\[ \langle S_j \rangle = N_1 \cos(K \cdot r_j) + N_2 \sin(K \cdot r_j) \]

Collinear spins: \( N_1 \times N_2 = 0 \)
Non-collinear spins: \( N_1 \times N_2 \neq 0 \)

States on both sides of critical point could be either (A) Insulators (B) Metals (C) Superconductors
SDWs in Mott insulators

La$_2$CuO$_4$

$\vec{K} = (\pi, \pi)$
Collinear spins

$\vec{K} = \left( \frac{4\pi}{3}, \frac{4\pi}{\sqrt{3}} \right)$
Non-collinear spins

“Disorder” the spins by enhancing quantum fluctuations in a variety of ways.....
Outline

A. “Dimerized” Mott insulators
   *Landau-Ginzburg-Wilson (LGW) theory.*

B. Kondo lattice models
   “Large” Fermi surfaces and the LGW SDW paramagnon theory.

C. Fractionalized Fermi liquids
   *Spin liquids and Fermi volume changing transitions with a topological order parameter.*

D. Deconfined quantum criticality
   *Berry phases and the transition from SDW to bond order.* (Talks by T. Senthil (N20.008) and L. Balents (N20.009))
(A) Magnetic quantum phase transitions in “dimerized” Mott insulators

Landau-Ginzburg-Wilson (LGW) theory:

Second-order phase transitions described by fluctuations of an order parameter associated with a broken symmetry
**Coupled Dimer Antiferromagnet**

\[ H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

\[ 0 \leq \lambda \leq 1 \]

$\lambda$ close to 0

Weakly coupled dimers
\( \lambda \) close to 0

Weakly coupled dimers

Paramagnetic ground state

\[ \langle \vec{S}_i \rangle = 0, \langle \vec{\phi} \rangle = 0 \]
\( \lambda \) close to 0

Weakly coupled dimers

\[
\begin{array}{c}
\text{Excitation: } S=1 \text{ triplon}
\end{array}
\]
$\lambda$ close to 0

Weakly coupled dimers

\[ \begin{align*}
\downarrow \uparrow - \uparrow \downarrow &= \frac{1}{\sqrt{2}} \left( \left| \uparrow \downarrow \right\rangle - \left| \downarrow \uparrow \right\rangle \right) 
\end{align*} \]

Excitation: $S=1$ triplon
$\lambda$ close to 0  

Weakly coupled dimers

\[
\text{Excitation: } S=1 \text{ triplon}
\]
\[ \lambda \text{ close to 0} \]

Weakly coupled dimers

\[ \begin{align*}
\downarrow \uparrow - \uparrow \downarrow &= 2 \\
\end{align*} \]

Excitation: \( S=1 \) triplon

\[ \begin{align*}
\text{state} &= \frac{1}{\sqrt{2}} \left( \left| \uparrow \downarrow \right> - \left| \downarrow \uparrow \right> \right) \\
\end{align*} \]
\( \lambda \) close to 0

Weakly coupled dimers

\[ \frac{1}{\sqrt{2}} \left( \left\lvert \uparrow \downarrow \right\rangle - \left\lvert \downarrow \uparrow \right\rangle \right) \]

Excitation: \( S=1 \) triplon
$\lambda$ close to 0

Weakly coupled dimers

Excitation: $S=1$ triplon (exciton, spin collective mode)

Energy dispersion away from antiferromagnetic wavevector

$$\varepsilon_p = \Delta + \frac{c_x^2 p_x^2 + c_y^2 p_y^2}{2\Delta}$$

$\Delta \rightarrow$ spin gap
Coupled Dimer Antiferromagnet

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

\[ 0 \leq \lambda \leq 1 \]

$\lambda$ close to 1

Weakly dimerized square lattice
Excitations:
2 spin waves (magnons)

$$\varepsilon_p = \sqrt{c_x^2 p_x^2 + c_y^2 p_y^2}$$

Ground state has long-range spin density wave (Néel) order at wavevector $K = (\pi, \pi)$

spin density wave order parameter: $\bar{\phi} = \eta_i \frac{\vec{S}_i}{S}$; $\eta_i = \pm 1$ on two sublattices
\[ \lambda_c = 0.52337(3) \]
M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama,

LGW theory for quantum criticality

Landau-Ginzburg-Wilson theory: write down an effective action for the antiferromagnetic order parameter $\bar{\phi}$ by expanding in powers of $\bar{\phi}$ and its spatial and temporal derivatives, while preserving all symmetries of the microscopic Hamiltonian.

$$S_{\phi} = \int d^2x d\tau \left[ \frac{1}{2} \left( (\nabla_x \bar{\phi})^2 + c^2 (\partial_\tau \bar{\phi})^2 + (\lambda_c - \lambda) \bar{\phi}^2 \right) + \frac{u}{4!} (\bar{\phi}^2)^2 \right]$$


For $\lambda < \lambda_c$ oscillations of $\bar{\phi}$ about $\bar{\phi} = 0$ lead to the following structure in the dynamic structure factor $S(p, \omega)$

$$\varepsilon(p) = \Delta + \frac{c^2 p^2}{2\Delta} ; \quad \Delta = \sqrt{\lambda_c - \lambda} / c$$

(B) Kondo lattice models

“Large” Fermi surfaces and the Landau-Ginzburg-Wilson spin-density-wave paramagnon theory
Kondo lattice

\[ H_K = \sum_{i<j} t_{ij} c_i^{\dagger} c_j + J_K \sum_i c_i^{\dagger} \vec{\tau}_{\sigma\sigma} c_i \cdot \vec{S}_{fi} + J \sum_{\langle ij \rangle} \vec{S}_{fi} \cdot \vec{S}_{fj} \]

At large \( J_K \), magnetic order is destroyed, and we obtain a non-magnetic Fermi liquid (FL) ground state

Luttinger’s Fermi volume on a $d$-dimensional lattice for the FL phase

Let $\nu_0$ be the volume of the unit cell of the ground state, $n_T$ be the total number density of electrons per volume $\nu_0$. (need not be an integer)

$$n_T = n_f + n_c = 1 + n_c$$

$$2 \times \frac{\nu_0}{(2\pi)^d}(\text{Volume enclosed by Fermi surface}) = n_T \pmod{2}$$

A “large” Fermi surface
Argument for the Fermi surface volume of the FL phase

Single ion Kondo effect implies $J_K \rightarrow \infty$ at low energies

Fermi liquid of $S=1/2$ holes with hard-core repulsion

Fermi surface volume $= -\left( \text{density of holes} \right) \mod 2$

$= -(1 - n_c) = (1 + n_c) \mod 2$
Doniach’s $T=0$ phase diagram for the Kondo lattice

Local moments choose some static spin arrangement

“Heavy” Fermi liquid with moments Kondo screened by conduction electrons. Fermi surface volume equals the Luttinger value.

SDW

$J_{Kc}$

FL

$J_K$
LGW theory for quantum critical point

Write down effective action for SDW order parameter $\bar{\phi}$

$S_\phi = \int \frac{d^d q d\omega}{(2\pi)^{d+1}} |\bar{\phi}(q, \omega)|^2 \left( q^2 + |\omega| + (J_K - J_{Kc}) \right) + \frac{u}{4} \int d^d r d\tau \left( \bar{\phi}^2 \right)^2$

$\bar{\phi}$ fluctuations are damped by mixing with fermionic quasiparticles near the Fermi surface

Fluctuations of $\bar{\phi}$ about $\bar{\phi} = 0 \Rightarrow$ the triplon is now a paramagnon

(C) Fractionalized Fermi liquids (FL*)

Spin liquids and Fermi volume changing transitions with a topological order parameter

Beyond LGW: quantum phases and phase transitions with emergent gauge excitations and fractionalization
Work in the regime with small $J_K$, and consider destruction of magnetic order by frustrating (RKKY) exchange interactions between $f$ moments.

Ground state has Neel order with $\tilde{\varphi} \neq 0$.
Work in the regime with small $J_K$, and consider destruction of magnetic order by frustrating (RKKY) exchange interactions between $f$ moments.

Destroy SDW order by perturbations which preserve full square lattice symmetry e.g. second-neighbor or ring exchange.
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Possible paramagnetic ground state with $\langle \bar{\phi} \rangle = 0$
Work in the regime with small $J_K$, and consider destruction of magnetic order by frustrating (RKKY) exchange interactions between $f$ moments.

Possible paramagnetic ground state with $\langle \bar{\phi} \rangle = 0$

Such a state breaks lattice symmetry and has $\langle \Psi_{\text{bond}} \rangle \neq 0$, where $\Psi_{\text{bond}}$ is the bond order parameter.
Work in the regime with small $J_K$, and consider destruction of magnetic order by frustrating (RKKY) exchange interactions between $f$ moments.

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Possible paramagnetic ground state with $\langle \bar{\phi} \rangle = 0$

Such a state breaks lattice symmetry and has $\langle \Psi_{\text{bond}} \rangle \neq 0$, where $\Psi_{\text{bond}}$ is the bond order parameter.

Bond order (and confinement) appear for collinear spins in $d=2$. 
Work in the regime with small $J_K$, and consider destruction of magnetic order by frustrating (RKKY) exchange interactions between $f$-moments

A *spin liquid* ground state with $\langle \bar{\phi} \rangle = 0$ and $\langle \Psi_{\text{bond}} \rangle = 0$

Excitations of the paramagnet with non-zero spin

\[ \langle \Psi_{\text{bond}} \rangle \neq 0 \]
Excitations of the paramagnet with non-zero spin

\[ \langle \Psi_{\text{bond}} \rangle \neq 0 \]
Excitations of the paramagnet with non-zero spin

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Excitations of the paramagnet with non-zero spin

\[ \langle \Psi_{\text{bond}} \rangle \neq 0 \]

\( S = 1 \) spinons \( f_\sigma \) are confined into a \( S = 1 \) triplon \( \bar{\varphi} \) by a confining compact U(1) gauge force

N. Read and S. Sachdev,

Excitations of the paramagnet with non-zero spin

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\( S = 1 \) spinons \( f_\sigma \) are confined into a \( S = 1 \) triplon \( \bar{\phi} \) by a confining compact U(1) gauge force


\( S = 1/2 \) spinons \( f_\sigma \) are deconfined and interact with \( Z_2 \) (non-collinear spins, \( d = 2,3 \)) or U(1) (collinear spins, \( d = 3 \)) gauge forces

Influence of conduction electrons + Local moments $f_{\sigma}$

$$H = \sum_{i<j} t_{ij} c_{i \sigma}^\dagger c_{j \sigma} + \sum_i \left( J_K c_{i \sigma}^\dagger \vec{\tau}_{\sigma \sigma'} c_{i \sigma'} \cdot \vec{S}_{fi} \right) + \sum_{i<j} J_H (i, j) \vec{S}_{fi} \cdot \vec{S}_{fj}$$

Determine the ground state of the quantum antiferromagnet defined by $J_H$, and then couple to conduction electrons by $J_K$

Choose $J_H$ so that ground state of antiferromagnet is a $Z_2$ or U(1) spin liquid
Influence of conduction electrons

Local moments $f_\sigma$

At $J_K = 0$ the conduction electrons form a Fermi surface on their own with volume determined by $n_c$.

Perturbation theory in $J_K$ is regular, and so this state will be stable for finite $J_K$.

So volume of Fermi surface is determined by $(n_T - 1) = n_c \, (\text{mod } 2)$, and does not equal the Luttinger value.

The (U(1) or $Z_2$) FL* state
A new phase: FL*

This phase preserves spin rotation invariance, and has a Fermi surface of *sharp* electron-like quasiparticles.

The state has "topological order" and associated neutral excitations. The topological order can be detected by the violation of Luttinger’s Fermi surface volume. It can only appear in dimensions $d > 1$

$$2 \times \frac{v_0}{(2\pi)^d} \left( \text{Volume enclosed by Fermi surface} \right)$$

$$= (n_T - 1) \left( \text{mod } 2 \right)$$

Phase diagram \((U(1), d=3)\)
Phase diagram \((U(1), d=3)\)

Fractionalized Fermi liquid with moments paired in a spin liquid. Fermi surface volume does not include moments and is unequal to the Luttinger value.

\(U(1) \text{ FL}^*\)

\(J_{Kc}\)

\(J_K\)
Phase diagram $(U(1), d=3)$

Fractionalized Fermi liquid with moments paired in a spin liquid. Fermi surface volume does not include moments and is unequal to the Luttinger value.

“Heavy” Fermi liquid with moments Kondo screened by conduction electrons. Fermi surface volume equals the Luttinger value.

$U(1) \text{ FL}^*$

$\text{FL}$
Phase diagram \((U(1), d=3)\)

Fractionalized Fermi liquid with moments paired in a spin liquid. Fermi surface volume does not include moments and is unequal to the Luttinger value.

“Heavy” Fermi liquid with moments Kondo screened by conduction electrons. Fermi surface volume equals the Luttinger value.

Sharp transition at \(T=0\) in \(d=3\) compact \(U(1)\) gauge theory; compactness “irrelevant” at critical point.
Phase diagram \((U(1), d=3)\)

No transition for \(T>0\) in \(d=3\) compact U(1) gauge theory; compactness essential for this feature.

Quantum Critical

Sharp transition at \(T=0\) in \(d=3\) compact U(1) gauge theory; compactness “irrelevant” at critical point.
Phase diagram \((\text{U}(1), d=3)\)

- Specific heat \(\sim T \ln T\)
- Violation of Wiedemann-Franz law

Quantum Critical

\(\text{U}(1) \text{ FL}^*\)

\(J_K - J_{Kc}\)
Phase diagram \((U(1), d=3)\)

\[
\text{Resistivity } \sim \frac{1}{\ln(1/T)}
\]

Quantum Critical

\(U(1) \text{ FL}^*\)

\(J_Kc\)

\(J_K\)
Phase diagram \((\mathbf{U}(1), d=3)\)
(D) Deconfined quantum criticality

Berry phases, bond order, and the breakdown of the LGW paradigm

All phases have conventional order, but gauge excitations and fractionalization emerge at the quantum critical point.

Talks by T. Senthil (N20.008) and L. Balents (N20.009)
Mott insulator with one $S=1/2$ spin per unit cell

Ground state has Neel order with $\langle \tilde{\phi} \rangle \neq 0$
Mott insulator with one $S=1/2$ spin per unit cell

Destroy Neel order by perturbations which preserve full square lattice symmetry e.g. second-neighbor or ring exchange. The strength of this perturbation is measured by a coupling $g$.

Small $g$ $\Rightarrow$ ground state has Neel order with $\langle \bar{\phi} \rangle \neq 0$

Large $g$ $\Rightarrow$ paramagnetic ground state with $\langle \bar{\phi} \rangle = 0$, $\langle \Psi_{\text{bond}} \rangle \neq 0$
Quantum theory for destruction of Neel order

**Ingredient missing from LGW theory:**
Spin Berry Phases

\[ e^{iS_A} \]
Quantum theory for destruction of Neel order

Ingredient missing from LGW theory:
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Quantum theory for destruction of Neel order

**Ingredient missing from LGW theory:**

**Spin Berry Phases**

\[ e^{iSA} \]
Quantum theory for destruction of Neel order
Quantum theory for destruction of Neel order

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points $a$
Quantum theory for destruction of Neel order

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points \( a \)

Recall \( \vec{\phi}_a = 2\eta_a \vec{S}_a \rightarrow \vec{\phi}_a = (0,0,1) \) in classical Neel state;

\( \eta_a \rightarrow \pm 1 \) on two square sublattices;

\[
(\mu = x, y, \tau)
\]
Quantum theory for destruction of Neel order

Discretize imaginary time: path integral is over fields on the sites of a cubic lattice of points \( a \)

Recall \( \vec{\phi}_a = 2\eta_a \vec{S}_a \rightarrow \vec{\phi}_a = (0,0,1) \) in classical Neel state;

\( \eta_a \rightarrow \pm 1 \) on two square sublattices ;

\( A_{a\mu} \rightarrow \text{half oriented area of spherical triangle} \)

formed by \( \vec{\phi}_a, \vec{\phi}_{a+\mu}, \) and an arbitrary reference point \( \vec{\phi}_0 \)

\( A_{a\mu} \) transforms like a compact U(1) gauge field

Quantum theory for destruction of Neel order

Ingredient missing from LGW theory:
Spin Berry Phases

\[ \exp \left( i \sum_a \eta_a A_{a\tau} \right) \]

Sum of Berry phases of all spins on the square lattice.
Quantum theory for destruction of Neel order

Partition function on cubic lattice

\[ Z = \prod_a \int d\vec{\phi}_a \delta(\vec{\phi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \vec{\phi}_a \cdot \vec{\phi}_{a+\mu} + i \sum_a \eta_a A_{a\tau} \right) \]

Modulus of weights in partition function: those of a classical ferromagnet at a “temperature” \( g \)

Small \( g \) ⇒ ground state has Neel order with \( \langle \vec{\phi} \rangle \neq 0 \)

Large \( g \) ⇒ paramagnetic ground state with \( \langle \vec{\phi} \rangle = 0 \)

Berry phases lead to large cancellations between different time histories

\[
Z = \prod_{a} \int d\bar{\phi}_a \delta(\bar{\phi}_a^2 - 1) \exp \left( \frac{1}{g} \sum_{a,\mu} \bar{\phi}_a \cdot \bar{\phi}_{a+\mu} + i \sum_{a} \eta_a A_{a\tau} \right)
\]

Neel order
\[\langle \bar{\phi} \rangle \neq 0\]

Bond order
\[\langle \Psi_{\text{bond}} \rangle \neq 0\]
Not present in LGW theory of \(\bar{\phi}\) order
Alternative formulation to describe transition: Express theory in terms of a complex spinor $z_{a\alpha}$, $\alpha = \uparrow, \downarrow$, with

$$\bar{\varphi}_a = z_{a\alpha}^* \bar{\sigma}_{\alpha\beta} z_{a\beta}$$

$$Z = \prod_a \int dz_{a\alpha} dA_{a\mu} \delta \left( |z_{a\alpha}|^2 - 1 \right)$$

$$\exp \left( \frac{1}{g} \sum_{a,\mu} z_{a\alpha}^* e^{iA_{a\mu}} z_{a+\mu,\alpha} + \text{c.c.} + i \sum_a \eta_a A_{a\tau} \right)$$

Theory of a second-order quantum phase transition between Neel and bond-ordered phases

At the quantum critical point:

- $A_\mu \rightarrow A_\mu + 2\pi$ periodicity can be ignored (Monopoles interfere destructively and are dangerously irrelevant).
- $S=1/2$ spinons $z_\alpha$, with $\bar{\phi} \sim z_\alpha^* \bar{\sigma}_{\alpha\beta} z_\beta$, are globally propagating degrees of freedom.

Second-order critical point described by emergent fractionalized degrees of freedom ($A_\mu$ and $z_\alpha$);
Order parameters ($\phi$ and $\Psi_{\text{bond}}$) are “composites” and of secondary importance


Phase diagram of $S=1/2$ square lattice antiferromagnet

Neel order

$\langle \vec{\phi} \rangle \sim \langle z^*_\alpha \vec{\sigma}_{\alpha\beta} z_{\beta} \rangle \neq 0$

Bond order $\langle \Psi_{\text{bond}} \rangle \neq 0$

(associated with condensation of monopoles in $A_\mu$),

$S = 1/2$ spinons $z_\alpha$ confined,

$S = 1$ triplon excitations

Second-order critical point described by

$$S_{\text{critical}} = \int d^2 x d\tau \left[ |(\partial_\mu - i A_\mu) z_\alpha|^2 + r |z_\alpha|^2 + \frac{u}{2} (|z_\alpha|^2)^2 + \frac{1}{4e^2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \right]$$

at its critical point $r = r_c$, where $A_\mu$ is non-compact

Conclusions

I. New FL* phase with a Fermi surface of electron-like quasiparticles (whose volume violates the Luttinger theorem), topological order, emergent gauge excitations, and neutral fractionalized quasiparticles.

Novel quantum criticality in the transition between the FL and FL* phases (and associated SDW and SDW* phases)
Conclusions

II. Theory of quantum phase transitions between magnetically ordered and paramagnetic states of Mott insulators:


B. S=1/2 square lattice: Berry phases induce bond order, and LGW theory breaks down. Critical theory is expressed in terms of emergent fractionalized modes, and the order parameters are secondary.
Conclusions

III. Deconfined quantum criticality in conducting systems?

Theory for FL-FL* transition could also apply to the FL-SDW transition between conventional phases.