Quantum phases of SYK models

Subir Sachdev
June 5, 2018
Integrable and Chaotic Quantum Dynamics: from Holography to Lattice,
Bled, Slovenia
Quantum matter with quasiparticles:

The quasiparticle idea is the key reason for the many successes of quantum condensed matter physics:

- Fermi liquid theory of metals, insulators, semiconductors
- Theory of superconductivity (pairing of quasiparticles)
- Theory of disordered metals and insulators (diffusion and localization of quasiparticles)
- Theory of metals in one dimension (collective modes as quasiparticles)
- Theory of the fractional quantum Hall effect (quasiparticles which are `fractions’ of an electron)
Quantum matter with quasiparticles:

- Quasiparticles are additive excitations: The low-lying excitations of the many-body system can be identified as a set \( \{n_\alpha\} \) of quasiparticles with energy \( \varepsilon_\alpha \)

\[
E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots
\]

In a lattice system of \( N \) sites, this parameterizes the energy of \( \sim e^{\alpha N} \) states in terms of poly(\( N \)) numbers.
Quasiparticles eventually collide with each other. Such collisions eventually lead to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{eq} \sim \frac{\hbar E_F}{(k_B T)^2}, \quad \text{as } T \to 0,$$

where $E_F$ is the Fermi energy.
1. Metal with quasiparticles
   Random matrix model of a `quantum dot'

2. Metal without quasiparticles
   SYK model of a `quantum dot'

3. Lattice models of SYK islands
   Theory of a strange metal

4. $Z_2$ Fractionalization in a SYK $t$-$J$ model

5. SYK $U(1)$ gauge theory
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5. SYK U(1) gauge theory
A simple model of a metal with quasiparticles

Pick a set of random positions
A simple model of a metal with quasiparticles

Place electrons randomly on some sites
A simple model of a metal with quasiparticles

Electrons move one-by-one randomly
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A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \ldots$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_{i} c_i^\dagger c_i = Q$$

$t_{ij}$ are independent random variables with $\overline{t_{ij}} = 0$ and $|t_{ij}|^2 = t^2$

Fermions occupying the eigenstates of a $N \times N$ random matrix
Infinite-range model with quasiparticles

Feynman graph expansion in $t_{ij}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = Q.$$ 

$G(\omega)$ can be determined by solving a quadratic equation.

$-\text{Im} \ G(\omega)$
Infinite-range model with quasiparticles

Now add weak interactions

\[
H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell
\]

\[J_{ij;kl}\] are independent random variables with \(\overline{U_{ij;kl}} = 0\) and \(\overline{|U_{ij;kl}|^2} = U^2\). We compute the lifetime of a quasiparticle, \(\tau_\alpha\), in an exact eigenstate \(\psi_\alpha(i)\) of the free particle Hamiltonian with energy \(\varepsilon_\alpha\). By Fermi’s Golden rule, for \(\varepsilon_\alpha\) at the Fermi energy

\[
\frac{1}{\tau_\alpha} = \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta))\delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta)
\]

\[
= \frac{\pi^3 U^2 \rho_0^2}{4} T^2
\]

where \(\rho_0\) is the density of states at the Fermi energy.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as \(\sim T^{-2}\) at the Fermi level.
A simple model of a metal with quasiparticles

Let $\varepsilon_\alpha$ be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest $NQ$ eigenvalues, upto the Fermi energy $E_F$. The density of states is $\rho(\omega) = (1/N) \sum_\alpha \delta(\omega - \varepsilon_\alpha)$. 
A simple model of a metal with quasiparticles

There are $2^N$ many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_{\alpha} \varepsilon_{\alpha},$$

where $n_{\alpha} = 0, 1$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $\varepsilon_{\alpha}$ have a level spacing $\sim 1/N$. 

Many-body level spacing $\sim 2^{-N}$

Quasiparticle excitations with spacing $\sim 1/N$
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The Sachdev-Ye-Kitaev (SYK) model

Pick a set of random positions
Place electrons randomly on some sites

The SYK model
The SYK model

Entangle electrons pairwise randomly
The SYK model

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The SYK model
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Entangle electrons pairwise randomly
This describes both a strange metal and a black hole!
The SYK model

(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large $N$ limit; T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. 53, 385 (1981))

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;kl} c_{i}^{\dagger} c_{j}^{\dagger} c_{k} c_{\ell} - \mu \sum_{i} c_{i}^{\dagger} c_{i}$$

$$c_{i} c_{j} + c_{j} c_{i} = 0 \quad , \quad c_{i}^{\dagger} c_{j}^{\dagger} + c_{j}^{\dagger} c_{i} = \delta_{ij}$$

$$Q = \frac{1}{N} \sum_{i} c_{i}^{\dagger} c_{i}$$

$U_{ij;kl}$ are independent random variables with $\overline{U_{ij;kl}} = 0$ and $|U_{ij;kl}|^2 = U^2$

$N \rightarrow \infty$ yields critical strange metal.

S. Sachdev and J. Ye, PRL 70, 3339 (1993)

Many-body level spacing \( \sim 2^{-N} = e^{-N \ln 2} \)

Non-quasiparticle excitations with spacing \( \sim e^{-Ns_0} \)

There are \( 2^N \) many body levels with energy \( E \), which do not admit a quasiparticle decomposition. Shown are all values of \( E \) for a single cluster of size \( N = 12 \). The \( T \to 0 \) state has an entropy \( S_{GPS} = Ns_0 \) with

\[
s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848 \ldots < \ln 2
\]

where \( G \) is Catalan’s constant, for the half-filled case \( Q = 1/2 \).

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB 63, 134406 (2001)

W. Fu and S. Sachdev, PRB 94, 035135 (2016)
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No quasiparticles!

\[
E \neq \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha, \beta} F_{\alpha \beta} n_\alpha n_\beta + \ldots
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The SYK model

Feynman graph expansion in $U_{ijk\ell}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)}, \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)
\]

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$$G(\tau = 0^-) = Q.$$ 

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \ldots , \quad G(z) = \frac{A}{\sqrt{z}}$$

where $A = e^{-i\pi/4}(\pi/U^2)^{1/4}$ at half-filling. The ground state is a non-Fermi liquid, with a continuously variable density $Q$. 

The SYK model

• $T = 0$ fermion Green’s function is incoherent: $G(\tau) \sim \tau^{-1/2}$ at large $\tau$. (Fermi liquids with quasiparticles have the coherent: $G(\tau) \sim 1/\tau$)

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- $T > 0$ Green’s function has conformal invariance
  $G \sim (T / \sin(\pi k_B T \tau / \hbar))^{1/2}$

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  A. Eberlein, V. Kasper, S. Sachdev, and J. Steinberg, PRB 96, 205123 (2017)
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- The last property indicates $\tau_{eq} \sim \hbar/(k_B T)$, and this has been found in a recent numerical study.

- The model exhibits eigenstate thermalization. Each eigenstate scrambles quantum information (as measured in the out-of-time-order correlation) in the fastest possible time of $\hbar/(2\pi k_B T(E))$.

S. Sachdev and J. Ye, PRL 70, 3339 (1993)

A. Georges and O. Parcollet PRB 59, 5341 (1999)

A. Eberlein, V. Kasper, S. Sachdev, and J. Steinberg, PRB 96, 205123 (2017)

J. Sonner and M. Vielma, arXiv:1707.08013
Quantum matter without quasiparticles:

- If there are no quasiparticles, then

\[ E \neq \sum_{\alpha} n_{\alpha} \varepsilon_{\alpha} + \sum_{\alpha, \beta} F_{\alpha \beta} n_{\alpha} n_{\beta} + \ldots \]
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• Systems without quasiparticles are the fastest possible in reaching local equilibrium, and all many-body quantum systems obey, as \( T \to 0 \)

\[ \tau_{\text{eq}} > C \frac{\hbar}{k_B T} . \]

  - In Fermi liquids \( \tau_{\text{eq}} \sim 1/T^2 \), and so the bound is obeyed as \( T \to 0 \).
  - This bound rules out quantum systems with e.g. \( \tau_{\text{eq}} \sim \hbar/(Jk_B T)^{1/2} \).
  - There is no bound in classical mechanics (\( \hbar \to 0 \)). By cranking up frequencies, we can attain equilibrium as quickly as we desire.

S. Sachdev,
Quantum Phase Transitions,
Cambridge (1999)
SYK models and black holes

- Black holes have an entropy proportional to their surface area, and a temperature, $T_H = \frac{\hbar c^3}{8\pi G M k_B}$.

- Black holes relax to thermal equilibrium in a time $\sim \frac{\hbar}{k_B T_H} = \frac{8\pi G M}{c^3}$.
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- Black holes relax to thermal equilibrium in a time \( \sim \frac{\hbar}{k_B T_H} = \frac{8\pi GM}{c^3} \).

- Black holes in \( d + 1 \) spatial dimensions are similar to a quantum system without quasiparticles in \( d \) spatial dimensions.
Holographic Metals and the Fractionalized Fermi Liquid

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(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a “small” Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, AdS$_2 \times \mathbb{R}^2$ physics of Reissner-Nordström black holes.

Einstein-Maxwell theory
+ cosmological constant

AdS$_2 \times \mathbb{T}^2$

$ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + d\vec{x}^2$

Gauge field: $A = (\mathcal{E}/\zeta)dt$

charge density $Q$

Black hole horizon

$\zeta = \infty$

$\zeta$

$\mathbb{T}^2$

$\vec{x}$
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5. SYK $U(1)$ gauge theory
SYK building blocks for a strange metal
SYK quantum islands of electrons with random hopping between them.

\[
H = \sum_{x} \sum_{i<j,k<l} U_{ijkl,x} c_{ix}^\dagger c_{jx} c_{kx} c_{lx} + \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij,xx'} c_{i,x}^\dagger c_{j,x,x'}
\]

\[
|U_{ijkl}|^2 = \frac{2U^2}{N^3} \quad |t_{ij,xx'}|^2 = t_0^2/N
\]

SYK building blocks for a strange metal

Mobile electrons (c) interacting with SYK quantum islands (f) with random exchange interactions. This yields the first model agreeing with magnetotransport in strange metals.

Mobile electrons (c) interacting with SYK quantum islands (f) with non-random exchange interactions.
Low ‘coherence’ scale

\[ E_c \sim \frac{t_0^2}{U} \]

Low ‘coherence’ scale

\[ E_c \sim \frac{t_0^2}{U} \]

For \( T < E_c \), the resistivity, \( \rho \), and entropy density, \( s \), are

\[
\rho = \frac{\hbar}{e^2} \left[ c_1 + c_2 \left( \frac{T}{E_c} \right)^2 \right]
\]

\[
s \sim s_0 \left( \frac{T}{E_c} \right)
\]
Low ‘coherence’ scale

\[ E_c \sim \frac{t_0^2}{U} \]

For \( E_c < T < U \), the resistivity, \( \rho \), and entropy density, \( s \), are

\[ \rho \sim \frac{h}{e^2} \left( \frac{T}{E_c} \right) , \quad s = s_0 \]
Superconductivity

Resistivity \sim \rho_0 + AT^\alpha


Physical Review B 81, 184519 (2010)
Figure: K. Fujita and J. C. Seamus Davis

$YBa_2Cu_3O_{6+x}$
“Fermi arcs” at low $p$
Conventional metal
Area enclosed by Fermi surface = 1 + p
Anomalous Criticality in the Electrical Resistivity of La$_{2-x}$Sr$_x$CuO$_4$

R. A. Cooper,$^1$ Y. Wang,$^1$ B. Vignolle,$^2$ O. J. Lipscombe,$^1$ S. M. Hayden,$^1$ Y. Tanabe,$^3$ T. Adachi,$^3$
Y. Koike,$^3$ M. Nohara,$^4$* H. Takagi,$^4$ Cyril Proust,$^2$ N. E. Hussey$^1$†

[Graph and diagram with axes labeled: $T$ (K) on the y-axis and Hole doping $\rho$ on the x-axis, with a color scale on the right side.]

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From the resistivity, they determined the value of the number $\alpha$ defined by

$$\rho(T) = \rho_0 + \alpha \frac{\hbar}{2e^2} \left( \frac{T}{T_F} \right)$$

where $T_F = (\pi \hbar^2 / k_B)(n/m^*)$ and $m^*$ is determined from the specific heat. This expression is obtained from the Drude form $\rho = m^* / (ne^2 \tau)$ and $\hbar/\tau = \alpha k_B T$. 

Universal $T$-linear resistivity and Planckian limit in overdoped cuprates

A. Legros$^{1,2}$, S. Benhabib$^3$, W. Tabis$^{3,4}$, F. Laliberté$^1$, M. Dion$^1$, M. Lizaire$^1$,

B. Vignolle$^3$, D. Vignolles$^3$, H. Raffy$^5$, Z. Z. Li$^5$, P. Auban-Senzier$^5$,

N. Doiron-Leyraud$^1$, P. Fournier$^{1,6}$, D. Colson$^2$, L. Taillefer$^{1,6}$, and C. Proust$^{3,6}$

The perfectly linear temperature dependence of the electrical resistivity observed as $T \to 0$ in a variety of metals close to a quantum critical point is a major puzzle of condensed matter physics. Here we show that $T$-linear resistivity as $T \to 0$ is a generic property of cuprates, associated with a universal scattering rate. We measured the low-temperature resistivity of the bi-layer cuprate Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and found that it exhibits a $T$-linear dependence with the same slope as in the single-layer cuprates Bi$_2$Sr$_2$CuO$_{6+\delta}$ (ref. 6), La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ (ref. 7) and La$_{2-x}$Sr$_x$CuO$_4$ (ref. 8), despite their very different Fermi surfaces and structural, superconducting and magnetic properties. We then show that the $T$-linear coefficient (per CuO$_2$ plane), $A_1 \Box$, is given by the universal relation $A_1 \Box T_F = \hbar / 2e^2$, where $e$ is the electron charge, $\hbar$ is the Planck constant and $T_F$ is the Fermi temperature. This relation, obtained by assuming that the scattering rate $1/\tau$ of charge carriers reaches the Planckian limit, where $\hbar/\tau = k_B T$, works not only for hole-doped cuprates but also for electron-doped cuprates, despite the different nature of their quantum critical point and strength of their electron correlations.
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<table>
<thead>
<tr>
<th>Material</th>
<th>$n$ (10$^{27}$ m$^{-3}$)</th>
<th>$m^*$ (m$_0$)</th>
<th>$A_1 / d$ (Ω / K)</th>
<th>$h / (2e^2 T_F)$ (Ω / K)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bi2212</td>
<td>$p = 0.23$</td>
<td>6.8</td>
<td>8.4 ± 1.6</td>
<td>8.0 ± 0.9</td>
<td>7.4 ± 1.4</td>
</tr>
<tr>
<td>Bi2201</td>
<td>$p \sim 0.4$</td>
<td>3.5</td>
<td>7 ± 1.5</td>
<td>8 ± 2</td>
<td>8 ± 2</td>
</tr>
<tr>
<td>LSCO</td>
<td>$p = 0.26$</td>
<td>7.8</td>
<td>9.8 ± 1.7</td>
<td>8.2 ± 1.0</td>
<td>8.9 ± 1.8</td>
</tr>
<tr>
<td>Nd-LSCO</td>
<td>$p = 0.24$</td>
<td>7.9</td>
<td>12 ± 4</td>
<td>7.4 ± 0.8</td>
<td>10.6 ± 3.7</td>
</tr>
<tr>
<td>PCCO</td>
<td>$x = 0.17$</td>
<td>8.8</td>
<td>2.4 ± 0.1</td>
<td>1.7 ± 0.3</td>
<td>2.1 ± 0.1</td>
</tr>
<tr>
<td>LCCO</td>
<td>$x = 0.15$</td>
<td>9.0</td>
<td>3.0 ± 0.3</td>
<td>3.0 ± 0.45</td>
<td>2.6 ± 0.3</td>
</tr>
<tr>
<td>TMTSF</td>
<td>$P = 11$ kbar</td>
<td>1.4</td>
<td>1.15 ± 0.2</td>
<td>2.8 ± 0.3</td>
<td>2.8 ± 0.4</td>
</tr>
</tbody>
</table>

Slope of $T$-linear resistivity vs Planckian limit in seven materials.
Universal $T$-linear resistivity and Planckian limit in overdoped cuprates

A. Legros$^{1,2}$, S. Benhabib$^3$, W. Tabis$^{3,4}$, F. Laliberté$^1$, M. Dion$^1$, M. Lizaire$^1$,

B. Vignolle$^3$, D. Vignolles$^3$, H. Raffy$^5$, Z. Z. Li$^5$, P. Auban-Senzier$^5$,

N. Doiron-Leyraud$^1$, P. Fournier$^{1,6}$, D. Colson$^2$, L. Taillefer$^{1,6}$, and C. Proust$^{3,6}$
Anomalous Criticality in the Electrical Resistivity of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$

R. A. Cooper,$^1$ Y. Wang,$^1$ B. Vignolle,$^2$ O. J. Lipscombe,$^1$ S. M. Hayden,$^1$ Y. Tanabe,$^3$ T. Adachi,$^3$
Y. Koike,$^3$ M. Nohara,$^4$* H. Takagi,$^4$ Cyril Proust,$^2$ N. E. Hussey$^1$†
1. Metal with quasiparticles
   Random matrix model of a `quantum dot'

2. Metal without quasiparticles
   SYK model of a `quantum dot'

3. Lattice models of SYK islands
   Theory of a strange metal

4. $Z_2$ Fractionalization in a SYK $t$-$J$ model

5. SYK U(1) gauge theory
Orthogonal metals

Fractionalize the electron $c_{i\alpha}$, $\alpha = \uparrow, \downarrow$ into an “orthogonal fermion” $f_{i\alpha}$ and an Ising spin $\sigma_i^z = \pm 1$:

$$c_{i\alpha} = \sigma_i^z f_{i\alpha}$$

This introduces a $\mathbb{Z}_2$ gauge invariance

$$\sigma_i^z \rightarrow \eta_i \sigma_i^z, \quad f_{i\alpha} \rightarrow \eta_i f_{i\alpha}$$

The orthogonal fermion, $f_\alpha$, carries both the spin and charge of the electron. The Ising matter field, $\sigma^z$, is ‘dark matter’ carrying only energy, and a $\mathbb{Z}_2$ gauge charge.
Orthogonal metals

Fractionalize the electron $c_{ip\alpha}$, on sites $i = 1 \ldots N$, with spin $\alpha = 1 \ldots M$ and orbital index $p = 1 \ldots M'$ into an "orthogonal fermion" $f_{i\alpha}$ and a real scalar $\phi_{ip}$:

$$c_{ip\alpha} = \phi_{ip} f_{i\alpha}$$

This introduces a $\mathbb{Z}_2$ gauge invariance

$$\phi_{ip} \rightarrow \eta_i \phi_{ip} \ , \ f_{i\alpha} \rightarrow \eta_i f_{i\alpha}$$

The orthogonal fermion $f_{i\alpha}$ carries both the spin and charge of the electron.

The scalar field, $\phi_{ip}$, is ‘dark matter’ carrying only energy, and a $\mathbb{Z}_2$ gauge charge.
A solvable model

We examine the $t$-$J$ model:

$$\mathcal{L} = \frac{1}{2g} \sum_{i,p} \left( \partial_\tau \phi_{ip} \right)^2 + \sum_{i,\alpha} f_{i\alpha}^\dagger \left( \frac{\partial}{\partial \tau} - \mu \right) f_{i\alpha}$$

$$+ \frac{1}{\sqrt{NM}} \sum_{i,j,p,\alpha} t_{ij} \phi_{ip} \phi_{jp} f_{i\alpha}^\dagger f_{j\alpha} + \frac{1}{\sqrt{NM}} \sum_{i>j,\alpha\beta} J_{ij} f_{i\alpha}^\dagger f_{i\beta} f_{j\beta}^\dagger f_{j\alpha},$$

with the scalar field obeying the fixed length constraint

$$\sum_{p=1}^{M'} \phi_{ip}^2 = M'.$$

With $t_{ij}$ and $J_{ij}$ independent random numbers with zero mean, $\mathcal{L}$ is solvable in the limit of large number of sites, $N$, followed by the limit of large $M$ and $M'$ at fixed

$$k \equiv \frac{M'}{M}. $$
A solvable model

$$\Sigma = \begin{array}{c}
\beta \\
\alpha
\end{array}_{i} + \begin{array}{c}
\beta \\
\alpha
\end{array}_{j} + \begin{array}{c}
p \\
\alpha
\end{array}_{i} + \begin{array}{c}
p \\
\alpha
\end{array}_{j} + \begin{array}{c}
p \\
\alpha
\end{array}_{i} + \begin{array}{c}
p \\
\alpha
\end{array}_{j}
$$

$$P = \begin{array}{c}
\alpha \\
p
\end{array}_{i} + \begin{array}{c}
\alpha \\
p
\end{array}_{j} + \begin{array}{c}
p \\
p
\end{array}_{i} + \begin{array}{c}
p \\
p
\end{array}_{j}
$$

Equations for the fermion Green's function $G$ and the boson Green's function $\chi$:

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)} \quad , \quad \Sigma(\tau) = -J^2G^2(\tau)G(-\tau) + k\tilde{t}^2G(\tau)\chi^2(\tau)$$

$$\chi(i\omega_n) = \frac{1}{\omega_n^2 + \chi_0^{-1} - P(i\omega_n) + P(i\omega_n = 0)} \quad , \quad P(\tau) = -2\tilde{t}^2G(\tau)G(-\tau)\chi(\tau)$$

where $\chi_0^{-1}$ is determined by solving $\chi(\tau = 0) = 1/g$, and $\tilde{t} = tJ$. 
A solvable model

Gapped boson
\[ \Delta_f = 1/4 \]

Gapless boson
\[ \Delta_b = k/(2(k + 2)) < 1/4 \]
\[ \Delta_f = 1/(k + 2) > 1/4 \]
A solvable model

\[ \langle \phi(\tau) \phi(0) \rangle \sim \frac{e^{-m|\tau|}}{\sqrt{\tau}} \]

\[ \langle f(\tau) f^\dagger(0) \rangle \sim \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta_f}} \]

Gapless boson
\[ \Delta_b = \frac{k}{(2(k + 2))} < \frac{1}{4} \]
\[ \Delta_f = \frac{1}{(k + 2)} > \frac{1}{4} \]

Gapped boson
\[ \Delta_f = \frac{1}{4} \]
A solvable model

\[ \langle \phi(\tau)\phi(0) \rangle \sim \frac{1}{|\tau|^{2\Delta_b}} \]
\[ \langle f(\tau)f^\dagger(0) \rangle \sim \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta_f}} \]

In gapless region, we always have the Fermi liquid form for the electron Green’s function \( \langle c(\tau)c^\dagger(0) \rangle \sim 1/\tau \), although \( \mathbb{Z}_2 \) charges remain deconfined. This is a consequences of the fixed point with a non-zero \( t \) term in the \( t-J \) model.
A solvable model

Gapped boson

\[ \Delta_f = \frac{1}{4} \]

Gapless boson

‘quasi-Higgs’

\[ \begin{align*}
\Delta_b &= \frac{k}{2(k + 2)} < \frac{1}{4} \\
\Delta_f &= \frac{1}{k + 2} > \frac{1}{4}
\end{align*} \]

Toy model of pseudogap

Toy model of overdoped region

\[ k \]

\[ g \]
A solvable model

In the overdoped region, Fermi liquid electron spectral function, with anomalies in other properties, match recent observations in cuprates (Hussey, Bozovic, Armitage, Taillefer...)

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$YBa_2Cu_3O_{6+x}$
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5. SYK $U(1)$ gauge theory
Fermions with random hopping coupled to a fluctuating U(1) gauge field

\[ H = -\frac{1}{(MN)^{1/2}} \sum_{ij=1}^{N} \sum_{\alpha=1}^{M} \left[ t_{\alpha\beta} e^{iA_{ij}} f_{i\alpha}^\dagger f_{j\beta} + (MN)^{1/2} \mu \delta_{ij} f_{i\alpha}^\dagger f_{i\alpha} \right] \]

\[ \ll t_{ij}^\alpha t_{ji}^\beta \gg = \ll |t_{ij}^\alpha|^2 \gg = t^2, \quad A_{ji} = -A_{ij}. \]
Fermions with random hopping coupled to a fluctuating $U(1)$ gauge field

\[
\Sigma(i\omega_n) = t^2 G(i\omega_n) + t^2 T \sum_{\Omega_m \neq 0} \frac{G(i\omega_n + i\Omega_m) - G(i\omega_n)}{\Pi(i\Omega_m) - \Pi(i\Omega_m = 0)},
\]

\[
\Pi(i\Omega_m) = t^2 T \frac{M}{N} \sum_{\omega_n} G(i\omega_n) G(i\omega_n + i\Omega_m), \quad G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}.
\]
Fermions with random hopping coupled to a fluctuating U(1) gauge field

\[ \Sigma(i\omega_n) = t^2 G(i\omega_n) + t^2 T \sum_{\Omega_m \neq 0} \frac{G(i\omega_n + i\Omega_m) - G(i\omega_n)}{\Pi(i\Omega_m) - \Pi(i\Omega_m = 0)}, \]

\[ \Pi(i\Omega_m) = t^2 T \frac{M}{N} \sum_{\omega_n} G(i\omega_n)G(i\omega_n + i\Omega_m), \quad G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}. \]

General low energy solution

\[ G(\tau > 0) = -\frac{C(\mathcal{E})}{t^{1-x}\tau^{1-x}}, \quad G(\tau < 0) = \frac{C(\mathcal{E})e^{-2\pi\mathcal{E}}}{t^{1-x}|\tau|^{1-x}}. \]

where \( \mathcal{E} \) is a parameter universally related to the filling fraction (\( \mathcal{E} = 0 \) at half-filling). The exponent \( x \) is the solution to

\[ \frac{(1/x - 2)(\cosh(2\pi\mathcal{E}) - \cos(\pi x))}{\tan(\pi x)\sin(\pi x)} = \frac{M}{N}. \]
Fermions with random hopping coupled to a fluctuating U(1) gauge field

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1. Non-Fermi liquid regime of SYK models with linear-in-$T$ resistivity

2. $\mathbb{Z}_2$ Fractionalization in a SYK $t$-$J$ model
   Overdoped state described by state with fractionalization but with a Fermi liquid electron spectral function

3. SYK $U(1)$ gauge theory