Quantum criticality, the cuprate superconductors, and the AdS/CFT correspondence

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arXiv:1001.1153

1. Coupled dimer antiferromagnets
   Introduction to quantum criticality

2. Theory of Ising-nematic ordering in the cuprate metals
   Strongly-coupled field theory

3. The AdS/CFT correspondence
   Phases of quantum matter at strong coupling
Outline

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   *Phases of quantum matter at strong coupling*
The cuprate superconductors
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state has long-range Néel order

Order parameter is a single vector field \( \vec{\phi} = \eta_i \vec{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\( \langle \vec{\phi} \rangle \neq 0 \) in Néel state.
TlCuCl$_3$
An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
Square lattice antiferromagnet

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Square lattice antiferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Weaken some bonds to induce spin entanglement in a new quantum phase
Square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state is a "quantum paramagnet" with spins locked in valence bond singlets

\[ = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
\[ \lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]
Quantum critical point with non-local entanglement in spin wavefunction

$$\rho = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$
Excitation spectrum in the paramagnetic phase
Excitation spectrum in the paramagnetic phase
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Excitation spectrum in the paramagnetic phase
Excitation spectrum in the paramagnetic phase
TlCuCl$_3$ at ambient pressure

FIG. 1. Measured neutron profiles in the $a^*c^*$ plane of TlCuCl$_3$
for $i=(1.35,0,0)$, $ii=(0,0,3.15)$ [r.l.u]. The spectrum at $T=1.5$ K

N. Cavadini, G. Heigold, W. Henggeler, A. Furrer, H.-U. Güdel, K. Krämer
Sharp spin 1 particle excitation above an energy gap (spin gap)

Excitation spectrum in the Néel phase
Excitation spectrum in the Néel phase

Spin waves

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Excitation spectrum in the Néel phase

Spin waves
Description using Landau-Ginzburg field theory

\[ S = \int d^2r d\tau \left[ (\partial_\tau \bar{\phi})^2 + c^2 (\nabla_r \bar{\phi})^2 + (\lambda - \lambda_c) \bar{\phi}^2 + u (\bar{\phi}^2)^2 \right] \]

\( \text{O}(3) \) order parameter \( \bar{\phi} \)

\( \lambda \)

\( \lambda_c \)

CFT3
Excitation spectrum in the paramagnetic phase

\[ V(\vec{\varphi}) = (\lambda - \lambda_c)\varphi^2 + u (\varphi^2)^2 \]

\[ \lambda > \lambda_c \]
Excitation spectrum in the paramagnetic phase

\[ V(\vec{\varphi}) = (\lambda - \lambda_c)\varphi^2 + u(\varphi^2)^2 \]

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Spin $S = 1$ “triplon”
Excitation spectrum in the paramagnetic phase

\[ V(\phi) = (\lambda - \lambda_c)\phi^2 + u(\phi^2)^2 \]

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Spin \( S = 1 \)

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$V(\vec{\phi}) = (\lambda - \lambda_c)\vec{\phi}^2 + u(\vec{\phi}^2)^2$

$\lambda > \lambda_c$

Spin $S = 1$

“triplon”
Excitation spectrum in the Néel phase
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Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

\[ V(\bar{\varphi}) = (\lambda - \lambda_c)\bar{\varphi}^2 + u(\bar{\varphi}^2)^2 \]

\[ \lambda < \lambda_c \]
Excitation spectrum in the Néel phase

Field theory yields spin waves ("Goldstone" modes) but also an additional longitudinal "Higgs" particle

\[ V(\bar{\varphi}) = (\lambda - \lambda_c)\bar{\varphi}^2 + u (\bar{\varphi}^2)^2 \]

\( \lambda < \lambda_c \)
TlCuCl$_3$ with varying pressure

Observation of $3 \rightarrow 2$ low energy modes, emergence of new Higgs particle in the Néel phase, and vanishing of Néel temperature at the quantum critical point.

Prediction of quantum field theory

Potential for $\bar{\phi}$ fluctuations: $V(\bar{\phi}) = (\lambda - \lambda_c)\bar{\phi}^2 + u (\bar{\phi}^2)^2$

Paramagnetic phase, $\lambda > \lambda_c$

Expand about $\bar{\phi} = 0$:

$V(\bar{\phi}) \approx (\lambda - \lambda_c)\bar{\phi}^2$

Yields 3 particles with energy gap $\sim \sqrt{(\lambda - \lambda_c)}$
Prediction of quantum field theory

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Yields 3 particles with energy gap $\sim \sqrt{(\lambda - \lambda_c)}$

Néel phase, $\lambda < \lambda_c$

Expand $\bar{\phi} = (0, 0, \sqrt{(\lambda_c - \lambda)/(2u)}) + \bar{\phi}_1$:

$V(\bar{\phi}) \approx 2(\lambda_c - \lambda)\bar{\phi}_{1z}^2$

Yields 2 gapless spin waves and one Higgs particle with energy gap $\sim \sqrt{2(\lambda_c - \lambda)}$
Prediction of quantum field theory

\[
\frac{\text{Energy of Higgs particle}}{\text{Energy of triplon}} = \sqrt{2}
\]

\[V(\varphi) = (\lambda - \lambda_c)\varphi^2 + u(\varphi^2)^2\]

\[\sqrt{2E(p < p_c)} \quad \text{unscaled}\]

\[\text{TlCuCl}_3\]
\[p_c = 1.07 \text{ kbar}\]
\[T = 1.85 \text{ K}\]

S. Sachdev, arXiv:0901.4103
The order parameter is given by:

\[ S = \int d^2r d\tau \left[ (\partial_\tau \varphi)^2 + c^2 (\nabla_r \varphi)^2 + (\lambda - \lambda_c) \varphi^2 + u (\varphi^2)^2 \right] \]

with the O(3) order parameter \( \varphi \).

The critical points are marked by \( \lambda_c \) and \( \lambda \). For \( \lambda > \lambda_c \), the system is in the symmetric phase, and for \( \lambda < \lambda_c \), it is in the broken phase. The state vector is

\[ \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Classical spin waves

Quantum critical

Dilute triplon gas

Neel order

Pressure in TlCuCl$_3$

Classical spin waves

Quantum critical

Dilute triplon gas

CFT3 at $T > 0$

Neel order

Pressure in TlCuCl$_3$

CFT3 at $T>0$

Quantum critical

Classical spin waves

Dilute triplon gas

Neel order

Pressure in TlCuCl$_3$

Strong coupling problem I: dynamics and transport at times \( > \hbar/(k_B T) \) where transport and damping constants are universally determined fundamental constants of nature.
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Doped square lattice antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Ground state has long-range Néel order

Order parameter is a single vector field \( \vec{\varphi} = \eta_i \vec{S}_i \)

\( \eta_i = \pm 1 \) on two sublattices

\( \langle \vec{\varphi} \rangle \neq 0 \) in Néel state.
Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface

K.M. Shen et al., Science 2005

M. Platé et al., PRL 2005

Smaller hole Fermi-pockets

Large hole Fermi surface
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![Phase Diagram Image](image-url)

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Large hole Fermi surface
STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$. 

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$O_N = Z_A + Z_B - Z_C - Z_D$

STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$.

$O_N^c(r, e = 1)$

Strong anisotropy of electronic states between $x$ and $y$ directions:

Electronic “Ising-nematic” order

$O_N = Z_A + Z_B - Z_C - Z_D$
Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface

Strange Metal

K.M. Shen et al., Science 2005

M. Platé et al., PRL 2005

Smaller hole Fermi-pockets

Large hole Fermi surface
Quantum criticality of Ising-nematic ordering

Fermi surface with full square lattice symmetry
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $x$ direction:
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $y$ direction:
Ising-nematic order parameter

\[ \phi \sim \int d^2 k \left( \cos k_x - \cos k_y \right) c_{k\sigma}^\dagger c_{k\sigma} \]

Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $x$ direction:
Ising order parameter $\phi > 0$. 

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Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $y$ direction:
Ising order parameter $\phi < 0$. 

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Quantum criticality of Ising-nematic ordering

Pomeranchuk instability as a function of coupling $r$

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$

$T_l \text{n}$

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

$T_c$

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Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
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Phase diagram as a function of $T$ and $r$

Classical $d=2$ Ising criticality

$T_{I-n}$

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

$D=2+1$ Ising criticality?
Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$

Quantum critical

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$

$T^*$

$T_{I-n}$

$T = 0$

$T_c$

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Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Fermi liquid theory

\[ S_{\text{FL}} = \int d\Omega_{\hat{n}} \int dx_{\perp} \psi_{\hat{n}a}^{\dagger}(x_{\perp}) \left( \frac{\partial}{\partial \tau} - iv_{F}(\hat{n}) \frac{\partial}{\partial x_{\perp}} \right) \psi_{\hat{n}a}(x_{\perp}) \]

Infinite number of 1+1 dimensional chiral fermions
Critical point is described by an infinite set of 2+1 dimensional field theories, one for each direction $\hat{q}$. 

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Critical point is described by an infinite set of 2+1 dimensional field theories, one for each direction $\hat{q}$. 
Non-Fermi liquid quantum critical point

Strong coupling problem II: Infinite number of 2+1 dimensional field theories at Ising-nematic quantum critical point

- Critical point is described by an infinite set of 2+1 dimensional field theories, one for each direction $\hat{q}$. 
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Field theories in $D$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

$$u \frac{dg}{du} = \beta(g)$$

where $u$ is the energy scale. The RG equation is local in energy scale, i.e., the RHS does not depend upon $u$. 

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Field theories in $D$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

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where $u$ is the energy scale. The RG equation is local in energy scale, i.e. the RHS does not depend upon $u$.

**Key idea:** ⇒ Implement $u$ as an extra dimension, and map to a local theory in $D+1$ dimensions.
At the RG fixed point, $\beta(g) = 0$, the $D$ dimensional field theory is invariant under the scale transformation

$$x^\mu \rightarrow x^\mu / b \quad , \quad u \rightarrow b u$$
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This is an invariance of the *metric* of the theory in $D + 1$ dimensions. The unique solution is

$$ds^2 = \left(\frac{u}{L}\right)^2 dx^\mu dx_\mu + L^2 \frac{du^2}{u^2}.$$ 

Or, using the length scale $z = L^2 / u$

$$ds^2 = L^2 \frac{dx^\mu dx_\mu + dz^2}{z^2}.$$ 

This is the space $\text{AdS}_{D+1}$, and $L$ is the AdS radius.
Figure 1: The extra (‘radial’) dimension of the bulk is the resolution scale of the field theory. The left figure indicates a series of block spin transformations labelled by a parameter $z$. The right figure is a cartoon of AdS space, which organizes the field theory information in the same way. In this sense, the bulk picture is a hologram: excitations with different wavelengths get put in different places in the bulk image.
AdS/CFT correspondence

The quantum theory of a black hole in a 3+1-dimensional negatively curved AdS universe is holographically represented by a CFT (the theory of a quantum critical point) in 2+1 dimensions.

3+1 dimensional AdS space

Maldacena, Gubser, Klebanov, Polyakov, Witten
AdS/CFT correspondence
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- **3+1 dimensional AdS space**
- **Quantum criticality in 2+1 dimensions**
- **Black hole temperature = temperature of quantum criticality**

Maldacena, Gubser, Klebanov, Polyakov, Witten

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Black hole entropy = entropy of quantum criticality

3+1 dimensional AdS space

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3+1 dimensional AdS space

Quantum criticality in 2+1 dimensions

Friction of quantum criticality = waves falling into black hole

Kovtun, Policastro, Son
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Strong coupling problem I: General solution of magneto-thermo-electric transport in quantum critical region.

C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son,

S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev,
AdS/CFT correspondence

3+1 dimensional AdS space

Quantum criticality in 2+1 dimensions
AdS/CFT correspondence

Move away from the quantum critical point to a system of matter at non-zero density: equivalent to adding an electrical charge to the black hole.

3+1 dimensional AdS space

Black hole with electrical charge

Finite density matter in 2+1 dimensions
AdS/CFT correspondence

Examine the free energy and Green’s function of a probe particle


3+1 dimensional AdS space

Black hole with electrical charge

Finite density matter in 2+1 dimensions
Green’s function of a fermion

\[ G(k, \omega) \approx \frac{1}{\omega - v_F(k - k_F) - i\omega \theta(k)} \]

Green’s function of a fermion

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Similar to our theory of the singular Fermi surface near the Ising-nematic quantum critical point


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Green’s function of a fermion

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Strong coupling problem II:
Suggestive and promising similarities between stringy and cond-mat results. Physical interpretation of string theory results remains unclear.

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Theories for the onset of Ising-nematic order (and spin density wave order) in metals are strongly coupled in two dimensions
Conclusions

The AdS/CFT offers promise in providing a new understanding of strongly interacting quantum matter at non-zero density