A simple model of many-particle entanglement: how it describes black holes and superconductors

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Subir Sachdev

Talk online: sachdev.physics.harvard.edu
A remarkable connections has emerged in modern physics between 
(A) the quantum theory of many strongly interacting particles (e.g. electrons in a crystal) 
and 
(B) the quantum theory of black holes.
A remarkable connections has emerged in modern physics between (A) the quantum theory of many strongly interacting particles (e.g. electrons in a crystal) and (B) the quantum theory of black holes.

Among the many remarkable features: gravity is completely negligible in (A), while gravitational forces are extremely strong in (B).
I will illustrate this connection using a simple example: the Sachdev-Ye-Kitaev (SYK) model which describes certain quantum many particle systems and certain black holes.
Quantum entanglement

Black holes
Black Holes

Objects so dense that light is gravitationally bound to them.

In Einstein’s theory, the region inside the black hole horizon is disconnected from the rest of the universe.

Horizon radius $R = \frac{2GM}{c^2}$

$G$ Newton’s constant, $c$ velocity of light, $M$ mass of black hole

For $M = \text{earth’s mass}$, $R \approx 9 \text{ mm}$!
Quantum Entanglement across a black hole horizon
Quantum Entanglement across a black hole horizon
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Quantum Entanglement across a black hole horizon

Hawking used this to show that black hole horizons have an entropy and a temperature (because to an outside observer, the state of the electron inside the black hole is an unknown)
• Black holes have an entropy and a temperature, $T_H$.
• The entropy, $S_{BH}$ is proportional to their surface area.

Bekenstein (1973)
Hawking (1974)
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All many-body quantum systems (without quantum gravity) have an entropy proportional to their volume! ?!? 

Bekenstein (1973)
Hawking (1974)
Thermodynamics of quantum black holes:

\[ \int \mathcal{D}g_{\mu\nu} \exp \left( -\frac{1}{\hbar} \mathcal{S}_{\text{Einstein gravity}}^{(d+1)}[g_{\mu\nu}] \right) \]

Quantum gravity: a summation over all possible configurations of spacetime, each weighted by a factor which is the exponential of (the ‘action’ of Einstein gravity)/(Planck’s constant)
Thermodynamics of quantum black holes:

$$\int \mathcal{D}g_{\mu\nu} \exp \left( -\frac{1}{\hbar} S_{\text{Einstein gravity}}^{(d+1)} \right)$$

In general, this summation is not well defined, because to the uncontrollably large number of spacetime configurations.
Thermodynamics of quantum black holes:

\[ \int \mathcal{D}g_{\mu\nu} \exp \left( -\frac{1}{\hbar} S_{\text{Einstein gravity}}^{(d+1)}(g_{\mu\nu}) \right) = \exp(S_{BH}) \times \left( \ldots \right) \]

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Gibbons, Hawking (1977)
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\[ = \exp(S_{BH}) \times \left( \ldots ???? \ldots \right) \]

Gibbons, Hawking (1977)

In general, this summation is not well defined, because to the uncontrollably large number of spacetime configurations. But in recent years, we have learnt how to evaluate the summation for Einstein gravity in some cases, and obtain the exact value of \ldots ???? \ldots thanks to connections to the Sachdev-Ye-Kitaev model.
Holography and duality

Thermodynamics of quantum black holes:

\[
\int \mathcal{D}g_{\mu\nu} \exp \left( -\frac{1}{\hbar} S_{\text{Einstein gravity}}^{(d+1)} [g_{\mu\nu}] \right)
= \exp \left( S_{BH} \right) \times \left( \text{Many body quantum theory in } d - 1 \text{ spatial dimensions without gravity} \right)
\]

Black holes are represented as a `hologram' by a quantum many-body system in one lower dimension.

**Duality:** a `change of variables' between the many-particle configurations and the metric of spacetime.
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_Duality:_ a `change of variables’ between the many-particle configurations and the metric of spacetime

Bekenstein (1973)  
Hawking (1974)  

Susskind, Maldacena…..
On September 14, 2015, LIGO detected the merger of two black holes, each weighing about 30 solar masses, with radii of about 100 km, 1.3 billion light years away! 0.1 seconds later!
The ring-down time $\frac{8\pi GM}{c^3} \sim 8$ milliseconds. Curiously, for essentially all types of black holes, the ring-down time equals $\frac{\hbar}{k_B T_H}$, $\hbar$ Planck’s constant, $k_B$ Boltzmann’s constant.
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LIGO
September 14, 2015

Vishveshwara (1970)
Black holes are represented as a `hologram' by a quantum many-body system in one lower dimension.

_Duality:_ a `change of variables' between the many-particle configurations and the metric of spacetime.
- Black holes have an entropy and a temperature, $T_H$
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a Planckian time $\sim \hbar / (k_B T_H)$.

The hologram of a black hole in $d$ dimensions is a quantum many-particle system in $(d - 1)$ dimensions which relaxes to thermal equilibrium in a Planckian time $\sim \hbar / (k_B T)$. 
Quantum entanglement

A simple many-particle (SYK) model

Black holes
Almost all many-electron systems are described by the quasiparticle concept: a quasiparticle is an “excited lump” in the many-electron state which responds just like an ordinary particle. The existence of quasiparticles implies limited many-particle entanglement.
Flowing quasiparticles scatter off each other in a typical scattering time $\tau$.

This time is much longer than a limiting ‘Planckian time’ $\frac{\hbar}{k_B T}$.

The long scattering time implies that quasiparticles are well-defined.
The Sachdev-Ye-Kitaev (SYK) model

Pick a set of random positions

Sachdev, Ye (1993); Kitaev (2015)
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

Place electrons randomly on some sites.
The SYK model

Place electrons randomly on some sites

Sachdev, Ye (1993); Kitaev (2015)
The SYK model

Sachdev, Ye (1993); Kitaev (2015)

Entangle electrons pairwise randomly
The SYK model

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Sachdev, Ye (1993); Kitaev (2015)
The complex SYK model

(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large N limit; T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. 53, 385 (1981))

\[ H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta} c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta} + \epsilon \sum_{\alpha} c_{\alpha}^{\dagger} c_{\alpha} \]

\[ c_{\alpha} c_{\beta} + c_{\beta} c_{\alpha} = 0, \quad c_{\alpha} c_{\beta}^{\dagger} + c_{\beta}^{\dagger} c_{\alpha} = \delta_{\alpha\beta} \]

\[ Q = \frac{1}{N} \sum_{\alpha} c_{\alpha}^{\dagger} c_{\alpha} \]

\( U_{\alpha\beta;\gamma\delta} \) are independent random variables with \( \overline{U_{\alpha\beta;\gamma\delta}} = 0 \) and \( |U_{\alpha\beta;\gamma\delta}|^2 = U^2 \) 

\( N \to \infty \) yields critical strange metal.
Complex multi-particle entanglement in the SYK model leads to a state without ‘quasiparticle’ excitations; i.e. multiple excitations cannot be built by composing an elementary set of ‘quasiparticle’ excitations.
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Many-body chaos and thermal equilibration in the shortest possible Planckian time $\sim \frac{\hbar}{k_B T}$. 
The complex SYK model

Feynman graph expansion in $U_{\alpha \beta; \gamma \delta}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

\[
G(i\omega) = \frac{1}{i\omega - \epsilon - \Sigma(i\omega)}, \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)
\]

\[
G(\tau = 0^-) = Q.
\]

Low energy solution with $q$-fermion Hamiltonian ($\Delta = 1/q$):

- \[
G(\tau) = -A e^{-2\pi \mathcal{E} T \tau} \left( \frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}, \quad 0 < \tau < 1/T,
\]
  where the ‘particle-hole asymmetry’ is determined by $\mathcal{E}$.

- There is a ‘Luttinger relation’ between $-\infty < \mathcal{E} < \infty$ and $0 < Q < 1$:

  \[
e^{2\pi \mathcal{E}} = \frac{\sin(\pi \Delta + \theta)}{\sin(\pi \Delta - \theta)}, \quad Q = \frac{1}{2} - \frac{\theta}{\pi} + \left( \Delta - \frac{1}{2} \right) \frac{\sin(2\theta)}{\sin(2\pi \Delta)}\]

- The entropy is also determined by $\mathcal{E}$: $dS/dQ = 2\pi \mathcal{E}$.

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  where the ‘particle-hole asymmetry’ is determined by $\epsilon$.

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Quantum entanglement

Black holes

Hologram?

A simple many-particle (SYK) model
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge.
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge. Zooming into the near-horizon region of a charged black hole at low temperature, yields a gravitational theory in one space and one time dimension.
SYK model and charged black holes

Horizon
Area yields
Bekenstein-Hawking
entropy $S_{BH}$

$\text{AdS}_2 \times S^2$

$\frac{dS_{BH}}{dQ} = 2\pi \mathcal{E}$,
where $Q$ is the charge density,
and $\mathcal{E}$ is the electric field on the horizon.

Sen (2005); Sachdev (2015)
SYK model and charged black holes

\( \text{AdS}_2 \times S^2 \)

1+1 spacetime dimensions

3+1 spacetime dimensions

Boundary graviton

\( \zeta \)

\( \vec{x} \)
Quantum gravity can be exactly solved in this region!
Thermodynamics of charged quantum black holes

\[ \int Dg_{\mu\nu} \exp \left( -\frac{1}{\hbar} S^{(3+1)}_{\text{Einstein–Maxwell theory}}[g_{\mu\nu}] \right) T \to 0, \]

\approx \int Dg_{\mu\nu} \exp \left( -\frac{1}{\hbar} S^{(1+1)}_{\text{Gravity of AdS}_2 \text{ and boundary}}[g_{\mu\nu}] \right) \]
SYK model and charged black holes

Thermodynamics of charged quantum black holes

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\approx \int \mathcal{D}g_{\mu\nu} \exp \left( -\frac{1}{\hbar} S_{\text{Gravity of AdS}_2 \text{ and boundary}}^{(1+1)}[g_{\mu\nu}] \right)
\]
\[
= \exp \left( S_{BH} \right) \times \exp \left( -\frac{1}{T} \times \text{Free energy of SYK model} \right)
\]

The hologram of the 1+1 dimensional gravity near the horizon of a charged black hole is the 0+1 dimensional SYK model

Sachdev (2010); Kitaev (2015); Sachdev (2015); Maldacena, Stanford, Yang (2016); Moitra, Trivedi, Vishal (2018); Gaikwad, Joshi, Mandal, Wadia (2018); Iliesiu, Turaci (2020)
Maxwell’s electromagnetism and Einstein’s general relativity allow black hole solutions with a net charge. The near-horizon $1+1$D-gravity theory is precisely that of the low T limit of the SYK models.
Quantum entanglement

A simple many-particle (SYK) model

Charged black holes

Copper-based superconductors
Antoine Georges: V21.00001. Bad Metals and Planckian Metals: DMFT, SYK and physical realisations, 3 PM, Thu Mar 18

Aavishkar Patel: B50.00002. Theories of Planckian dissipation in strange metals, 12:06 PM, Mon Mar 15
Alexander Wietek, Henry Shackleton: R44.00005. SYK criticality and spin glass order in the random doped Heisenberg model, 8:48 AM, Thu Mar 18

Maria Tikhanovskaya: R44.00011. Excitation spectra of quantum matter without quasiparticles II: random t-J model, 10:00 AM, Thu Mar 18

Session R44, Thu 8 AM:

Haoyu Guo, Grigory Tarnopolsky: R44.00001. Excitation spectra of quantum matter without quasiparticles I: Sachdev-Ye-Kitaev models, 8:00 AM, Thu Mar 18

Philipp Dumitrescu, Nils Wentzell, Olivier Parcollet: R44.00002. Deconfined metallic criticality and Sachdev-Ye-Kitaev physics of spin-1/2 electrons at finite doping, 8:12 AM, Thu Mar 18

Alexander Wietek, Henry Shackleton: R44.00005. SYK criticality and spin glass order in the random doped Heisenberg model, 8:48 AM, Thu Mar 18

Maria Tikhanovskaya: R44.00011. Excitation spectra of quantum matter without quasiparticles II: random t-J model, 10:00 AM, Thu Mar 18

Darshan Joshi: C43.00008. Anomalous density fluctuations in a random t-J model, 4:24 PM, Mon Mar 15

Chenyuan Li: E44.00014. Critical Anomalous Metal Near Superconductivity in a Random Hubbard Model, 10:36 AM, Tue Mar 16
High temperature superconductors

YBa$_2$Cu$_3$O$_{6+x}$
Insulating antiferromagnet
Antiferromagnet doped with hole density $p$
Real-space view

$p$ mobile holes in a background of fluctuating spins

Baskaran, Anderson (1988)
$p$ mobile holes in a background of fluctuating spins

Baskaran, Anderson (1988)
Real-space view

$p$ mobile holes in a background of fluctuating spins

Baskaran, Anderson (1988)
$p$ mobile holes in a background of fluctuating spins

Baskaran, Anderson (1988)
Strange Metal

Pseudogap Metal

Fermi liquid

AF insulator

Temperature (K)

Hole doping, $\rho$

$T_N$

$T^*$

$T_c$

$p_c$
Fermi Liquid

Strange Metal

Pseudogap Metal

“Large” Fermi surface

Fermi liquid

AF insulator

$T_N$

$T^*$

$T_c$

$\rho_c$
Fermi liquid
Strange Metal
Pseudogap Metal
“Small” Fermi surface
Fermi liquid

Temperature (K)

AF insulator

Superconductor

Hole doping, \( p \)

\( T_N \)

\( T^* \)

\( T_c \)

\( p_c \)
Angle-dependent magnetoresistance in Nd-LSCO near $p = p_c \approx 0.23$.

\[
\frac{1}{\tau} = \frac{1}{\tau_{\text{aniso}}(\vec{k})} + \frac{\alpha}{\hbar} k_B T
\]
Angle-dependent magnetoresistance in Nd-LSCO near $p = p_c \approx 0.23$.

\[ \frac{1}{\tau} = \frac{1}{\tau_{\text{aniso}}(\vec{k})} + \frac{\alpha}{\hbar} k_B T \]
Random $t$-$J$ model
Random $t-J$ model
Random $t$-$J$ model
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Random $t$-$J$ model
Random $t$-$J$ model
Random $t$-$J$ model
\[ \mathcal{N}(\epsilon) = \frac{1}{N} \sum_{\lambda} \delta(\epsilon - \epsilon_\lambda) \sum_{i,j} \langle \lambda | i \rangle \langle c_{i\sigma}^\dagger c_{j\sigma} \rangle \langle j | \lambda \rangle \]

where \(|\lambda\rangle\) are one-particle eigenstates of the \(t_{ij}\). In a Fermi liquid, the Luttinger identity implies that \(\mathcal{N}(\epsilon)\) has a discontinuity at the free particle Fermi energy \(\epsilon_F\). \((D(\epsilon)\) is the Wigner semi-circle density of states.)
\[ N(\epsilon) = \frac{1}{N} \sum_{\lambda} \delta(\epsilon - \epsilon_\lambda) \sum_{i,j,\sigma} \langle \lambda | i \rangle \langle c_{i\sigma}^\dagger c_{j\sigma} | j \rangle \langle j | \lambda \rangle \]

where \( |\lambda\rangle \) are one-particle eigenstates of the \( t_{ij} \). In a Fermi liquid, the Luttinger identity implies that \( N(\epsilon) \) has a discontinuity at the free particle Fermi energy \( \epsilon_F \). (\( D(\epsilon) \) is the Wigner semi-circle density of states.)
Dynamic spin susceptibility

Probability to flip an electron spin while absorbing energy $\hbar \omega$

Near criticality, the model is predicted to exhibit SYK-like criticality with a non-zero extensive entropy and a graviton in the holographic dual \[ \Phi \rightarrow 0 \text{ (which general-} \]
Dynamic spin susceptibility

Probability to flip an electron spin while absorbing energy $\hbar \omega$

Spin susceptibility and other properties match those of an ordinary metal $p > p_c$
**Dynamic spin susceptibility**

Probability to flip an electron spin while absorbing energy $\hbar \omega$

---

Critical spin susceptibility matches the SYK model!

Planckian dissipation in time $\sim \hbar/(k_B T)$, and frequency dependence $\sim \text{sgn}(\omega) [1 - C\gamma|\omega| + \ldots]$ matches contribution of boundary graviton.
The SYK spectral weight has a leading large-frequency behavior near the critical point at low frequencies. We subtract a function in the thermodynamic limit. To do this, we first show for several values of doping in Fig. 3. Using the Lanczos algorithm on an 18-site cluster, is predicted that the ordered Fermi liquid is expected to have a low-frequency gap at low frequency is visible, and the spectral function is described well by a combination of SYK and a low-frequency hump. A critical prediction given in Ref. [2020], averaged over 100 disorder realizations on an 18-site cluster, is that the spectral function is largely independent of system size. No gap at low frequency is visible, and the critical value of doping near the SYK prediction is stable for small values of doping, up to a critical temperature which eventually vanishes. Therefore, the spin glass contribution to the spectral function is largely independent of system size.

Near criticality, the model is predicted to exhibit SYK-like criticality with a non-zero extensive entropy and a disordered Fermi liquid phase for all non-zero values of doping. With increasing doping, the magnitude of this peak is reduced, and the low-frequency behavior closely resembles the rescaled spectral weight of the low-frequency peak is non-zero, indicating spin glass order. With increasing doping, the magnitude of this peak is reduced, and the low-frequency behavior closely resembles the rescaled spectral weight of the low-frequency peak is non-zero, indicating spin glass order.

Critical point

Dynamic spin susceptibility

Probability to flip an electron spin while absorbing energy $\hbar \omega$

SYK criticality can be understood by the fractionalization of the electron into ‘partons’ carrying its spin and charge. These partons obey an SYK-like model.
Quantum Monte Carlo of $N = \infty$ model
Quantum Monte Carlo of $N = \infty$ model
Quantum Monte Carlo of $N = \infty$ model

\[ -Z \cdot \text{Im}[\Sigma_{\text{extrap}}(0)] \]

\[ \frac{1}{\tau^*} = (1.4 \pm 0.1) \frac{k_B T}{\hbar} \]

P. Dumitrescu, N. Wentzell, O. Parcollet, A. Georges, arXiv:2103.08607; R44.00002, 8:12 AM, Thu Mar 18

Antoine Georges: V21.0000, 3 PM, Thu Mar 18
Random Yukawa model

Fermi surface of ψ fermions coupled to a critical boson φ (representing an order parameter or a gauge field)

\[ S = \int d\tau \int_{\mathbf{k}} \sum_i \left[ \psi_i^\dagger(\mathbf{k}, \tau) \left( \partial_\tau - \varepsilon_\mathbf{k} - \mu \right) \psi_i(\mathbf{k}, \tau) \right] \]

\[ + \int d\tau \int_{\mathbf{x}} \sum_\ell \phi_\ell \left( -\partial_\tau^2 - \nabla_\mathbf{r}^2 + m^2 \right) \phi_\ell \]

\[ - \int d\tau \int_{\mathbf{x}} \sum_{i,j,\ell=1}^N \frac{g_{i\ell}(\mathbf{x})}{N} \psi_i^\dagger(\mathbf{x}, \tau) \psi_j(\mathbf{x}, \tau) \phi_\ell(\mathbf{x}, \tau) \]

\[ g_{i\ell}(\mathbf{x}) g_{j\ell}(\mathbf{x}) = g^2 \delta_{ii'} \delta_{jj'} \delta_{\ell\ell'} \delta^2(\mathbf{x} - \mathbf{x}') \]

Yields Planckian transport in \( d = 2 \)

with \( \frac{1}{\tau^*} = \frac{\pi k_B T}{2 \hbar} \times \frac{\ln \ln T}{\ln T} \)
Quantum entanglement

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Charged black holes

Copper-based superconductors
Quantum entanglement

A simple many-particle (SYK) model

Charged black holes

2D quantum gravity

Copper-based superconductors
Quantum entanglement

A simple many-particle (SYK) model

Charged black holes

SYK criticality of random $t$-$J$ and random Yukawa models

Copper-based superconductors
Complex multi-particle entanglement leads to quantum systems without quasiparticle excitations.

Many-body chaos and thermal equilibration in the shortest possible Planckian time $\sim \frac{\hbar}{k_B T}$. 