Entanglement, fermion signs and the holographic strange metals.

Jan Zaanen
Von Neumann entropy: entanglement and fields.

Bipartite vN entropy: measures entanglement = quantum information of Bell pairs

\[ |Bell\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad \Rightarrow S_{vN} = \sqrt{2} \]

\[ |Pro\,d\rangle = \frac{1}{2} (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \quad \Rightarrow S_{vN} = 0 \]

Field theory: infinite number of “qubits”, not two.

Divide space in two:

\[ \rho_A = Tr_B \rho, \quad S_{vN,A} = Tr[\rho_A \ln \rho_A] \]
“Space bipartite” Von Neumann entropy: success!

Divide space in two: \[ \rho_A = Tr_B \rho, \quad S_A = Tr[\rho_A \ln \rho_A] \]

1+1 D CFT’s (Calabrese-Cardy): \[ S_A = \frac{c}{3} \log L \quad c = \text{central charge} \]

Topological insulators (Kitaev-Preskil, Levin-Wen):
\[ S_A = \alpha \Sigma - \gamma + \cdots, \quad \Sigma = L^{d-1}, \quad \gamma = \log D \quad D = \text{total quantum dimension.} \]

Conformal fields (Klebanov etc.):
\[ S_A = \alpha \Sigma - F \]

Entanglement spectrum (Haldane):
\[ S_A = -\sum_\lambda p_\lambda \ln p_\lambda, \quad p_\lambda = \frac{e^{-\xi_\lambda}}{\sum_\lambda e^{-\xi_\lambda}} \]

Measures the edge/surface states without a surface!
“Space bipartite” Von Neumann entropy: compressible systems.

Divide space in two: \( \rho_A = Tr_B \rho, \quad S_A = Tr[\rho_A \ln \rho_A] \)

\[
S_A = -\sum_{\lambda} p_\lambda \ln p_\lambda, \quad p_\lambda = \frac{e^{-\xi_\lambda}}{\sum_{\lambda} e^{-\xi_\lambda}}
\]

Bosonic field theory: scales with surface area.

\[
S_{A,bosons} = \alpha \Sigma + \cdots, \quad \Sigma = L^{d-1}
\]

Single point in momentum space with massless excitations.

Fermi-liquid: extra log, longer range!

\[
S_{A,FL} \propto \Sigma \log \Sigma + \cdots
\]

d-1 dimensional surface of massless excitations in momentum space = Fermi surface.
AdS/CFT: entanglement entropy and the minimal surface.

\[ \rho_A = Tr_B[\rho] \quad S_{vN} = Tr[\rho_A \ln \rho_A] \]

The spatial bipartite entanglement entropy in the boundary is dual to the area of the minimal surface in the bulk, bounded by the cut in the space of the boundary (Takayanagi-Ryu conjecture).
Holographic strange metal entanglement entropy.

Geometry in Einstein-Maxwell-Dilaton bulk:

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

“Hyperscaling violation” in boundary theory:

\[ x_i \rightarrow \xi x_i, \quad t \rightarrow \xi t, \quad ds \rightarrow \xi^d ds \]

Thermal entropy:

\[ S \propto T^{(d-\theta)/z} \]

Entanglement entropy (Sachdev, Huijse, Swingle):

<table>
<thead>
<tr>
<th>[ S_{vN} \propto \Sigma ]</th>
<th>[ \theta &lt; d - 1 ]</th>
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<tbody>
<tr>
<td>[ S_{vN} \propto \Sigma \ln \Sigma ]</td>
<td>[ \theta = d - 1 ]</td>
</tr>
<tr>
<td>[ S_{vN} \propto \Sigma^{\theta/(d-1)} ]</td>
<td>[ \theta &gt; d - 1 ]</td>
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Looks much like a deconfined Fermi-liquid, hidden from the gauge singlet UV propagators!

But this is longer ranged !?
The Reissner-Nordstrom AdS2 horizon: local quantum criticality.

Near-horizon geometry of the extremal RN black hole:

- Space directions: flat, codes for simple Galilean invariance in the boundary.

- Time-radial(=scaling) direction: emergent AdS$_2$, codes for emergent temporal scale invariance!

(Fermion) spectral functions:

“Algebraic Pseudogap metal”

\[ A(k,\omega) \propto G''_{AdS_2}(k,\omega) \propto \omega^{2\nu_k} \]

\[ \nu_k = \frac{1}{\sqrt{6}} \sqrt{k^2 + \frac{1}{\xi^2}} \]
Holographic very strange metal entanglement entropy

Ground state entropy: \( S(T = 0) \propto \mu N^2 \) Needs both finite density and large \( N \).

RN metal holographic entanglement entropy:
\[
\theta = d \left( 1 - \frac{1}{z} \right), \quad z \to \infty \quad \text{(Sachdev)}
\]
\[
S_{vN} \propto l^d \quad \text{Volume scaling!}
\]

For infinite \( l \) this turns into the \( T=0 \) entropy:
\[
S_{T=0} = Tr[\rho \ln \rho]
\]

Counting excitations:

Massless point in momentum space = bosonic CFT’s:
\[
S_{vN} \propto l^{d-1}
\]

Massless surface in momentum space = Fermi liquid:
\[
S_{vN} = \text{(# points on FS)} \cdot (\text{CFT}^2 S_{vN})
\]
\[
S_{vN} \propto (l^{d-1})(\ln l)
\]

Massless volume in momentum space = RN metal:
\[
S_{vN} = \text{(#points in k space)} \cdot (\text{CFT}^1 S_{vN})
\]
\[
S_{vN} \propto l^d
\]
Fermions at a finite density: the sign problem.

Imaginary time first quantized path-integral formulation

\[ Z = \text{Tr} \exp(-\beta \hat{H}) = \int dR \rho(R, R; \beta) \]

\[ R = (r_1, \ldots, r_N) \in \mathbb{R}^{Nd} \]

\[ \rho_{B/F}(R, R; \beta) = \frac{1}{N!} \sum_{\mathcal{P}} (\pm 1)^{\mathcal{P}} \rho_{D}(R, \mathcal{P}R; \beta) \]

\[ = \frac{1}{N!} \sum_{\mathcal{P}} (\pm 1)^{\mathcal{P}} \int_{\mathcal{R} \to \mathcal{P}R} \mathcal{D}R(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left( \frac{m}{2} \ddot{R}^2(\tau) + V(R(\tau)) \right) \right\} \]

Boltzmannons or Bosons:
- integrand non-negative
- probability of equivalent classical system: (crosslinked) ringpolymers

Fermions:
- negative Boltzmann weights
- non probablistic: NP-hard problem (Troyer, Wiese)!!!
A universal phase diagram

High Tc superconductors

Heavy fermions

Iron superconductors (?)
Fermions at a finite density: land of opportunity?

Dealing with fermions at finite density we just know two serious (mathematical) operations:

1. Declare them to be a Fermi-gas that you might want to hit with uncontrolled perturbation theory.

2. Declare that they run into a BCS/Hartree-Fock “bosonic” instability.

But behind the “sign brick wall” there might be a vast landscape of new states of truly extreme quantum matter waiting to be discovered!

Is this what the holographic strange metals try to tell to us?

But first some other glimpses behind the sign brick wall …
Von Neumann entropy: entanglement versus quantum statistics.

“Entanglement” entropy = quantum information of Bell pairs.

$|Bell\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \Rightarrow S_{vN} = \sqrt{2}$

Fermi gas vN entropy is “longer” ranged: $S_{vN,FL} \propto \Sigma \log \Sigma + \cdots$

Is the Fermi gas a “better” quantum information resource??

Von Neumann entropy can speak with two tongues:
it scrambles quantum-statistical non-locality and quantum information.

Fermi-Dirac is classical non-local information:
“Mottness in momentum space”.

Quantum statistics and first quantized path integrals

\[ \rho_{B/F}(R, R; \beta) = \frac{1}{N!} \sum_p (\pm 1)^p \rho_D(R, pR; \beta) \]

\[ = \frac{1}{N!} \sum_p (\pm 1)^p \int_{R \rightarrow pR} DR(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left( \frac{m}{2} \dot{R}^2(\tau) + V(R(\tau)) \right) \right\} \]

Bose condensation: Partition sum dominated by infinitely long cycles

Fermions: infinite cycles set in at \( T_F \), but cycles with length \( w \) and \( w + 1 \) cancel each other approximately. Free energy pushed to \( E_F \)!
The nodal hypersurface

Antisymmetry of the wave function

$$\Psi(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_N) = -\Psi(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_N)$$

Free Fermions

$$\Psi_0(R) \sim \det (e^{ik_ir_j})_{ij}$$

Pauli hypersurface

$$P = \bigcup_{i \neq j} P_{ij}$$

$$P_{ij} = \{R \in \mathbb{R}^{Nd} | r_i = r_j\}$$

$$\dim P = Nd - d$$

Nodal hypersurface

$$\Omega = \{R \in \mathbb{R}^{Nd} | \Psi(R) = 0\}$$

$$\dim \Omega = Nd - 1$$

Test particle
Constrained path integrals

Formally we can solve the sign problem!!

$$\rho_F(R, R'; \beta) = \frac{1}{N!} \sum_{p, \text{even}} \int_{\rho \in \Gamma(R, P)} D\rho(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left( \frac{m}{2} \dot{R}^2(\tau) + V(R(\tau)) \right) \right\}$$

$$\Gamma(R, R') = \{ \gamma : R \to R' | \rho_F(R, R(\tau); \tau) \neq 0 \}$$

Self-consistency problem:
Path restrictions depend on $\rho_F$!
Ceperley path integral: Fermi gas in momentum space

Single particle propagator:

\[ g(k, k', \tau) = 2\pi \delta(k - k')e^{-\frac{|k|^2\tau}{2\hbar m}} \]

single particle momentum conserved

N particle density matrix:

\[ \rho_F(K, K', \tau) = \frac{1}{N!} \sum_{\mathcal{P}} (-1)^P \prod_{i=1}^{N} g(k_i, k'_{p(i)}, \tau) \]

\[ = \frac{1}{N!} e^{-\sum_{i=1}^{N} \frac{|k_i|^2\tau}{2\hbar m}} \sum_{\mathcal{P}} (-1)^P \prod_{i=1}^{N} 2\pi \delta(k_i - k'_{p(i)}) \]

\[ = \prod_{k_1 \neq k_2 \neq \ldots \neq k_N} 2\pi \delta(k_i - k'_i)e^{-\frac{|k_i|^2\tau}{2\hbar m}} \]

‘harmonic potential’
Fermi gas = cold atom Mott insulator in harmonic trap!

\[ \rho_F(K,K'';\tau) = \prod_{k_1 \neq k_2 \neq \ldots \neq k_N} 2\pi\delta(k_i - k_i')e^{\frac{-|k_i|^2\tau}{2\hbar m}} \]

Mottness: traffic jam non-locality

This car is stuck
By von Neumann cut it might look “quantum” entangled …

Because of this ..
The origin of the long range Fermi-gas “entanglement” entropy

Mott insulating Cold atoms in real space

Extreme non-locality in k space

Fourier transformation

Change one particle coordinate and the nodal surface changes everywhere.

Fermi gas: Mottness in k space.

Extreme non-locality in real space.

The constrained path integral leaves no doubt:

\[ S_{A,FL} \propto \Sigma \log \Sigma + \cdots \]

is a consequence of Fermi-Dirac statistics which is in turn a carrier of purely classical non-local information.
Cuprates start as doped Mott-insulators

\[ H = t \sum_{ij} c_{i\sigma}^+ c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \]

Mott insulator

Doped Mott insulator

\[ H_{t-J} = t \sum_{ij} \tilde{c}_{i\sigma}^+ \tilde{c}_{j\sigma} + J \sum_{ij} \tilde{S}_i \cdot \tilde{S}_j \]
Mott insulator: the vanishing of Fermi-Dirac statistics

Mott-insulator: the electrons become distinguishable, stay at home principle!

Spins live in tensor-product space. “Spin signs” are like hard core bosons in a magnetic field, can be gauged away on a bipartite lattice (“Marshall signs”)

\[ Z_{t-J} = \sum_c \tau_c \mathcal{Z}[c] \]

\[ \mathcal{Z}[c] \geq 0 \quad \tau_c = +1 \]
Doped Mott-insulator: Weng statistics

t-J model: spin up is background, spin down’s (‘spinons’) and holes (‘holons’) are hard core bosons.

Exact Partition sum: \[ Z_{t-J} = \sum_c \tau_c Z[c] \]
\[ Z[c] = \left( \frac{2t}{J} \right)^{M_h[c]} \sum_n \frac{\left(\frac{\beta J}{2}\right)^n}{n!} \delta_{n,M_h+M_{\uparrow1}+M_Q} \]

The sign of any term is set by:

\[ \tau_c \equiv (-1)^{N_{\downarrow}h[c] + N_{\uparrow}h[c]} \]

- \( N_{\downarrow}h[c] \) The (fermionic) number of holon exchanges
- \( N_{\uparrow}h[c] \) The number of spinon-holon ‘collisions’

arXiv: 0802.0273
Anderson Localization without quenched disorder!

“Numerically exact” DMRG of one hole t-J model on n-leg ladders: remnant signs cause Anderson localization of the hole on the perfect lattice!

Zhu et al., subm. Science
Constrained path integrals

Formally we can solve the sign problem!!

\[
\rho_F(R, R'; \beta) = \frac{1}{N!} \sum_{\mathcal{P}, \text{even}} \int_{\gamma \in \Gamma(R, \mathcal{P}R)} DR(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left( \frac{m}{2} \dot{R}^2(\tau) + V(R(\tau)) \right) \right\}
\]

\[
\Gamma(R, R') = \{ \gamma : R \rightarrow R' | \rho_F(R, R(\tau); \tau) \neq 0 \}
\]

Self-consistency problem:
Path restrictions depend on \( \rho_F \)!
Fermi-energy: confinement energy imposed by local geometry

\[ l^2(\tau) = \langle (r_i(\tau) - r_i(0))^2 \rangle = 2dD\tau = 2d\frac{\hbar}{2m} \tau \]

\[ l^2(\tau_c) \sim r_s^2 \rightarrow \tau_c \sim \frac{1}{2d} \frac{2m}{\hbar} n^{-2/d} \]

\[ \hbar \omega_c = \frac{\hbar}{\tau_c} \sim d \frac{\hbar^2}{2m} n^{2/d} \sim E_F \]

Fermi surface encoded globally: \( \rho_F = Det(e^{ikr_j}) = 0 \)

Change in coordinate of one particle changes the nodes everywhere

Finite T: \( \rho_F = (4\pi \lambda \beta)^{-dN/2} Det \left[ \exp \left( -\frac{(r_i - r_j)^2}{4\lambda\tau} \right) \right] \)

\[ \lambda = \hbar^2 / (2M) \]

Non-locality length: \( \lambda_{nl} = v_F \tau_{inel} = v_F \left( \frac{E_F}{k_B T} \right) \left( \frac{\hbar}{k_B T} \right) \)

Average node to node spacing

\[ \sim r_s = \left( \frac{V}{N} \right)^{1/d} = n^{-1/d} \]
Key to fermionic quantum criticality

At the QCP scale invariance, no $E_F$ Nodal surface has to become fractal !!!
Vacuum structure

Long time, zero temperature:

\[ \rho_F(R, R(\tau); \tau \rightarrow \infty) = \Psi^*(R)\Psi(R(\infty)) \]

IR fermionic information encoded in the ground state wavefunction.

Need the answer: the wave function!
Hydrodynamic backflow

Classical fluid: incompressible flow

Feynman-Cohen: mass enhancement in $^4$He

Wave function ansatz for "foreign" atom moving through He superfluid with velocity small compared to sound velocity:

$$g(r) \sim \frac{kr}{r^3} \rightarrow \Psi = \phi \exp[ik \left( r_A + \sum_{i \neq A} \frac{r_i - r_A}{r_i^3} \right)]$$

Backflow wavefunctions in Fermi systems

$$\psi_{bf}(R) \sim \text{Det} \left( e^{ik_i \tilde{r}_j} \right)_{ij}$$

$$\tilde{r}_j = r_j + \sum_{l(\neq j)} \eta(r_{ji})(r_j - r_l)$$

Widely used for node fixing in QMC

→ Significant improvement of variational GS energies
Frank’s fractal nodes ...

Feynman’s fermionic backflow wavefunction:

\[ \psi_{bf}(\mathbf{R}) \sim \text{Det} \left( e^{i\mathbf{k} \cdot \mathbf{\tilde{r}}_j} \right)_{ij} \]

\[ \mathbf{\tilde{r}}_j = \mathbf{r}_j + \sum_{i(\neq j)} \eta(r_{ji})(\mathbf{r}_j - \mathbf{r}_i) \]

\[ \eta(r) = \frac{a^3}{r^3 + r_0^3} \]

Frank Krüger
Fractal dimension of the nodal surface

Calculate the correlation integral $C(r) \sim r^\nu$ on random $d=2$ dimensional cuts

$$Nd - 1 < D_H = N \nu_d < Nd$$
$$d - \frac{1}{N} < \nu_d < d$$

$N = 13$: $\nu = 1.976 \pm 0.012$

$\rightarrow D_H = 25 + (0.69 \pm 0.16)$

$N = 29$: $\nu = 1.982 \pm 0.008$

$\rightarrow D_H = 57 + (0.48 \pm 0.23)$

Backflow turns nodal surface into a fractal !!!
Fermionic quantum phase transitions in the heavy fermion metals

\[ m^* = \frac{1}{E_F} \]

\[ E_F \rightarrow 0 \Rightarrow m^* \rightarrow \infty \]
Turning on the backflow

Nodal surface has to become fractal !!!

Try backflow wave functions

\[ \psi_{bf}(\mathbf{R}) \sim \text{Det} \left( e^{ik_i \tilde{r}_j} \right)_{ij} \]

\[ \tilde{r}_j = \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l) \]

\[ \eta(r) = \frac{a^3}{r^3 + r_0^3} \]

Collective (hydrodynamic) regime:

\[ a \gg r_s \]
MC calculation of $n(k)$

\[
\frac{m}{m^*} \propto \left(1 - \frac{a}{a_c}\right)^3
\]

Divergence of effective mass as $a \to a_c$
Fractal nodes versus conformal fermion matter

Is the fractal geometry of the nodal surface necessary condition for the finite density conformal fermion liquid?

Credible but no proof …

Is the hydrodynamical back flow system representative?

Suspicious: hidden free system …

Computations that can be done (hard work in progress):

Bare fermion propagators: quasi-local quantum criticality?

vN entropy: the fractal nodes encode more strongly non-local “statistical” information than “Fermi-gas Mottness” !?
The full density matrix: a trace too far.

Field theory: the density matrix is the n-point propagator.

\[ \rho(\tilde{X}, \tilde{X}'(\tau); \tau) \equiv \langle \overline{\Psi}(\tilde{x},0)\Psi(\tilde{x}',\tau) \rangle, \quad \tau \to \infty \]

\[ \overline{\Psi}(\tilde{X}) = \overline{\psi}(\tilde{x}_1)\overline{\psi}(\tilde{x}_2)\cdots\overline{\psi}(\tilde{x}_{N-1})\overline{\psi}(\tilde{x}_N) \]

The von Neumann entropy departs from this information.

\[ \rho_A = Tr_B[\rho] \quad S_{vN} = Tr[\rho_A \ln \rho_A] \]

But by taking the traces it scrambles fermion statistics information and quantum information.
The nodal surface and the dictionary.

Diagnosing generalized (non Fermi-Dirac) fermion statistics:

$$\rho\left(\tilde{X}, \tilde{X}'(\tau); \tau\right) \equiv \left\langle \overline{\Psi}\left(\tilde{X}, 0\right) \Psi\left(\tilde{X}', \tau\right)\right\rangle = 0, \quad \tau \to \infty$$

$$\overline{\Psi}\left(\tilde{X}\right) = \overline{\psi}\left(\tilde{x}_1\right) \overline{\psi}\left(\tilde{x}_2\right) \cdots \overline{\psi}\left(\tilde{x}_{N-1}\right) \overline{\psi}\left(\tilde{x}_N\right)$$

One needs an answer first, the nodal surface is diagnostics:

Are the holographic strange metals characterized by unconventional (non Fermi-gas) nodal structure?
The nodal surface and the dictionary (I).

Computing the n-point propagator holographically:

\[ \rho(\tilde{X}, \tilde{X}'; \tau; \tau) \equiv \langle \overline{\Psi}(\tilde{X},0)\Psi(\tilde{X}',\tau) \rangle = 0, \quad \tau \to \infty \]

\[ \overline{\Psi}(\tilde{X}) = \overline{\psi}(\tilde{x}_1)\overline{\psi}(\tilde{x}_2)\cdots\overline{\psi}(\tilde{x}_{N-1})\overline{\psi}(\tilde{x}_N) \]

A first guess: identify \( \psi(\tilde{x}) \) with the O(1/N) gauge singlet fermions.

Liu, McGreevy, … : in this order AdS/CFT turns into an effective weak-weak duality with the ramification that n-point propagators are described by weak coupling (bulk) diagrams, inserting fully dressed (strong coupling) single fermion propagators

\[ g(\tilde{x}, \tilde{x}'; \tau) = \langle \overline{\psi}(\tilde{x},0)\psi(\tilde{x}',\tau) \rangle \]
The nodal surface and the dictionary (II).

Computing the n-point propagator holographically:

\[
\rho(\vec{X}, \vec{X}'(\tau); \tau) \equiv \left\langle \overline{\Psi}(\vec{X}, 0)\Psi(\vec{X}', \tau) \right\rangle = 0, \quad \tau \to \infty
\]

\[
\overline{\Psi}(\vec{X}) = \overline{\psi}(\vec{x}_1)\overline{\psi}(\vec{x}_2)\cdots\overline{\psi}(\vec{x}_{N-1})\overline{\psi}(\vec{x}_N)
\]

But we would like to know how this acts out for the “deconfined stuff behind the horizon” of the large N limit, look for color ….

The vN entropy (full trace) shows that the information is there.

Conundrum: what is the dictionary entry yielding nodal surface information?
Summary …

Fermions at finite density: the fermion signs are wrecking established mathematical machinery, but it leaves room for BIG surprises.

Is AdS/CFT revealing such surprises? The AdS\textsuperscript{2} strange metals, the emergent Fermi liquids, the holographic superconductors.

The entanglement entropy knows about “fermion signs” but is ambiguous: in the Fermi gas it reveals Fermi-Dirac = classical non-local information.

General measure of non FD fermion statistics: Ceperley’s nodal surface. When signs are the cause, the nodal surfaces of the temporally conformal AdS/CFT strange metals have to be fractals!

What is the dictionary entry for nodal surface information?
Fermions, information and the constrained path integral.


Mottness: fermion statistics which is not Fermi-Dirac.


Fermions and conformal invariance: the fractal nodal surface.