Magnetic phases and critical points of insulators and superconductors

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• cond-mat/0109419

*Quantum Phase Transitions*

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Talks online: [Google Sachdev](#)
What is a quantum phase transition?

Non-analyticity in ground state properties as a function of some control parameter $g$.

True level crossing:
Usually a *first*-order transition

Avoided level crossing which becomes sharp in the infinite volume limit:
*second*-order transition
Why study quantum phase transitions?

- Theory for a quantum system with strong correlations: describe phases on either side of $g_c$ by expanding in deviation from the quantum critical point.
- Critical point is a novel state of matter without quasiparticle excitations
- Critical excitations control dynamics in the wide quantum-critical region at non-zero temperatures.

Important property of ground state at $g=g_c$:
- temporal and spatial scale invariance;
- characteristic energy scale at other values of $g$: $\Delta \sim |g - g_c|^{\nu}$
Outline

I. **Quantum Ising chain**

II. Coupled Dimer Antiferromagnet  
   A. Coherent state path integral  
   B. Quantum field theory near critical point

III. Coupled dimer antiferromagnet in a magnetic field  
    Bose condensation of “triplons”

IV. Magnetic transitions in superconductors  
    Quantum phase transition in a background  
    Abrikosov flux lattice

V. Antiferromagnets with an odd number  
   of $S=1/2$ spins per unit cell.  
   **Class A:** Compact U(1) gauge theory: collinear spins,  
   bond order and confined spinons in $d=2$  
   **Class B:** $Z_2$ gauge theory: non-collinear spins, RVB,  
   visons, topological order, and deconfined spinons

VI. Conclusions
I. Quantum Ising Chain

Degrees of freedom: $j = 1 \ldots N$ qubits, $N$ "large"

$$|\uparrow\rangle_j, |\downarrow\rangle_j$$

or

$$|\rightarrow\rangle_j = \frac{1}{\sqrt{2}}(|\uparrow\rangle_j + |\downarrow\rangle_j)$$

$$|\leftarrow\rangle_j = \frac{1}{\sqrt{2}}(|\uparrow\rangle_j - |\downarrow\rangle_j)$$

Hamiltonian of decoupled qubits:

$$H_0 = -Jg \sum_j \sigma_j^x$$
Coupling between qubits:

\[ H_1 = -J \sum_j \sigma_j^z \sigma_{j+1}^z \]

\[
\left( |\rightarrow\rangle_j \langle \leftarrow | + |\leftarrow \rangle_j \langle \rightarrow | \right) \left( |\rightarrow\rangle_{j+1} \langle \leftarrow | + |\leftarrow \rangle_{j+1} \langle \rightarrow | \right)
\]

Prefers neighboring qubits are either \( |\uparrow\rangle_j \uparrow \rangle_{j+1} \) or \( |\downarrow\rangle_j \downarrow \rangle_{j+1} \) (not entangled)

Full Hamiltonian

\[
H = H_0 + H_1 = -J \sum_j \left( g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)
\]

leads to entangled states at \( g \) of order unity
Weakly-coupled qubits ($g \gg 1$)

Ground state:

$$\left| G \right> = \left| \cdots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \right>$$

$$-\frac{1}{2g} \left| \cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftrightarrow \leftrightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \right> - \cdots$$

Lowest excited states:

$$\left| \ell_j \right> = \left| \cdots \rightarrow \rightarrow \rightarrow \rightarrow \leftrightarrow_j \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \right> + \cdots$$

Coupling between qubits creates “flipped-spin” quasiparticle states at momentum $p$

$$\left| p \right> = \sum_j e^{ipx_j/\hbar} \left| \ell_j \right>$$

Excitation energy $\varepsilon(p) = \Delta + 4J \sin^2 \left( \frac{pa}{2\hbar} \right) + O\left( g^{-1} \right)$

Excitation gap $\Delta = 2gJ - 2J + O\left( g^{-1} \right)$

Entire spectrum can be constructed out of multi-quasiparticle states
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum $p$

$$Z\delta(\omega - \epsilon(p))$$

Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to a Lorentzian of width $1/\tau_\phi$, where the phase coherence time $\tau_\phi$ is given by

$$\frac{1}{\tau_\phi} = \frac{2k_B T}{\pi\hbar} e^{-\Delta/k_B T}$$

Strongly-coupled qubits ($g \ll 1$)

Ground states:

$$|G \uparrow\rangle = |\cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots\rangle$$

$$-\frac{g}{2}|\cdots \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots\rangle - \cdots$$

Second state $|G \downarrow\rangle$ obtained by $\uparrow \iff \downarrow$

$|G \downarrow\rangle$ and $|G \uparrow\rangle$ mix only at order $g^N$

Lowest excited states: domain walls

$$|d_j\rangle = |\cdots \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \cdots\rangle + \cdots$$

Coupling between qubits creates new “domain-wall” quasiparticle states at momentum $p$

$$|p\rangle = \sum_j e^{ipx_j/\hbar} |d_j\rangle$$

Excitation energy $\varepsilon(p) = \Delta + 4Jg \sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^2\right)$

Excitation gap $\Delta = 2J - 2gJ + O\left(g^2\right)$

Ferromagnetic moment

$$N_0 = \langle G | \sigma^z | G \rangle \neq 0$$
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)
while transferring energy $\hbar \omega$ and momentum $p$

$S(p, \omega) = N_0^2 (2\pi)^2 \delta(\omega) \delta(p)$

Structure holds to all orders in $g$

At $T > 0$, motion of domain walls leads to a finite \textit{phase coherence time} $\tau_{\varphi}$,
and broadens coherent peak to a width $1/\tau_{\varphi}$ where

$$\frac{1}{\tau_{\varphi}} = \frac{2 k_B T}{\pi \hbar} e^{-\Delta/k_B T}$$

Entangled states at $g$ of order unity

“Flipped-spin” Quasiparticle weight $Z$

$Z \sim (g - g_c)^{1/4}$


Ferromagnetic moment $N_0$

$N_0 \sim (g_c - g)^{1/8}$


Excitation energy gap $\Delta$

$\Delta \sim |g - g_c|$
Dynamic Structure Factor $S(p, \omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)
while transferring energy $\hbar\omega$ and momentum $p$

Critical coupling ($g = g_c$)

$$S(p, \omega) \sim \left(\omega^2 - c^2 p^2\right)^{-7/8}$$

No quasiparticles --- dissipative critical continuum
\[ H_I = -J \sum_i \left( g \sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \]

\[ \chi(\omega) = \frac{i}{\hbar} \sum_k \int_0^\infty dt \left\langle \left[ \sigma_j^z(t), \sigma_k^z(0) \right] \right\rangle e^{i\omega t} \]

\[ = \frac{A}{T^{7/4} \left( 1 - i\omega / \Gamma_R + \ldots \right)} \]

\[ \Gamma_R = \left( 2 \tan \frac{\pi}{16} \right) k_B T \frac{k}{\hbar} \]

\[ \left\langle \sigma_j^z \sigma_k^z \right\rangle \sim \frac{1}{|j-k|^{1/4}} \]


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VI. Conclusions
II. Coupled Dimer Antiferromagnet


\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

\[ 0 \leq \lambda \leq 1 \]
Square lattice antiferromagnet

Experimental realization: $La_2CuO_4$

Ground state has long-range magnetic (Neel) order

$$\langle \vec{S}_i \rangle = (-1)^{i_x+i_y} N_0 \neq 0$$

Excitations: 2 spin waves (magnons) $\varepsilon_p = \sqrt{c_x^2 p_x^2 + c_y^2 p_y^2}$
$\lambda$ close to 0

Weakly coupled dimers

\begin{align*}
\hbar = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\end{align*}

Paramagnetic ground state

\[ \left\langle \vec{S}_i \right\rangle = 0 \]
\[ \lambda \text{ close to 0} \]

Weakly coupled dimers

\[ \downarrow \uparrow - \uparrow \downarrow = 2 \]

Excitation: \( S=1 \) triplon (exciton, spin collective mode)

Energy dispersion away from antiferromagnetic wavevector

\[ \varepsilon_p = \Delta + \frac{c_x^2 p_x^2 + c_y^2 p_y^2}{2\Delta} \]

\( \Delta \rightarrow \text{spin gap} \)
\( \lambda \) close to 0

Weakly coupled dimers

\[
\frac{1}{\sqrt{2}} (|↑↓⟩ - |↓↑⟩)
\]

\( S = \frac{1}{2} \) spinons are confined by a linear potential into a \( S = 1 \) triplon
Quantum paramagnet $S_N = 0$ in cuprates?
II.A Coherent state path integral

Path integral for quantum spin fluctuations

**Key ingredient: Spin Berry Phases**
II.A Coherent state path integral

Path integral for quantum spin fluctuations

**Key ingredient: Spin Berry Phases**
II.A Coherent state path integral


Path integral for a single spin

\[ Z = \text{Tr}\left( e^{-H[S]/T} \right) \]

\[ = \int \mathcal{D}N(\tau) \delta(N^2 - 1) \exp\left( -iS \int A_\tau(\tau) d\tau - \int d\tau H\left[ SN(\tau) \right] \right) \]

\[ A_\tau(\tau)d\tau = \text{Oriented area of triangle on surface of unit sphere bounded by } N(\tau), N(\tau + d\tau), \text{ and a fixed reference } N_0 \]

Action for lattice antiferromagnet

\[ N_j(\tau) = \eta_j n(x_j, \tau) + L(x_j, \tau) \]

\[ \eta_j = \pm 1 \text{ identifies sublattices} \]

\[ n \text{ and } L \text{ vary slowly in space and time} \]
Integrate out $L$ and take the continuum limit

$$Z = \int \mathcal{D}n(x, \tau) \delta(n^2 - 1) \exp \left( -iS \sum_j \int \eta_j A_{\tau}(x_j, \tau) d\tau \right) - \frac{1}{2g} \int d^2 x d\tau \left( (\partial_{\tau} n)^2 + c^2 (\nabla_x n)^2 \right)$$

$\eta_j = \pm 1$ identifies sublattices

Discretize spacetime into a cubic lattice

$$Z = \prod_a \int dn_a \delta(n_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a, \mu} n_a \cdot n_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)$$

$A_{a\mu} \rightarrow$ oriented area of spherical triangle formed by $n_a$, $n_{a+\mu}$, and an arbitrary reference point $n_0$
Integrate out $L$ and take the continuum limit

$$Z = \int \mathcal{D}n(x,\tau) \delta(n^2 - 1) \exp \left( -iS \sum_j \eta_j A_\tau(x_j,\tau) d\tau \right)$$

$$-\frac{1}{2g} \int d^2x d\tau \left( (\partial_\tau n)^2 + c^2 (\nabla_x n)^2 \right)$$

$\eta_j = \pm 1$ identifies sublattices

Discretize spacetime into a cubic lattice

$$Z = \prod_a \int d n_a \delta(n_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} n_a \cdot n_{a+\mu} \right)$$

**Berry phases can be neglected for coupled dimer antiferromagnet** (justified later)

Quantum path integral for two-dimensional quantum antiferromagnet

$\Leftrightarrow$ Partition function of a classical three-dimensional ferromagnet at a “temperature” $g$

Quantum transition at $\lambda = \lambda_c$ is related to classical Curie transition at $g = g_c$
II.B Quantum field theory for critical point

\( \lambda \) close to \( \lambda_c \) : use “soft spin” field

\[
S_b = \int d^2 x d\tau \left[ \frac{1}{2} \left( (\nabla_x \phi_\alpha)^2 + c^2 (\partial_\tau \phi_\alpha)^2 + (\lambda_c - \lambda) \phi_\alpha^2 \right) + \frac{u}{4!} (\phi_\alpha^2)^2 \right]
\]

\( \phi_\alpha \rightarrow \) 3-component antiferromagnetic order parameter

Oscillations of \( \phi_\alpha \) about zero (for \( \lambda < \lambda_c \))

\( \rightarrow \) spin-1 collective mode

\[
\text{Im} \chi(p, \omega) = \Delta + \frac{c^2 p^2}{2\Delta}
\]

\[
\Delta = \sqrt{\lambda_c - \lambda} / c
\]
Dynamic spectrum at the critical point

\[ \text{Im} \chi(p, \omega) \sim \left( \omega^2 - c^2 p^2 \right)^{-\frac{2-\eta}{2}} \]

No quasiparticles --- dissipative critical continuum
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VI. Conclusions
\[
\left< S_j \right> = N_1 \cos(\vec{K} \cdot \vec{r}_j) + N_2 \sin(\vec{K} \cdot \vec{r}_j)
\]

Collinear spins: \( N_1 \times N_2 = 0 \)
Non-collinear spins: \( N_1 \times N_2 \neq 0 \)

\( \left< S_j \right> = 0 \)

Quantum critical point

Evolution of phase diagram in a magnetic field

Both states are insulators
Effect of a field on paramagnet

Energy of zero momentum triplon states

\[ \Delta \]

Bose-Einstein condensation of \( S_z = 1 \) triplon
III. Phase diagram in a magnetic field.

\[ H \]
\[ g\mu_B H = \Delta \]

SDW

Spin singlet state with a spin gap

1 Tesla = 0.116 meV

Related theory applies to double layer quantum Hall systems at \( \nu=2 \)
III. Phase diagram in a magnetic field.

Zeeman term leads to a uniform precession of spins

\[ |\partial_\tau \phi_\alpha|^2 \Rightarrow (\partial_\tau \phi_\alpha^* - i\varepsilon_{\alpha\sigma\rho} H_\sigma \phi_\rho)(\partial_\tau \phi_\alpha - i\varepsilon_{\alpha\beta\gamma} H_\beta \phi_\gamma) \]

Take \( H \) oriented along the \( z \) direction. Then

\[ (\lambda_c - \lambda)(\phi_x^2 + \phi_y^2) \Rightarrow (\lambda_c - \lambda - H^2)(\phi_x^2 + \phi_y^2) \]

For \( \lambda > \lambda_c \), \( \phi_x \sim \sqrt{\lambda - \lambda_c + H^2} \), while for \( \lambda < \lambda_c \), \( H_c = \Delta \sim \sqrt{\lambda_c - \lambda} \)

1 Tesla = 0.116 meV

Related theory applies to double layer quantum Hall systems at \( \nu=2 \)
**III. Phase diagram in a magnetic field.**

Zeeman term leads to a uniform precession of spins

\[ |\partial_x \phi_\alpha|^2 \Rightarrow (\partial_x \phi_\alpha^* - i \varepsilon_{\alpha\sigma\rho} H_{\sigma} \phi_\rho) (\partial_x \phi_\alpha - i \varepsilon_{\alpha\beta\gamma} H_{\beta} \phi_\gamma) \]

Take \( H \) oriented along the \( z \) direction. Then

\[ (\lambda_c - \lambda)(\phi_x^2 + \phi_y^2) \Rightarrow (\lambda_c - \lambda - H^2)(\phi_x^2 + \phi_y^2). \]

For \( \lambda > \lambda_c \), \( \phi_x \sim \sqrt{\lambda - \lambda_c + H^2} \), while for \( \lambda < \lambda_c \), \( H_c = \Delta \sim \sqrt{\lambda_c - \lambda} \)

Elastic scattering intensity

\[ I[H] = I[0] + a \left( \frac{H}{J} \right)^2 \]

1 Tesla = 0.116 meV

Related theory applies to double layer quantum Hall systems at \( \nu = 2 \)
III. Phase diagram in a magnetic field.
III. Phase diagram in a magnetic field.

At very large $H$, magnetization saturates.
III. Phase diagram in a magnetic field.

Respulsive interactions between triplons can lead to magnetization plateau at any rational fraction

\[ \sum_{i<j} J_{ij} S_i S_j \]
III. Phase diagram in a magnetic field.

Quantum transitions in and out of plateau are Bose-Einstein condensations of “extra/missing” triplons.
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VI. Conclusions
\[ \langle S_j \rangle = N_1 \cos(\overline{K} \cdot \vec{r}_j) + N_2 \sin(\overline{K} \cdot \vec{r}_j) \]
\[ = \text{Re} \left[ \Phi e^{i\overline{K} \cdot \vec{r}_j} \right] \]
\[ \Phi = N_1 - iN_2 \]

Collinear spins: \( N_1 \times N_2 = 0 \)
Non-collinear spins: \( N_1 \times N_2 \neq 0 \)

We have so far considered the case where both states are insulators.
\[ \langle S_j \rangle = N_1 \cos(\vec{K} \cdot \vec{r}_j) + N_2 \sin(\vec{K} \cdot \vec{r}_j) \]

\[ = \text{Re} \left[ \Phi e^{i\vec{K} \cdot \vec{r}_j} \right] \]

\[ \Phi = N_1 - iN_2 \]

Collinear spins: \( N_1 \times N_2 = 0 \)
Non-collinear spins: \( N_1 \times N_2 \neq 0 \)

Now both sides have a "background" superconducting (SC) order
Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

(additional commensurability effects near $\delta=0.125$)

Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

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Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

Superconductor with $T_{c,\text{min}} = 10$ K

(additional commensurability effects near $\delta=0.125$)

Collinear magnetic (spin density wave) order

\[ \langle S_j \rangle = N_1 \cos(K \cdot r_j) + N_2 \sin(K \cdot r_j) \]

Collinear spins

\( K = (\pi, \pi) \); \( N_2 = 0 \)

\( K = (3\pi/4, \pi) \); \( N_2 = 0 \)

\( K = (3\pi/4, \pi) \);
\( N_2 = (\sqrt{2} - 1)N_1 \)
Interplay of SDW and SC order in the cuprates

T=0 phases of LSCO

Superconductor with $T_{c,\text{min}} = 10$ K

Use simplest assumption of a direct second-order quantum phase transition between SC and SC+SDW phases
If $\vec{K}$ does not exactly connect two nodal points, critical theory is as in an insulator.

Otherwise, new theory of coupled excitons and nodal quasiparticles.

Magnetic transition in a $d$-wave superconductor

\[ S = \int d^2 r d \tau \left[ \left| \nabla_r \Phi_\alpha \right|^2 + c^2 \left| \partial_\tau \Phi_\alpha \right|^2 + V(\Phi_\alpha) \right] \]

Similar terms present in action for SDW ordering in the insulator.

Coupling to the $S=1/2$ Bogoliubov quasiparticles of the $d$-wave superconductor

Trilinear "Yukawa" coupling

\[ \int d^2 r d \tau \Phi_\alpha \Psi \Psi \]

is prohibited unless ordering wavevector is fine-tuned.

\[ \kappa \sum_\alpha \int d^2 r d \tau |\Phi_\alpha|^2 \Psi^\dagger \Psi \]

is allowed

Scaling dimension of $\kappa = (1/\nu - 2) < 0 \Rightarrow$ irrelevant.
Neutron scattering measurements of dynamic spin correlations of the superconductor (SC) in a magnetic field


Peaks at \((0.5,0.5) \pm (0.125,0)\) and \((0.5,0.5) \pm (0,0.125)\) \(\Rightarrow\) dynamic SDW of period 8

Neutron scattering off La\(_{2-\delta}\)Sr\(_\delta\)CuO\(_4\) (\(\delta = 0.163\), *SC phase*) at low temperatures in \(H=0\) (red dots) and \(H=7.5\)T (blue dots)
Neutron scattering measurements of dynamic spin correlations of the superconductor (SC) in a magnetic field


Peaks at $(0.5,0.5) \pm (0.125,0)$ and $(0.5,0.5) \pm (0,0.125)$
⇒ dynamic SDW of period 8

Neutron scattering off La$_{2-\delta}$Sr$_\delta$CuO$_4$ ($\delta = 0.163$, *SC phase*) at low temperatures in $H=0$ (red dots) and $H=7.5T$ (blue dots).

Dominant effect of magnetic field: 
Abrikosov flux lattice

Spatially averaged superflow kinetic energy

\[ \sim \left\langle v_s^2 \right\rangle \sim \frac{H}{H_{c2}} \ln \frac{3H_{c2}}{H} \]
Effect of magnetic field on SDW+SC to SC transition (extreme Type II superconductivity)

\[ \Phi_\alpha = N_{1\alpha} - i N_{2\alpha} \]

Quantum theory for dynamic and critical spin fluctuations

\[
S_b = \int d^2 r \int_0^{1/T} d\tau \left[ |\nabla_r \Phi_\alpha|^2 + c^2 |\partial_\tau \Phi_\alpha|^2 + s |\Phi_\alpha|^2 + \frac{g_1}{2} \left( |\Phi_\alpha|^2 \right)^2 + \frac{g_2}{2} |\Phi_\alpha|^4 \right]
\]

Static Ginzburg-Landau theory for non-critical superconductivity

\[
S_c = \int d^2 r d\tau \left[ \frac{\nu}{2} |\Phi_\alpha|^2 |\psi|^2 \right]
\]

\[
F_{GL} = \int d^2 r \left[ -|\psi|^2 + \frac{|\psi|^4}{2} + \left| (\nabla_r - iA) \psi \right|^2 \right]
\]

\[
Z[\psi(r)] = \int D\Phi(r, \tau) e^{-F_{GL} - S_b - S_c}
\]

\[
\frac{\delta \ln Z[\psi(r)]}{\delta \psi(r)} = 0
\]

Effect of magnetic field on SDW+SC to SC transition

Quantum theory for dynamic and critical spin fluctuations

Static Ginzburg-Landau theory for non-critical superconductivity
Triplon wavefunction in bare potential $V_0(x)$

Bare triplon potential

$$V_0(r) = s + V |\psi(r)|^2$$

Energy

Spin gap $\Delta$

Vortex cores
Wavefunction of lowest energy triplon $\Phi_\alpha$

after including triplon interactions: $V(\mathbf{r}) = V_0(\mathbf{r}) + g \langle |\Phi_\alpha(\mathbf{r})|^2 \rangle$

Bare triplon potential

$V_0(\mathbf{r}) = s + V |\psi(\mathbf{r})|^2$

Strongly relevant repulsive interactions between excitons imply that triplons must be extended as $\Delta \to 0$.


Phase diagram of SC and SDW order in a magnetic field

The suppression of SC order appears to the SDW order as a uniform effective "doping" $\delta$:

$$\delta_{\text{eff}}(H) = \delta - C \frac{H}{H_{c2}} \ln \left( \frac{3H_{c2}}{H} \right)$$

Phase diagram of SC and SDW order in a magnetic field

Elastic scattering intensity

\[ I[H, \delta] \approx I[0, \delta_{\text{eff}}] \]

\[ \approx I[0, \delta] + a \frac{H}{H_{c2}} \ln \left( \frac{3H_{c2}}{H} \right) \]

\[ \delta_{\text{eff}}(H) = \delta_c \quad \Rightarrow \quad H \sim \frac{(\delta - \delta_c)}{\ln\left(1/(\delta - \delta_c)\right)} \]

Structure of long-range SDW order in SC+SDW phase


$$\delta |f_0|^2 \propto H \ln(1/H)$$

$$S(k, \omega) = (2\pi)^3 \delta(\omega) \sum_G |f_G|^2 \delta(k-G) + \cdots$$

$G \rightarrow$ reciprocal lattice vectors of vortex lattice.

$k$ measures deviation from SDW ordering wavevector $K$
Interplay of SDW and SC order in the cuprates

Superconductor with $T_{c,\text{min}} = 10$ K

Use simplest assumption of a direct second-order quantum phase transition between SC and SC+SDW phases

Follow intensity of elastic Bragg spots in a magnetic field
Neutron scattering of La$_{2-x}$Sr$_x$CuO$_4$ at $x=0.1$


Phase diagram of a superconductor in a magnetic field

Neutron scattering observation of SDW order enhanced by superflow.

\[ \delta_{\text{eff}} (H) = \delta_c \Rightarrow H \sim \frac{(\delta - \delta_c)}{\ln(1/(\delta - \delta_c))} \]

Prediction: SDW fluctuations enhanced by superflow and bond order pinned by vortex cores (no spins in vortices) should be observable in STM.

\( S = 1 \) triplon energy

\[ \varepsilon (H) = \varepsilon (0) - \frac{H}{\ln \left( \frac{3H^2}{H^*} \right)} \]

K. Park and S. Sachdev, Physical Review B 64, 184510 (2001);
E. Demler, S. Sachdev, and Ying Zhang, Phys. Rev. Lett. 87, 067202 (2001);
Vortex-induced LDOS of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ integrated from 1meV to 12meV

Our interpretation: LDOS modulations are signals of bond order of period 4 revealed in vortex halo

See also:

Fourier Transform of Vortex-Induced LDOS map

Distances in k-space have units of $2\pi/a_0$
$a_0=3.83\ \text{Å}$ is Cu-Cu distance

Spectral properties of the STM signal are sensitive to the microstructure of the charge order

Measured energy dependence of the Fourier component of the density of states which modulates with a period of 4 lattice spacings


Theoretical modeling shows that this spectrum is best obtained by a modulation of bond variables, such as the exchange, kinetic or pairing energies.

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   A. Coherent state path integral
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    Bose condensation of “triplons”

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V. Antiferromagnets with an odd number
    of S=1/2 spins per unit cell
   Class A: Compact U(1) gauge theory: collinear spins,
            bond order and confined spinons in \( d=2 \)
   Class B: \( Z_2 \) gauge theory: non-collinear spins, RVB,
            visons, topological order, and deconfined spinons

VI. Conclusions
V. Order in Mott insulators

**Magnetic order**

\[
\langle S_j \rangle = N_1 \cos(\vec{K} \cdot \vec{r}_j) + N_2 \sin(\vec{K} \cdot \vec{r}_j)
\]

**Class A. Collinear spins**

\(\vec{K} = (\pi, \pi) ; \ N_2 = 0\)

\(\vec{K} = (3\pi/4, \pi) ; \ N_2 = 0\)

\(\vec{K} = (3\pi/4, \pi) ; \ N_2 = (\sqrt{2} - 1) N_1\)
V. Order in Mott insulators

**Magnetic order**

\[
\langle S_j \rangle = N_1 \cos(\bar{K} \cdot \bar{r}_j) + N_2 \sin(\bar{K} \cdot \bar{r}_j)
\]

**Class A. Collinear spins**

**Key property**

Order specified by a single vector \( N \).

Quantum fluctuations leading to loss of magnetic order should produce a paramagnetic state with a vector \( S=1 \) quasiparticle excitation.
Class A: Collinear spins and compact U(1) gauge theory

Write down path integral for quantum spin fluctuations

Key ingredient: Spin Berry Phases

$e^{iSA}$
Class A: Collinear spins and compact U(1) gauge theory

Write down path integral for quantum spin fluctuations

**Key ingredient: Spin Berry Phases**
Class A: Collinear spins and compact U(1) gauge theory

\( S=1/2 \) square lattice antiferromagnet with non-nearest neighbor exchange

\[
H = \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

Include Berry phases after discretizing coherent state path integral on a cubic lattice in spacetime

\[
Z = \prod_a \int d\mathbf{n}_a \delta(\mathbf{n}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \mathbf{n}_a \cdot \mathbf{n}_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{a\tau} \right)
\]

\( \eta_a \to \pm 1 \) on two square sublattices; 
\( \mathbf{n}_a \sim \eta_a \vec{S}_a \to \text{Neel order parameter;} \)
\( A_{a\mu} \to \text{oriented area of spherical triangle} \)
formed by \( \mathbf{n}_a, \mathbf{n}_{a+\mu}, \) and an arbitrary reference point \( \mathbf{n}_0 \)
Spin-wave theory about Neel state receives minor modifications from Berry phases.

\[ Z = \prod_a \int d\mathbf{n}_a \delta(\mathbf{n}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \mathbf{n}_a \cdot \mathbf{n}_{a+\mu} - \frac{i}{2} \sum_a \eta_a A_{\alpha\tau} \right) \]

**Small** \( g \) \( \rightarrow \) Spin-wave theory about Neel state receives minor modifications from Berry phases.

**Large** \( g \) \( \rightarrow \) Berry phases are crucial in determining structure of "quantum-disordered" phase with \( \langle n_a \rangle = 0 \)

*Integrate out* \( n_a \) *to obtain effective action for* \( A_{a\mu} \)
Change in choice of $n_0$ is like a “gauge transformation”

$$A_{a\mu} \rightarrow A_{a\mu} - \gamma_{a+\mu} + \gamma_a$$

($\gamma_a$ is the oriented area of the spherical triangle formed by $n_a$ and the two choices for $n_0$).

The area of the triangle is uncertain modulo $4\pi$, and the action is invariant under

$$A_{a\mu} \rightarrow A_{a\mu} + 4\pi$$

These principles strongly constrain the effective action for $A_{a\mu}$ which provides description of the large $g$ phase
Simplest large $g$ effective action for the $A_{a\mu}$

$$Z = \prod_{a,\mu} \int dA_{a\mu} \exp\left(\frac{1}{2e^2} \sum \cos\left(\frac{1}{2} (\Delta_{\mu} A_{av} - \Delta_{v} A_{a\mu}) \right) - \frac{i}{2} \sum \eta_a A_{a\tau} \right)$$

with $e^2 \sim g^2$

This is compact QED in $d+1$ dimensions with static charges $\pm 1$ on two sublattices.

This theory can be reliably analyzed by a duality mapping.

$d=2$: The gauge theory is always in a confining phase and there is bond order in the ground state.

$d=3$: A deconfined phase with a gapless “photon” is possible.

Exact duality transform on periodic Gaussian ("Villain") action for compact QED yields

\[ Z = \sum_{\{h_j\}} \exp \left( -\frac{e^2}{2} \sum_j (\Delta_\mu h_j - \Delta_\mu x_j)^2 \right) \]

with \( h_j \) integer.
Height model in 2+1 dimensions with ‘offsets’ \( x_j = 0, 1/4, 1/2, 3/4 \) on the four dual sublattices.
For large $e^2$, low energy height configurations are in exact one-to-one correspondence with dimer coverings of the square lattice.

$\implies$ 2+1 dimensional height model is the path integral of the **Quantum Dimer Model**

There is no roughening transition for three dimensional interfaces, which are smooth for all couplings.

$\implies$ There is a definite average height of the interface

$\implies$ **Ground state has bond order.**
V. Order in Mott insulators

**Paramagnetic states** \( \langle S_j \rangle = 0 \)

**Class A. Bond order and spin excitons in \( d=2 \)**

\[
\frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle \right)
\]

\( S=1/2 \) spinons are confined by a linear potential into a \( S=1 \) spin triplon

Spontaneous bond-order leads to vector \( S=1 \) spin excitations

Bond order in a frustrated $S=1/2$ XY magnet


First *large scale* numerical study of the destruction of Neel order in a $S=1/2$ antiferromagnet with full square lattice symmetry

\[
H = 2J \sum_{\langle ij \rangle} \left( S_i^x S_j^x + S_i^y S_j^y \right) - K \sum_{\langle ijkl \rangle} \left( S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+ \right)
\]
Outline

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V.B Order in Mott insulators

**Magnetic order**

\[
\langle S_j \rangle = N_1 \cos(\vec{K} \cdot \vec{r}_j) + N_2 \sin(\vec{K} \cdot \vec{r}_j)
\]

Class B. Noncollinear spins

\[
\vec{K} = \left(\frac{3\pi}{4}, \pi\right)
\]


\[
\vec{K} = \left(\frac{4\pi}{3}, \frac{4\pi}{\sqrt{3}}\right)
\]

\[
N_2^2 = N_1^2 \quad \text{and} \quad N_1 \cdot N_2 = 0
\]
V.B Order in Mott insulators

Magnetic order

\[ \langle S_j \rangle = N_1 \cos(\tilde{K} \cdot \tilde{r}_j) + N_2 \sin(\tilde{K} \cdot \tilde{r}_j) \]

Class B. Noncollinear spins

\[ N_2^2 = N_1^2, \quad N_1 \cdot N_2 = 0 \]

Solve constraints by expressing \( N_{1,2} \) in terms of two complex numbers \( z^\uparrow, z^\downarrow \)

\[
N_1 + iN_2 = \begin{pmatrix}
2z^\uparrow z^\downarrow \\
\left( z^\downarrow + z^\uparrow \right)
\end{pmatrix} \quad i\left( z^\downarrow + z^\uparrow \right)
\]

Order in ground state specified by a spinor \((z^\uparrow, z^\downarrow)\) (modulo an overall sign).
This spinor can become a \( S=1/2 \) spinon in paramagnetic state.

Order parameter space: \( S_3/Z_2 \)

Physical observables are invariant under the \( Z_2 \) gauge transformation \( z_a \rightarrow \pm z_a \)

V.B Order in Mott insulators

**Magnetic order** \[ \langle S_j \rangle = N_1 \cos(\vec{K} \cdot \vec{r}_j) + N_2 \sin(\vec{K} \cdot \vec{r}_j) \]

**Class B. Noncollinear spins**

**Vortices associated with** \( \pi_1(S_3/Z_2) = Z_2 \) (\textit{visons})

Such vortices (visons) can also be defined in the phase in which spins are “quantum disordered”. A \( Z_2 \) spin liquid with deconfined spinons must have \textit{visons suppressed}.

Model effective action and phase diagram

\[ S = -J \sum_{\langle ij \rangle} \sigma_{ij} \bar{z}_{\alpha i} z_{\alpha j} + \text{h.c.} - K \prod \sigma_{ij} \]


\[ \sigma_{ij} \rightarrow Z_2 \text{ gauge field} \]

First order transition

Magnetically ordered

Confined spinons

Free spinons and topological order

V.B Order in Mott insulators

Paramagnetic states \( \langle S_j \rangle = 0 \)

Class B. Topological order and deconfined spinons

A topologically ordered state in which vortices associated with \( \pi_1(S_3/Z_2) = \mathbb{Z}_2 \) ["visons"] are gapped out. This is an RVB state with deconfined \( S = 1/2 \) spinons \( z_a \)

Recent experimental realization: Cs₂CuCl₄


V.B  Order in Mott insulators

**Paramagnetic states**  \( \langle S_j \rangle = 0 \)

Class B. Topological order and deconfined spinons

Direct description of topological order with valence bonds

Number of valence bonds cutting line is conserved modulo 2. Changing sign of each such bond does not modify state. This is equivalent to a \( Z_2 \) gauge transformation with \( z_a \rightarrow -z_a \) on sites to the right of dashed line.

V.B Order in Mott insulators

**Paramagnetic states** \[ \langle S_j \rangle = 0 \]

Class B. Topological order and deconfined spinons

Direct description of topological order with valence bonds

Number of valence bonds cutting line is conserved modulo 2. Changing sign of each such bond does not modify state. This is equivalent to a \( \mathbb{Z}_2 \) gauge transformation with \( z_a \rightarrow -z_a \) on sites to the right of dashed line.

V.B Order in Mott insulators

**Paramagnetic states** \( \langle S_j \rangle = 0 \)

Class B. Topological order and deconfined spinons

Direct description of topological order with valence bonds

Terminating the line creates a plaquette with \( \mathbb{Z}_2 \) flux at the X --- a *vison*.

Effect of flux-piercing on a topologically ordered quantum paramagnet

N. E. Bonesteel,

G. Misguich, C. Lhuillier,
M. Mambrini, and P. Sindzingre,

\[ |D\rangle = \sum_{D} a_D |D\rangle \]

\[ \Phi \]

\[
\sum_{i} \frac{1}{2} \left( \hat{L}_x^{-2} \hat{L}_x^{-1} \hat{L}_x \right) \]

\[
\Psi = \sum_{D} a_D |D\rangle
\]
Effect of flux-piercing on a topologically ordered quantum paramagnet

N. E. Bonesteel,

G. Misguich, C. Lhuillier,
M. Mambrini, and P. Sindzingre,

\[ |\Psi\rangle = \sum_D a_D |D\rangle \]

After flux insertion \(|D\rangle \Rightarrow \)

Number of bonds \((-1)^{\text{cutting dashed line}} |D\rangle\);  

Equivalent to inserting a *vison* inside hole of the torus.

This leads to a ground state degeneracy.
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VI. Cuprates are best understood as doped class A Mott insulators.
1. **Pairing order of BCS theory (SC)**

(Bose-Einstein) condensation of \(d\)-wave Cooper pairs

**Orders (possibly fluctuating) associated with proximate Mott insulator in class A**

2. **Collinear magnetic order (CM)**

3. **Bond/charge/stripe order (B)**

(couples strongly to half-breathing phonons)

Evidence cuprates are in class A
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• Neutron scattering shows collinear magnetic order co-existing with superconductivity


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• Proximity of $Z_2$ Mott insulators requires stable $hc/e$ vortices, vison gap, and Senthil flux memory effect

Evidence cuprates are in class A

- Neutron scattering shows collinear magnetic order co-existing with superconductivity
- Proximity of $Z_2$ Mott insulators requires stable $hc/e$ vortices, vison gap, and Senthil flux memory effect
- Non-magnetic impurities in underdoped cuprates acquire a $S=1/2$ moment
Effect of static non-magnetic impurities (Zn or Li)

Spinon confinement implies that free \( S=1/2 \) moments form near each impurity

\[
\chi_{\text{impurity}}(T \to 0) = \frac{S(S+1)}{3k_B T}
\]
Spatially resolved NMR of Zn/Li impurities in the superconducting state

Inverse local susceptibility in YBCO


\[ \chi_{\text{impurity}}(T \to 0) = \frac{S(S+1)}{3k_B T} \]

Measured \( \chi_{\text{impurity}}(T \to 0) = \frac{S(S+1)}{3k_B T} \) with \( S = 1/2 \) in underdoped sample.

This behavior does not emerge out of BCS theory.

Evidence cuprates are in class A

- Neutron scattering shows collinear magnetic order co-existing with superconductivity
- Proximity of $Z_2$ Mott insulators requires stable $hc/e$ vortices, vison gap, and Senthil flux memory effect
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Evidence cuprates are in class A

• Neutron scattering shows collinear magnetic order co-existing with superconductivity

• Proximity of $Z_2$ Mott insulators requires stable $hc/e$ vortices, vison gap, and Senthil flux memory effect

• Non-magnetic impurities in underdoped cuprates acquire a $S=1/2$ moment

• Tests of phase diagram in a magnetic field
Phase diagram of a superconductor in a magnetic field

Neutron scattering observation of SDW order enhanced by superflow.

\[ \delta_{\text{eff}}(H) = \delta_c \quad \Rightarrow \quad H \sim \frac{(\delta - \delta_c)}{\ln\left(\frac{1}{(\delta - \delta_c)}\right)} \]

Phase diagram of a superconductor in a magnetic field

Neutron scattering observation of SDW order enhanced by superflow.

\[ \delta_{\text{eff}}(H) = \delta_c \implies H \sim \frac{(\delta - \delta_c)}{\ln(1/(\delta - \delta_c))} \]

Possible STM observation of predicted bond order in halo around vortices

K. Park and S. Sachdev *Physical Review B* 64, 184510 (2001);
VI. Doping Class A

Doping a paramagnetic bond-ordered Mott insulator

systematic $\text{Sp}(N)$ theory of translational symmetry breaking, while preserving spin rotation invariance.

A phase diagram

- **Pairing order of BCS theory (SC)**
- **Collinear magnetic order (CM)**
- **Bond order (B)**

Vertical axis is any microscopic parameter which suppresses CM order