The phase diagram of the cuprates and the quantum phase transitions of metals in two dimensions

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Outline

1. Phase diagram of the cuprates
   Quantum criticality of the competition between antiferromagnetism and superconductivity

2. Influence of an applied magnetic field
   Theoretical predictions and experimental tests

3. Theory of spin density wave ordering in a metal
   Order parameter at zero wavevector

4. Theory of Ising-nematic ordering in a metal
   Order parameter at zero wavevector
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   *Order parameter at zero wavevector*

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Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface.
Crossovers in transport properties of hole-doped cuprates

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Strange metal: quantum criticality of optimal doping critical point at $x = x_m$?
Only candidate quantum critical point observed at low $T$

Spin density wave order present below a quantum critical point at $x = x_s$ with $x_s \approx 0.12$ in the La series of cuprates
Antiferromagnetism

Fermi surface

d-wave superconductivity
Antiferromagnetism

Fermi surface

d-wave superconductivity
The electron spin polarization obeys

$$\langle \vec{S}(\mathbf{r}, \tau) \rangle = \varphi(\mathbf{r}, \tau)e^{iK \cdot \mathbf{r}}$$

where \( K \) is the ordering wavevector.
Hole-doped cuprates

Increasing SDW order

Hole-doped cuprates

Increasing SDW order

Hole-doped cuprates

Increasing SDW order

Hot spots

Hole-doped cuprates

Fermi surface breaks up at hot spots into electron and hole “pockets”

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Fermi surface breaks up at hot spots into electron and hole “pockets”

Electron pockets in the Fermi surface of hole-doped high-$T_c$ superconductors

David LeBoeuf$^1$, Nicolas Doiron-Leyraud$^1$, Julien Levallois$^2$, R. Daou$^1$, J.-B. Bonnemaison$^1$, N. E. Hussey$^3$, L. Balicas$^4$, B. J. Ramshaw$^5$, Ruixing Liang$^{5,6}$, D. A. Bonn$^{5,6}$, W. N. Hardy$^{5,6}$, S. Adachi$^7$, Cyril Proust$^2$ & Louis Taillefer$^{1,6}$

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Evidence for small Fermi pockets

Fermi liquid behaviour in an underdoped high Tc superconductor

Suchitra E. Sebastian, N. Harrison, M. M. Altarawneh, Ruixing Liang, D. A. Bonn, W. N. Hardy, and G. G. Lonzarich

arXiv:0912.3022

FIG. 2: Magnetic quantum oscillations measured in YBa$_2$Cu$_3$O$_{6+x}$ with $x \approx 0.56$ (after background polynomial subtraction). This restricted interval in $B = |B|$ furnishes a dynamic range of $\sim 50$ dB between $T = 1$ and 18 K. The actual $T$ values are provided in Fig. 3.
Theory of quantum criticality in the cuprates

Fluctuating Fermi pockets

Strange Metal

Large Fermi surface

Increasing SDW order

T*

Spin density wave (SDW)

Underlying SDW ordering quantum critical point in metal at $x = x_m$
Evidence for connection between linear resistivity and stripe-ordering in a cuprate with a low $T_c$

- Magnetic field of upto 35 T used to suppress superconductivity
- Identifies $x_m \approx 0.24$

Linear temperature dependence of resistivity and change in the Fermi surface at the pseudogap critical point of a high-$T_c$ superconductor
Antiferromagnetism

Fermi surface

d-wave superconductivity
Theory of quantum criticality in the cuprates

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Underlying SDW ordering quantum critical point

in metal at $x = x_m$
Theory of quantum criticality in the cuprates

Onset of $d$-wave superconductivity hides the critical point $x = x_m$
Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

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Large Fermi surface

d-wave superconductor

Spin density wave (SDW)

Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$.


Theory of quantum criticality in the cuprates


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Theory of quantum criticality in the cuprates

Fluctuating, paired Fermi pockets

Strange Metal

Magnetic quantum criticality

Spin gap

Thermally fluctuating SDW

Spin density wave (SDW)

T*

\( T^* \)

\( \lambda_c \)

\( \lambda \)

\( x = x_s \)

\( x = x_m \)

Competition between SDW order and superconductivity moves the actual quantum critical point to \( x = x_s < x_m \).
Theory of quantum criticality in the cuprates

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Large Fermi surface

d-wave superconductor

Spin density wave (SDW)


Competition between SDW order and superconductivity moves the actual quantum critical point to $x = x_s < x_m$. 

Thursday, February 25, 2010
Theory of quantum criticality in the cuprates

Physics of competition: \textit{d}-wave SC and SDW “eat up” same pieces of the large Fermi surface.


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Small Fermi pockets with pairing fluctuations

Large Fermi surface

Strange Metal

Fluctuating, paired Fermi pockets

Magnetic quantum criticality

Spin gap

d-wave SC

Thermally fluctuating SDW

$T^*$

$\mathcal{X}_S$
E. Demler, S. Sachdev and Y. Zhang, 
Nd$_{2-x}$Ce$_x$CuO$_4$

E. Demler, S. Sachdev and Y. Zhang, 
Field-induced transition between magnetically disordered and ordered phases in underdoped La$_{2-x}$Sr$_x$CuO$_4$

B. Khaykovich, S. Wakimoto, R. J. Birgeneau, M. A. Kastner, Y. S. Lee, P. Smeibidl, P. Vorderwisch, and K. Yamada

FIG. 1. (a) A fragment of the theoretical phase diagram, adopted from Refs. 4 and 20. The vertical axis is the magnetic field and the horizontal axis is the coupling strength between superconductivity and magnetic order. (b) Field dependence of the magnetic Bragg peak corresponding to the incommensurate SDW peak at $Q=(1.125,0.125,0)$. Every point is measured after field cooling at $T=1.5$ K. The data are fitted to $I=I_0+A|H-H_c|^{2\beta}$ above $H_c$ as explained in the text. Spectrometer configuration: 45-60-Be—S—Be-60-open; cold Be filters were used before and after the sample to eliminate contamination from high-energy neutrons; $E=4$ meV.

Neutron scattering experiments on Nd$_{2-x}$Ce$_x$CuO$_4$ show that at low fields $x_s = 0.14$, while quantum oscillations at high fields show that $x_m = 0.165$. 

Thursday, February 25, 2010
Nd$_{2-x}$Ce$_x$CuO$_4$

Strange Metal

Large Fermi surface

Small Fermi pockets with pairing fluctuations

Thermally fluctuating SDW

Magnetic quantum criticality

Spin gap

Fluctuating, paired Fermi pockets

d-wave SC

SC+ SDW

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

\( T^* \)

\( x_m \)

\( x_s \)

M
Similar phase diagram for CeRhIn$_5$

Fluctuating, paired Fermi pockets

Magnetic quantum criticality

Spin gap

Thermally fluctuating SDW

SDW (Small Fermi pockets)

"Normal" (Large Fermi surface)

Large Fermi surface

Strange Metal

d-wave SC

SC

SC+

SDW

T*

T
Strange Metal

Large Fermi surface

Small Fermi pockets with pairing fluctuations

d-wave SC

T_{sdw}

SC+ SDW

SDW (Small Fermi pockets)

M

"Normal" (Large Fermi surface)

Thursday, February 25, 2010
Similar phase diagram for the pnictides

Strange
Metal

Large Fermi
surface

Fluctuating,
paired Fermi
pockets

d-wave
SC

SDW

(SC+ SDW)

SDW
(Small Fermi
pockets)

"Normal"
(Large Fermi
surface)

T*

T_{sdw}

T

x

H

Thursday, February 25, 2010
Remnants of SDW order for \( x_s < x < x_m \)

For incommensurate ordering, the SDW order parameter consists of 2 complex 3-component vectors \( \Phi_x, \Phi_y \):

\[
\langle \hat{S}(\mathbf{r}, \tau) \rangle = \Phi_x(\mathbf{r}, \tau)e^{i\mathbf{K}_x \cdot \mathbf{r}} + \Phi_y(\mathbf{r}, \tau)e^{i\mathbf{K}_y \cdot \mathbf{r}} + \text{c.c.}
\]

where \( \mathbf{K}_x = (\pi(1 - \vartheta), \pi) \) and \( \mathbf{K}_y = (\pi, \pi(1 - \vartheta)) \), with \( \vartheta = 1/4 \) near 1/8 doping.
SDW correlations also Ising nematic order \( \phi \propto |\Phi_x|^2 - |\Phi_y|^2 \), which can be long-ranged, with SDW and VBS/CDW order all short ranged. This implies of preferential enhancement of electronic exchange/pairing energies along the \( x \) or \( y \) directions.

Remnants of SDW order for $x_s < x < x_m$

SDW correlations also Ising nematic order $\phi \propto |\Phi_x|^2 - |\Phi_y|^2$, which can be long-ranged, with SDW and VBS/CDW order all short ranged. This implies of preferential enhancement of electronic exchange/pairing energies along the $x$ or $y$ directions.


Onset of superconductivity disrupts SDW order, but VBS/CDW/Ising-nematic ordering can survive.

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Strange Metal

Large Fermi surface

Fluctuating, paired Fermi pockets

d-wave

SC

T_c

T_{sdw}

VBS and/or Ising nematic order

SDW

Insulator

"Normal" (Large Fermi surface)

H_c2

SDW

(Small Fermi pockets)

T^*

T

H

x

x_m

x_s

Thursday, February 25, 2010
"Normal" (Large Fermi surface)

SDW (Small Fermi pockets)

SDW Insulator

Fluctuating, paired Fermi pockets

SC

SDW

d-wave

VBS and/or Ising nematic order

Thursday, February 25, 2010
Large Fermi surface breaks up into electron and hole pockets

Hole-doped cuprates

Increasing SDW order

Hole pockets

Electron pockets

\( \phi \) fluctuations act on the large Fermi surface


Start from the “spin-fermion” model

\[ Z = \int \mathcal{D}c_\alpha \mathcal{D}\varphi \exp (-S) \]

\[ S = \int d\tau \sum_k c_k^\dagger (\frac{\partial}{\partial \tau} - \varepsilon_k) c_k \]

\[ - \lambda \int d\tau \sum_i c_i^\dagger \varphi_i \cdot \bar{\sigma} c_i \alpha \beta e^{iK \cdot r_i} \]

\[ + \int d\tau d^2 r \left[ \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \right] \]
\[
\mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell
\]

\[
\mathbf{v}_{1}^{\ell=1} = (v_x, v_y), \quad \mathbf{v}_{2}^{\ell=1} = (-v_x, v_y)
\]
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]
\[ L_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i v_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i v_2 \cdot \nabla_r) \psi_{2\alpha} \]
\[ L_f = \psi_{1\alpha}^{l \dagger} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha}^l + \psi_{2\alpha}^{l \dagger} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha}^l \]

Order parameter: \[ L_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]
\[ \mathcal{L}_f = \psi_{1\alpha} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:
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“Yukawa” coupling:
\[ \mathcal{L}_c = -\lambda \varphi \cdot \left( \psi_{1\alpha} \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha} \bar{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]
\[ \mathcal{L}_f = \psi_{1\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_{\mathbf{r}}) \psi_{1\alpha} + \psi_{2\alpha}^{\dagger} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_{\mathbf{r}}) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_{\mathbf{r}} \varphi)^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

"Yukawa" coupling: \[ \mathcal{L}_c = -\lambda \varphi \cdot \left( \psi_{1\alpha}^{\dagger} \bar{\sigma}_{\alpha\beta} \psi_{2\beta}^{\dagger} + \psi_{2\alpha}^{\dagger} \bar{\sigma}_{\alpha\beta} \psi_{1\beta}^{\dagger} \right) \]

Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain non-local corrections to \( \mathcal{L}_\varphi \)

\[ \mathcal{L}_\varphi = \frac{1}{2} \varphi^2 \left[ \mathbf{q}^2 + \gamma |\omega| \right] / 2 \quad ; \quad \gamma = \frac{2}{\pi v_x v_y} \]

Exponent \( z = 2 \) and mean-field criticality (upto logarithms)
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:

\[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

“Yukawa” coupling:

\[ \mathcal{L}_c = -\lambda \bar{\varphi} \cdot \left( \psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^\dagger + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^\dagger \right) \]

**Hertz-Moriya-Millis (HMM) theory**

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Exponent \( z = 2 \) and mean-field criticality (upto logarithms)

But, higher order terms contain an infinite number of marginal couplings .......

\[ \mathcal{L}_f = \psi^\dagger_{1\alpha} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_{1\alpha} + \psi^\dagger_{2\alpha} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi \cdot \left( \psi^\dagger_{1\alpha} \tilde{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi^\dagger_{2\alpha} \tilde{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

Perform RG on both fermions and \( \varphi \), using a local field theory.
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - iv_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - iv_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter: \[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \varphi')^2 + \frac{\zeta}{2} (\partial_\tau \varphi')^2 + \frac{s}{2} \varphi'^2 + \frac{u}{4} \varphi'^4 \]

“Yukawa” coupling: \[ \mathcal{L}_c = -\lambda \varphi' \cdot \left( \psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta}^\dagger + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta}^\dagger \right) \]

Under the rescaling \( x' = xe^{-\ell}, \tau' = \tau e^{-z\ell} \), the spatial gradients are fixed if the fields transform as

\[ \varphi' = e^{(d+z-2)\ell/2} \varphi ; \quad \psi' = e^{(d+z-1)\ell/2} \psi. \]

Then the Yukawa coupling transforms as

\[ \lambda' = e^{(4-d-z)\ell/2} \lambda \]

For \( d = 2 \), with \( z = 2 \) the Yukawa coupling is invariant, and the bare time-derivative terms \( \zeta, \tilde{\zeta} \) are irrelevant.
\[ \mathcal{L}_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - iv_1^l \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - iv_2^l \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:

\[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\zeta}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

“Yukawa” coupling:

\[ \mathcal{L}_c = -\bar{\varphi} \cdot (\psi_{1\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \bar{\sigma}_{\alpha\beta} \psi_{1\beta}^\dagger) \]

With \( z = 2 \) scaling, \( \zeta \) is irrelevant.
So we take \( \zeta \to 0 \)
(watch for dangerous irrelevancy).
\[ \mathcal{L}_f = \psi^{\dagger}_1 \left( \zeta \partial_{\tau} - i \mathbf{v}_1 \cdot \nabla_r \right) \psi_1 + \psi^{\dagger}_2 \left( \zeta \partial_{\tau} - i \mathbf{v}_2 \cdot \nabla_r \right) \psi_2 \]

Order parameter:
\[ \mathcal{L}_{\varphi} = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_{\tau} \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling:
\[ \mathcal{L}_c = -\varphi \cdot \left( \psi^{\dagger}_1 \bar{\sigma}_{\alpha \beta} \psi_2 + \psi^{\dagger}_2 \bar{\sigma}_{\alpha \beta} \psi_1 \right) \]

Set \( \varphi \) wavefunction renormalization by keeping co-efficient of \((\nabla_r \varphi)^2\) fixed (as usual).
\[ \mathcal{L}_f = \psi_1^{\dagger \alpha} (\zeta \partial_\tau - i \mathbf{v}_1 \cdot \nabla_r) \psi_1^\alpha + \psi_2^{\dagger \alpha} (\zeta \partial_\tau - i \mathbf{v}_2 \cdot \nabla_r) \psi_2^\alpha \]

Order parameter:  
\[ \mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \bar{\varphi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \bar{\varphi})^2 + \frac{s}{2} \bar{\varphi}^2 + \frac{u}{4} \bar{\varphi}^4 \]

"Yukawa" coupling:  
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Set fermion wavefunction renormalization by keeping Yukawa coupling fixed.

\[ L_f = \psi_{1\alpha}^\dagger (\zeta \partial_\tau - i v_1 \cdot \nabla_r) \psi_{1\alpha} + \psi_{2\alpha}^\dagger (\zeta \partial_\tau - i v_2 \cdot \nabla_r) \psi_{2\alpha} \]

Order parameter:  \[ L_\varphi = \frac{1}{2} (\nabla_r \varphi)^2 + \frac{\zeta}{2} (\partial_\tau \varphi)^2 + \frac{s}{2} \varphi^2 + \frac{u}{4} \varphi^4 \]

“Yukawa” coupling:  \[ L_c = -\varphi \cdot \left( \psi_{1\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \tilde{\sigma}_{\alpha\beta} \psi_{1\beta} \right) \]

We find consistent two-loop RG factors, as \( \zeta \to 0 \), for the velocities \( v_x, v_y \), and the wavefunction renormalizations.

**Consistency check:** the expression for the boson damping constant, \( \gamma = \frac{2}{\pi v_x v_y} \), is preserved under RG.
RG flow can be computed a $1/N$ expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{dl} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$
RG flow can be computed a 1/N expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{dl} = -\frac{3\alpha^2}{\pi N 1 + \alpha^2}$$

The velocities flow as

$$\frac{1}{v_x} \frac{dv_x}{dl} = \frac{A(\alpha) + B(\alpha)}{2} \quad ; \quad \frac{1}{v_y} \frac{dv_y}{dl} = -\frac{A(\alpha) + B(\alpha)}{2}$$

$$A(\alpha) \equiv \frac{3}{\pi N} \frac{\alpha}{1 + \alpha^2}$$

$$B(\alpha) \equiv \frac{1}{2\pi N} \left( \frac{1}{\alpha} - \alpha \right) \left( 1 + \left( \frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$
RG flow can be computed a $1/N$ expansion (with $N$ fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{d\ell} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$

The anomalous dimensions of $\bar{\varphi}$ and $\psi$ are

$$\eta_{\bar{\varphi}} = \frac{1}{2\pi N} \left( \frac{1}{\alpha} - \alpha + \left( \frac{1}{\alpha^2} + \alpha^2 \right) \tan^{-1} \frac{1}{\alpha} \right)$$

$$\eta_{\psi} = -\frac{1}{4\pi N} \left( \frac{1}{\alpha} - \alpha \right) \left( 1 + \left( \frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \rightarrow 0 \text{ logarithmically in the infrared.} \]

Dynamical Nesting

\[ v_1 \quad v_2 \]

Bare Fermi surface
\( \alpha = \frac{v_y}{v_x} \rightarrow 0 \) logarithmically in the infrared.

**Dynamical Nesting**
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \to 0 \text{ logarithmically in the infrared.} \]

Dynamical Nesting

Bare Fermi surface
RG-improved Migdal-Eliashberg theory

\[ \alpha = \frac{v_y}{v_x} \rightarrow 0 \text{ logarithmically in the infrared.} \]

**Dynamical Nesting**

Dressed Fermi surface
RG-improved Migdal-Eliashberg theory

\[ \alpha = \nu_y / \nu_x \rightarrow 0 \text{ logarithmically in the infrared.} \]

In $\varphi$ SDW fluctuations, characteristic $q$ and $\omega$ scale as

\[ q \sim \omega^{1/2} \exp \left( -\frac{3}{64\pi^2} \left( \frac{\ln(1/\omega)}{N} \right)^3 \right). \]

However, $1/N$ expansion cannot be trusted in the asymptotic regime.
New infra-red singularities as $\zeta \to 0$ at higher loops
(Breakdown of Migdal-Eliashberg)

$\bar{\phi}$ propagator

$$\frac{1}{N \left( q^2 + \gamma |\omega| \right)}$$

fermion propagator

$$\frac{1}{\mathbf{v} \cdot \mathbf{q} + i\zeta \omega + i \frac{1}{N \sqrt{\gamma v}} \sqrt{\omega F} \left( \frac{v^2 q^2}{\omega} \right)}$$
New infra-red singularities as $\zeta \to 0$ at higher loops
(Breakdown of Migdal-Eliashberg)

\[ \bar{\phi} \text{ propagator} \]

\[ \frac{1}{N} \frac{1}{(q^2 + \gamma|\omega|)} \]

\[ \text{fermion propagator} \]

\[ \mathbf{v} \cdot \mathbf{q} + i\zeta \omega + i \frac{1}{N\sqrt{\gamma}v} \sqrt{\omega}F \left( \frac{v^2q^2}{\omega} \right) \]

⚠️ Dangerous
New infra-red singularities as $\zeta \to 0$ at higher loops
(Breakdown of Migdal-Eliashberg)

Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$

Actual order $\sim \frac{1}{N^0}$
Double line representation

- A way to compute the order of a diagram.
- Extra powers of N come from the Fermi-surface

\[ G(\omega, \vec{k}) = \frac{1}{-\Sigma_1(\omega, \vec{k}) - \vec{v} \cdot \vec{k}} \]
\[ \Sigma_1 \sim \frac{1}{N} \]

- What are the conditions for all propagators to be on the Fermi surface?
- Concentrate on diagrams involving a single pair of hot-spots
- Any bosonic momentum may be (uniquely) written as

\[ \vec{q} = \vec{k}_1 - \vec{k}_2 \quad \vec{k}_1 \in \text{FS of } \psi_1 \quad \vec{k}_2 \in \text{FS of } \psi_2 \]

New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Singularities as $\zeta \to 0$ appear when fermions in closed blue and red line loops are exactly on the Fermi surface

Actual order $\sim \frac{1}{N^0}$
New infra-red singularities as $\zeta \to 0$ at higher loops (Breakdown of Migdal-Eliashberg)

Actual order $\sim \frac{1}{N^0}$

Graph is planar after turning fermion propagators also into double lines by drawing additional dotted single line loops for each fermion loop

Sung-Sik Lee, arXiv:0905.4532
A consistent analysis requires resummation of all planar graphs.
Outline

1. Phase diagram of the cuprates
   Quantum criticality of the competition between antiferromagnetism and superconductivity

2. Influence of an applied magnetic field
   Theoretical predictions and experimental tests

3. Theory of spin density wave ordering in a metal
   Order parameter at zero wavevector

4. Theory of Ising-nematic ordering in a metal
   Order parameter at zero wavevector
1. Phase diagram of the cuprates
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Quantum criticality of Pomeranchuk instability

\[ \langle \phi \rangle \neq 0 \]

\[ \langle \phi \rangle = 0 \]

Pomeranchuk instability as a function of coupling \( r \)
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

\[ S_\phi = \int d^2 x d\tau \left[ (\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (r - r_c) \phi^2 + u \phi^4 \right] \]
Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

\[ S_{\phi} = \int d^2x d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (r - r_c)\phi^2 + u\phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \]

\[ \equiv \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k) c_{k\alpha} \]
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

\[ S_{\phi c} = -\gamma \int d\tau \phi \sum_{\alpha=1}^{N_f} \sum_{k} (\cos k_x - \cos k_y) c_{k\alpha}^{\dagger} c_{k\alpha} \]

for spatially independent \( \phi \)

\[ \langle \phi \rangle > 0 \quad \text{and} \quad \langle \phi \rangle < 0 \]
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

$$S_{\phi_c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha}$$

for spatially dependent $\phi$

$$\langle \phi \rangle > 0 \quad \text{and} \quad \langle \phi \rangle < 0$$
Quantum criticality of Pomeranchuk instability

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (r - r_c)\phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k) c_{k\alpha} \]

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]
A $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$.

Expand fermion kinetic energy at wavevectors about $\vec{k}_0$. 
\[ \mathcal{L} = \psi_+^\dagger \left( \zeta \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \zeta \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- 
\]
\[- \lambda \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g} \left( \partial_y \phi \right)^2 + \frac{r}{2} \phi^2 \]

**Theory of Ising-nematic transition**
Emergent “Galilean invariance” at low energy \((s = \pm)\):

\[
\phi(x, y) \rightarrow \phi(x, y + \theta x), \quad \psi_s(x, y) \rightarrow e^{-is\left(\frac{\theta}{2} y + \frac{\theta^2}{4} x\right)}\psi_s(x, y + \theta x)
\]

which implies for the fermion Green’s function

\[
G(q_x, q_y) = G(sq_x + q_y^2).
\]
Emergent “Galilean invariance” at low energy ($s = \pm$):

$$\phi(x, y) \to \phi(x, y + \theta x), \quad \psi_s(x, y) \to e^{-is\left(\frac{\theta}{2}y + \frac{\theta^2}{4}x\right)}\psi_s(x, y + \theta x)$$

which implies for the fermion Green’s function

$$G(q_x, q_y) = G(sq_x + q_y^2).$$

Line of singularities in momentum space on the “hot” Fermi surface $sq_x + q_y^2 = 0$. 
Critical point is described by an infinite set of 2+1 dimensional field theories, one for each direction \( \hat{q} \).

Contrast with "Fermi surface bosonization" methods where there are an infinite set of 1+1 dimensional field theories, one for each direction \( \hat{q} \).

Our approach leads to a redundant description of underlying degrees of freedom. The "Galilean symmetry" ensures consistency of redundant description.
• Critical point is described by an *infinite* set of 2+1 dimensional field theories, one for each direction $\hat{q}$.

• Contrast with “Fermi surface bosonization” methods where there are an infinite set of 1+1 dimensional field theories, one for each direction $\hat{q}$. 
• Critical point is described by an infinite set of 2+1 dimensional field theories, one for each direction $\hat{q}$.

• Contrast with “Fermi surface bosonization” methods where there are an infinite set of 1+1 dimensional field theories, one for each direction $\hat{q}$.

• Our approach leads to a redundant description of underlying degrees of freedom. The “Galilean symmetry” ensures consistency of redundant description.
\( \bullet (q'_x, q'_y) \)

\[
q'_x = q_x - \kappa_x + 2\kappa_y(q_y - \kappa_y)
\]
\[
q'_y = q_y - \kappa_y
\]

where \( \vec{\kappa}_1 = (\kappa_x, \kappa_y) \) and \( \kappa_x + \kappa_y^2 = 0 \).
\[ \begin{align*}
q'_x &= q_x - \kappa_x + 2\kappa_y(q_y - \kappa_y) \\
q'_y &= q_y - \kappa_y,
\end{align*} \]

where \( \vec{\kappa}_1 = (\kappa_x, \kappa_y) \) and \( \kappa_x + \kappa_y^2 = 0 \).

Note \( q'_x + q'_y^2 = q_x + q_y^2 \): ensures compatibility of redundant 2+1 dimensional field theories.
\[ \mathcal{L} = \psi_+^\dagger (\zeta \partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\zeta \partial_\tau + i \partial_x - \partial_y^2) \psi_- \\
- \lambda \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g} (\partial_y \phi)^2 + \frac{r}{2} \phi^2 \]

After tuning the single parameter \( r \sim \lambda - \lambda_c \), and sending \( \zeta \to 0 \), \( \mathcal{L} \) describes a critical theory with no coupling constants. There is a separate copy of this critical theory for each direction \( \hat{q} \). This theory has 2 independent exponents \( z \) and \( \eta \), and the correlation length and susceptibility exponents are given by

\[ \nu = \frac{1}{z - 1} ; \quad \gamma = 1 \]

The fermion and order parameter Green’s functions obey the scaling forms

\[ G(q, \omega) = \xi^{2-\eta} \Phi_{\psi} \left( (q_x + q_y^2) \xi^2, \omega \xi^z \right) ; \quad D(q, \omega) = \xi^{z-1} \Phi_{\phi} \left( q_y \xi, \omega \xi^z \right) \]

We have computed the exponents to three loops, and find \( z = 3 \) and \( \eta = 0.06824 \) at this order.
All planar graphs of $\psi_+$ alone are as important as the leading term

Computations in the $1/N$ expansion

All planar graphs of $\psi_+$ alone are as important as the leading term

Graph mixing $\psi_+$ and $\psi_-$ is $O(N^{3/2})$ (instead of $O(N)$), violating genus expansion

Conclusions

Identified quantum criticality in cuprate superconductors with a critical point at optimal doping associated with onset of spin density wave order in a metal.

Elusive optimal doping quantum critical point has been “hiding in plain sight”.

It is shifted to lower doping by the onset of superconductivity.
Conclusions

Theories for the onset of spin density wave and Ising-nematic order in metals are strongly coupled in two dimensions.