rho_Q \rangle \neq 0$ for certain $Q$

Ginzburg-Landau-Wilson approach to multiple order parameters:

\[ F = F_{sc} \left[ \Psi_{sc} \right] + F_{\text{charge}} \left[ \rho_Q \right] + F_{\text{int}} \]

\[ F_{sc} \left[ \Psi_{sc} \right] = r_1 \left| \Psi_{sc} \right|^2 + u_1 \left| \Psi_{sc} \right|^4 + \cdots \]

\[ F_{\text{charge}} \left[ \rho_Q \right] = r_2 \left| \rho_Q \right|^2 + u_2 \left| \rho_Q \right|^4 + \cdots \]

\[ F_{\text{int}} = v \left| \Psi_{sc} \right|^2 \left| \rho_Q \right|^2 + \cdots \]

Distinct symmetries of order parameters permit couplings only between their energy densities

Predictions of LGW theory

First order transition

Superconductor

Charge-ordered insulator

Coexistence (Supersolid)

Superconductor

Charge-ordered insulator

"Disordered"

Superconductor

Charge-ordered insulator

\[ \langle \Psi_{sc} \rangle \]

\[ \langle \rho_q \rangle \]

\[ r_1 - r_2 \]

\[ \langle \Psi_{sc} \rangle = 0, \langle \rho_q \rangle = 0 \]
Predictions of LGW theory

\[ \langle \Psi_{sc} \rangle \]
Superconductor

First order transition

\[ \langle \rho_Q \rangle \]
Charge-ordered insulator

\[ r_1 - r_2 \]

Coexistence
(Supersolid)

\[ \langle \Psi_{sc} \rangle \]
Superconductor

\[ \langle \Psi_{sc} \rangle = 0, \langle \rho_Q \rangle = 0 \]
"Disordered"
(\neq \text{topologically ordered})

\[ \langle \rho_Q \rangle \]
Charge-ordered insulator

\[ r_1 - r_2 \]
Excitations of the superfluid: **Vortices**

The circulation of a vortex is quantized:

\[ \oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint \nabla \theta \cdot d\mathbf{r} = n \frac{\hbar}{m} \]

where \( n \) is an integer.
Observation of quantized vortices in rotating ultracold Na

Quantized fluxoids in $\text{YBa}_2\text{Cu}_3\text{O}_{6+y}$


In superconductors, vortices carry quantized magnetic flux:

$$\int \mathbf{B} \cdot d\mathbf{S} = n \frac{hc}{2e}$$
Excitations of the superfluid: **Vortices**

**Central question:**
In two dimensions, we can view the vortices as point particle excitations of the superfluid. What is the quantum mechanics of these “particles”? 
In ordinary fluids, vortices experience the Magnus Force

\[ F_M = (\text{mass density of air}) \cdot (\text{velocity of ball}) \cdot (\text{circulation}) \]
For a vortex in a superfluid, this is

\[
\mathbf{F}_M = (m \rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \oint \mathbf{v}_s \cdot d\mathbf{r} \right)
\]

\[
= n \hbar \rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z}
\]

where \( \rho = \) number density of bosons

\( \mathbf{v}_s = \) local velocity of superfluid

\( \mathbf{r}_v = \) position of vortex
For a vortex in a superfluid, this is

\[ \mathbf{F}_M = (m\rho) \left( \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \right) \left( \int \mathbf{v}_s \cdot d\mathbf{r} \right) \]

\[ = n\hbar \rho \left( \mathbf{v}_s - \frac{d\mathbf{r}_v}{dt} \right) \times \hat{z} \]

\[ = n \left( \mathbf{E} + \frac{d\mathbf{r}_v}{dt} \times \mathbf{B} \right) \]

where \( \mathbf{E} = \rho \mathbf{v}_s \times \hat{z} \) and \( \mathbf{B} = -\hbar \rho \hat{z} \)

\textbf{Dual picture:}

The vortex is a quantum particle with dual “electric” charge \( n \), moving in a dual “magnetic” field of strength = \( \hbar \times \text{(number density of Bose particles)} \)
- The vortices are quantum particles moving in a periodic potential with the symmetry of the square lattice, and in the presence of a dual “magnetic” field of strength = $h \rho$, where $\rho$ is the number density of bosons per unit cell.

- The vortex motion can be described by the effective Hofstadter Hamiltonian:

$$\mathcal{H}_v = -t \sum_{\langle ij \rangle} (e^{iA_{ij}} \varphi_i^* \varphi_j + \text{c.c.})$$

where $\varphi_i$ is an operator which annihilates a vortex particle at site $i$ of a square lattice.

\[ A_1 + A_2 + A_3 + A_4 = 2\pi f \]

where $f$ is the boson filling fraction.
Bosons at filling fraction $f = 1$

- At $f=1$, the “magnetic” flux per unit cell is $2\pi$, and the vortex does not pick up any phase from the boson density.

- The effective dual “magnetic” field acting on the vortex is zero, and the corresponding component of the Magnus force vanishes.
Bosons at rational filling fraction $f=p/q$

Quantum mechanics of the vortex “particle” in a periodic potential with $f$ flux quanta per unit cell

Space group symmetries of Hofstadter Hamiltonian:

$T_x, T_y$ : Translations by a lattice spacing in the $x, y$ directions

$R$ : Rotation by 90 degrees.

Magnetic space group:

\[
T_x T_y = e^{2\pi i f} T_y T_x ; \\
R^{-1} T_y R = T_x ; \\
R^{-1} T_x R = T_y^{-1} ; \\
R^4 = 1
\]

The low energy vortex states must form a representation of this algebra
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

Hofstadter spectrum of the quantum vortex “particle” with field operator $\varphi$

At filling $f=p/q$, there are $q$ species of vortices, $\varphi_\ell$ (with $\ell=1…q$), associated with $q$ degenerate minima in the vortex spectrum. These vortices realize the smallest, $q$-dimensional, representation of the magnetic algebra.

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i\ell f} \varphi_\ell$$

$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i\ell mf}$$
Vortices in a superfluid near a Mott insulator at filling $f = p/q$

The $q \varphi_\ell$ vortices characterize both superconducting and VBS/CDW orders

Superconductor/insulator: $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$
Vortices in a superfluid near a Mott insulator at filling \( f = p/q \)

The \( q \varphi_{\ell} \) vortices characterize both
superconducting and VBS/CDW orders

VBS order:
Status of space group symmetry determined by
density operators \( \rho \_Q \) at wavevectors \( Q_{mn} = \frac{2\pi p}{q}(m, n) \)

\[
\rho_{mn} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_{\ell}^* \varphi_{\ell+n} e^{2\pi i\ell mf}
\]

\[
T_x : \rho_Q \rightarrow \rho_Q e^{iQ \cdot \hat{x}} \quad ; \quad T_y : \rho_Q \rightarrow \rho_Q e^{iQ \cdot \hat{y}}
\]

\[
R : \rho(Q) \rightarrow \rho(RQ)
\]
Vortices in a superfluid near a Mott insulator at filling $f = \frac{p}{q}$

- The excitations of the superfluid are described by the quantum mechanics of $q$ flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.
Mott insulators obtained by “condensing” vortices

Spatial structure of insulators for $q=2$ ($f=1/2$)
Field theory with projective symmetry
Spatial structure of insulators for $q=4$ ($f=1/4$ or $3/4$)

$a \times b$ unit cells;
$q/a'q/b'ab/q'$
all integers
Vortices in a superfluid near a Mott insulator at filling $f=p/q$

- The excitations of the superfluid are described by the quantum mechanics of $q$ flavors of low energy vortices moving in zero dual "magnetic" field.

- The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.
The excitations of the superfluid are described by the quantum mechanics of $q$ flavors of low energy vortices moving in zero dual "magnetic" field.

The orientation of the vortex in flavor space implies a particular configuration of VBS order in its vicinity.

Any pinned vortex must pick an orientation in flavor space: this induces a halo of VBS order in its vicinity.
Vortex-induced LDOS of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ integrated from 1meV to 12meV at 4K.

Vortices have halos with LDOS modulations at a period $\approx 4$ lattice spacings.


Measuring the inertial mass of a vortex

The spatial extent of the LDOS modulations measures the region over which the vortex executes its zero-point motion. The size of this region can be determined by solving the equations of motion

\[ m_v \frac{d^2 \mathbf{r}}{dt^2} = F_M \]

and so is determined by the inertial vortex mass \( m_v \).
Measuring the inertial mass of a vortex

*Preliminary* estimates for the BSCCO experiment:

- Inertial vortex mass $m_v \approx 10m_e$
- Vortex magnetoplasmon frequency $\nu_p \approx 1$ THz = 4 meV

Future experiments can directly detect vortex zero point motion by looking for resonant absorption at this frequency.

Vortex oscillations can also modify the electronic density of states.
Superfluids near Mott insulators

The Mott insulator has average Cooper pair density, $f = p/q$ per site, while the density of the superfluid is close (but need not be identical) to this value.

- Vortices with flux $\hbar/(2e)$ come in multiple (usually $q$) “flavors”

- The lattice space group acts in a projective representation on the vortex flavor space.

- These flavor quantum numbers provide a distinction between superfluids: they constitute a “quantum order”

- Any pinned vortex must chose an orientation in flavor space. This necessarily leads to modulations in the local density of states over the spatial region where the vortex executes its quantum zero point motion.