Fermi surfaces and gauge-gravity duality

Aspen Center for Physics, Feb 1-5, 2011

Lecture notes
arXiv:1010.0682
arXiv: 1012.0299

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Compressible quantum matter

- Consider a continuum quantum system with a globally conserved \( U(1) \) charge \( Q \) (the “electron density”) in spatial dimension \( d > 1 \).
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There are only a few established examples of such phases in condensed matter physics. However, they appear naturally as duals of gravitational theories, and we want to interpret them in the gauge theory.
The Fermi surface

All known examples of such phases have a *Fermi surface*: the locus of zero energy singularities in momentum space in the two-point correlator of fermions carrying charge $Q$.

$$G_{\text{fermion}}^{-1}(k = k_F, \omega = 0) = 0.$$  

**Luttinger relations:** The “volume (area)” $A$ enclosed by the Fermi surface has a simple linear relation to $\langle Q \rangle$. 

Area $A$
The Fermi Liquid (FL)

Most common example: electrons with short-range interactions (or screened long-range interactions), which are adiabatically connected to the non-interacting limit. The electron Green’s function $G_c$ has a pole which crosses zero energy at $k = k_F$, and the Fermi surface has the same area as the non-interacting case.

$$L = \bar{c} \left( \partial_a - \mu \delta_{at} \right) \gamma^a c + 4 \text{ Fermi terms}$$

$$A = \langle \bar{c} \gamma^t c \rangle = \langle Q \rangle$$

$$G_c = \frac{1}{\omega - v_F (k - k_F) + i\omega^2}$$
Outline

1. Fermion-boson mixtures
   *Luttinger relations with bosons*

2. Gauge theories
   *The fractionalized Fermi liquid (FL*) phase*

3. Connections to semi-holographic theories
   *Low energy theory of FL and FL* phases*

4. Solvable models with infinite-range hopping
   *Mapping to gravity duals on AdS$_2 \times \mathbb{R}^d$*
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Introduce a second “flavor” of electrons, $f$, and allow $c$ to decay into $f$ and a boson $b$

\[
\mathcal{L} = \bar{c} (\partial_a - \mu \delta_{at}) \gamma^a c + \bar{f} (\partial_a - (\mu + \mu_g) \delta_{at}) \gamma^a f + |(\partial_a + \mu_g \delta_{at}) b|^2 + s |b|^2 + \lambda (\bar{c} f b + \text{c.c.}) + \ldots
\]

Two conserved charges:

\[
Q = \bar{c} \gamma^t c + \bar{f} \gamma^t f
\]
\[
Q_g = \bar{f} \gamma^t f - \bar{b} \partial_t b
\]

Friday, February 4, 2011
There is a Luttinger relation for each conserved U(1) charge. However, the boson, $b$ cannot have a Fermi surface in its Green’s function, and so there is no area associated with it, although the boson density is *included* in the Luttinger relation

\[
A_c + A_f = \langle \bar{c} \gamma^t c \rangle + \langle \bar{f} \gamma^t f \rangle = \langle Q \rangle \\
A_f = \langle \bar{f} \gamma^t f \rangle - \langle \bar{b} \partial_t b \rangle = \langle Q_g \rangle
\]

The $b$ bosons have bound with $f$ fermions to form $c$ “molecules”

If $U_g(1)$ is spontaneously broken by condensation of $b$, only the Luttinger relation associated with $U(1)$ applies. Also the $c$ and $f$ flavors can now mix ("hybridize"), and so form only a single Fermi surface when the mixing is large enough.

\[ A = \langle \bar{c} \gamma^t c \rangle + \langle \bar{f} \gamma^t f \rangle = \langle Q \rangle \]
Phase diagram of boson-fermion mixture

\[ \mathcal{A} = \langle Q \rangle \]

Superfluid: \( \langle b \rangle \neq 0 \)
\( \text{U}_g(1) \) broken
\( \text{U}(1) \) unbroken

\[ \mathcal{A}_c = \langle Q - Q_g \rangle \]

Normal: \( \langle b \rangle = 0 \)
\( \text{U}(1) \times \text{U}_g(1) \) unbroken

\[ \mathcal{A}_f = \langle Q_g \rangle \]

\[ \mathcal{L} = \bar{c} (\partial_a - \mu \delta_{at}) \gamma^a c + \bar{f} (\partial_a - (\mu + \mu_g) \delta_{at}) \gamma^a f \]
\[ + |(\partial_a + \mu_g \delta_{at})b|^2 + s|b|^2 + \lambda(\bar{c}f b + \text{c.c.}) + \ldots \]

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   Mapping to gravity duals on $AdS_2 \times R^d$
• Now gauge $U_g(1)$ by a dynamical (“emergent”) gauge field $A_a$. 

$$\mathcal{L} = \bar{c} \left( \partial_a - \mu \delta_{at} \right) \gamma^a c + \bar{f} \left( \partial_a - (\mu + \mu_g) \delta_{at} \right) \gamma^a f$$

$$+ \left| \left( \partial_a + \mu_g \delta_{at} \right) b \right|^2 + s \left| b \right|^2 + \lambda (\bar{c} f b + \text{c.c.}) + \ldots$$
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\[ \mathcal{L} = \bar{c} (\partial_a - \mu \delta_{at}) \gamma^a c + \bar{f} (\partial_a - i A_a + (\mu + \mu_g) \delta_{at}) \gamma^a f \\
+ |(\partial_a + i A_a + \mu_g \delta_{at})b|^2 + s|b|^2 + \lambda (\bar{c} f b + c.c.) + \ldots \]
• Now gauge $U_g(1)$ by a dynamical (“emergent”) gauge field $A_a$.

• Longitudinal gauge fluctuations are screened by the $f$ fermions. But transverse gauge fluctuations remain unscreened, and are Landau-damped by excitations near the $f$ Fermi surface. The theory of a Fermi surface coupled to transverse gauge fluctuations is strongly coupled in two spatial dimensions.

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• The overdamped transverse gauge modes lead to "non-Fermi liquid" broadening of the fermion pole near the $f$ Fermi surface.

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• The overdamped transverse gauge modes lead to “non-Fermi liquid” broadening of the fermion pole near the $f$ Fermi surface.

• The gauge-neutral $c$ fermion also acquires a weaker non-Fermi liquid broadening.

$$\mathcal{L} = \overline{c} \left( \partial_a - \mu \delta_{at} \right) \gamma^a c + \overline{f} \left( \partial_a - iA_a + (\mu + \mu_g) \delta_{at} \right) \gamma^a f$$
$$\quad + \left| \left( \partial_a + iA_a + \mu_g \delta_{at} \right) b \right|^2 + s \left| b \right|^2 + \lambda (\overline{c} f b + \text{c.c.}) + \ldots$$
• However, the locations of the Fermi surfaces are well defined, and the Luttinger relations apply as before.

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Phase diagram of $U_g(1)$ gauge theory

Higgs/confining phase: Fermi liquid (FL)

Deconfined phase: Fractionalized Fermi liquid (FL*)

$\mathcal{L} = \bar{c} (\partial_a - \mu \delta_{at}) \gamma^a c + \bar{f} (\partial_a - iA_a + (\mu + \mu_g) \delta_{at}) \gamma^a f$

$+ |(\partial_a + iA_a + \mu_g \delta_{at})b|^2 + s|b|^2 + \lambda(\bar{c}f b + c.c.) + \ldots$

Phase diagram of gauge theories with global U(1) charge density

The same phase diagram applies when $U_g(1)$ is generalized to non-Abelian or discrete gauge groups. It also applies when the $f$ Fermi surface is gapped by the lattice potential, or by pairing of the $f$ fermions.

Phase diagram of gauge theories with global U(1) charge density

Deconfined phase: Fractionalized Fermi liquid (FL*)

Needed: AdS/CFT description of these phases and of the quantum phase transition between them

Higgs/confining phase: Fermi liquid (FL)

\[ \mathcal{L} = \bar{c} (\partial_a - \mu \delta_{at}) \gamma^a c + \bar{f} (\partial_a - iA_a + (\mu + \mu_g) \delta_{at}) \gamma^a f 
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Low energy theory of FL* phase

Integrate out gapped $b$ bosons, and write theory of $c$ electrons as

$$\mathcal{L}_c = \bar{c} (\partial_a - \mu \delta_{at}) c + \lambda (\bar{c} F + \bar{F} c)$$

- The $c$ are the observable gauge-invariant fermions ("probe UV fermions").
- The $F$ are composite gauge-invariant operators ("IR fermions"), carrying global U(1) charge, which probe the strongly-coupled physics of the $f$ Fermi surface coupled to a fluctuating gauge field

$$F \sim (\bar{f} f) c$$

Note this also involves a contribution from the "probe sector" (the factor of $c$), so the UV and IR are not completely decoupled.
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Now $b$ is condensed; so write $b = e^{-i\vartheta}$ and integrate out the $f$ fermions. Then the $c$ electrons ("UV fermions") obey

$$\mathcal{L}_c = \bar{c}(\partial_a - \mu \delta_{at} - \Sigma_f - i A_{a}^{\text{ext}})c + \Pi_f(\tilde{A}_a) + (\partial_a \vartheta - \tilde{A}_a + A_{a}^{\text{ext}})^2$$

Here $A_{a}^{\text{ext}}$ is an external source coupling to the global $U(1)$ charge, $\tilde{A}_a = A_a + A_{a}^{\text{ext}}$, and $\Sigma_f$ and $\Pi_f$ are the self-energy and effective action generated by integrating out the $f$. 
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This structure is similar to the semi-holographic theory of Nickel and Son (arXiv:1009.3094).
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Solution of lattice models

Place theory on lattice, integrate out $b$ and $A_\alpha$, to obtain Kondo lattice Hamiltonian

$$H = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j} J_H(i,j) \vec{S}_i \cdot \vec{S}_j + J_K \sum_{i} \vec{S}_i \cdot c_{i\alpha}^\dagger \vec{\sigma}_{\alpha \beta} c_{i\beta}$$

where $\vec{S}_i = f_{i\alpha}^\dagger \vec{\sigma}_{\alpha \beta} f_{i\beta}$

Fully-connected cluster

Bethe lattice for large coordination

Solution of lattice models

\[ \mathcal{L} = \mathcal{L}_{\text{imp}} [c_0, f_0] + c_0^\dagger F_{\text{bulk}} + F_{\text{bulk}}^\dagger c_0 + \mathcal{L}_{\text{bulk}} \]

Has to be combined with a *self-consistency condition* between correlators on the impurity and the bulk.

Obtain both FL and FL* phases;

**properties of the FL* phase:**

- The ground state has a non-zero entropy density
- The correlations of the bulk are local ($z = \infty$)
- The correlations in time have a conformal structure with scaling dimension $\Delta$ (as in the boundary of $\text{AdS}_2$)
- Imposition of the self-consistency condition between impurity and boundary yields the scaling dimension $\Delta = 1$, the 'marginal Fermi liquid' value.

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These features, and the resulting fermion correlator and transport properties, coincide with those obtained (for general $\Delta$) using the holographic $\text{AdS}_2 \times \mathbb{R}^d$ theory defined on the extremal horizon of the Reissner-Nordstrom black hole (T. Faulkner, H. Liu, J. McGreevy and D. Vegh, arXiv:0907.2694)


- The correlations are local ($z = \infty$).

- The correlations in time have a conformal structure with scaling dimension $\Delta$ (as in the boundary of $\text{AdS}_2$).

- Imposition of the self-consistency condition between impurity and boundary yields the scaling dimension $\Delta = 1$, the ‘marginal Fermi liquid’ value.


Compressible quantum matter is characterized by Fermi surfaces.

Fermi surfaces can be removed from the Luttinger count if the fermions acquire gauge charges.

Phases of a strongly-coupled gauge theory contains Fermi liquid (FL) and fractionalized Fermi liquid (FL*) phases: attractive candidate for a dual gravity theory.
Conclusions

Mean field Kondo lattice models capture the physics of holographic metals with a AdS$_2 \times \mathbb{R}^d$ geometry

Needed: Holographic theory without a factorized geometry, in which the probe fermions and the strongly-coupled IR refer to the same degrees of freedom.