The world of long-range entanglement – from new states of quantum matter to a unification of gauge inter. and Fermi statistics

Xiao-Gang Wen, Perimeter Institute/MIT, Feb, 2013
Quantum phases ← Symmetry

there are much more than four phases:
\[
\text{different phases} = \text{different symmetry breaking}
\]

\[
\begin{align*}
\text{(a)} & \quad \begin{array}{c}
A \\
\phi
\end{array} & \quad \begin{array}{c}
A \\
\phi
\end{array} & \quad \begin{array}{c}
A \\
\phi
\end{array} \\
& \quad \text{ε}_g & \quad \text{ε}_{g_c} & \quad \text{ε}_{g'}
\end{align*}
\]

From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D. **The math foundation is group theory**
Quantum phases ← q-Topology

there are much more than symmetry-breaking phases:

• Example:
  - Quantum Hall states \( \sigma_{xy} = \frac{m}{n} \frac{e^2}{h} \)
  - Spin liquid states

• FQH states and spin-liquid states have no symmetry breaking, no crystal order, no spin order, ... so they must have a new order – topological order

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The world of long-range entanglement – from new states of quantum matter to a unification of gauge interactions and Fermi statistics.
What is topological order?

Two aspects:
- Macroscopic definition
- Microscopic picture

For example,
- crystal order is defined/probed by X-ray diffraction:
Macroscopic definition of topological orders

Topological order can be defined “experimentally” through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** \( D_g \) Wen 89

\[
\begin{align*}
\text{Deg.}=1 & & \text{Deg.}=D_1 & & \text{Deg.}=D_2 \\
g=0 & & g=1 & & g=2 \\
\end{align*}
\]

(2) **Non-Abelian geometric’s phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation \( T \): \( |\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = T_{\alpha\beta} |\Psi_\beta\rangle \)

- 90° rotation \( S \): \( |\Psi_\alpha\rangle \rightarrow |\Psi''_\alpha\rangle = S_{\alpha\beta} |\Psi_\beta\rangle \)

- \( T, S \), define topological order “experimentally”.

- \( T, S \) is a universal probe for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.

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The world of long-range entanglement – from new states of quantum matter to a unification of gauge interactions and Fermi statistics.
Symmetry-breaking/topological orders through experiments

<table>
<thead>
<tr>
<th>Order</th>
<th>Experiment</th>
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</thead>
<tbody>
<tr>
<td>Crystal order</td>
<td>X-ray diffraction</td>
</tr>
<tr>
<td>Ferromagnetic order</td>
<td>Magnetization</td>
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<tr>
<td>Anti-ferromagnetic order</td>
<td>Neutron scattering</td>
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<tr>
<td>Superconducting order</td>
<td>Zero-resistance &amp; Meissner effect</td>
</tr>
<tr>
<td>Topological order (Global dancing pattern)</td>
<td>Topological degeneracy, non-Abelian geometric phase</td>
</tr>
</tbody>
</table>

- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields.
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** $T, S$ define a completely new class of order – topologically **order**.
- $T, S$ determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12
What is the microscopic picture of topological order?

represent an experimental definition of topological order.

• But what is the microscopic understanding of topological order?
• Zero-resistance and Meissner effect \(\rightarrow\) experimental definition of superconducting order.
• A microscopic picture of superconducting order: electron-pair condensation

Bardeen-Cooper-Schrieffer 57

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What is the microscopic picture of topological order?

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect → experimental definition of superconducting order.
- A microscopic picture of superconducting order: **electron-pair condensation**
  Bardeen-Cooper-Schrieffer 57

- A microscopic picture of topological order: **long-range entanglements**
  Chen-Gu-Wen 10
  (defined by local unitary trans. and motivated by topological entanglement entropy).
  Kitaev-Preskill 06, Levin-Wen 06
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle =$ direct-product state $\rightarrow$ unentangled (classical)
- $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow$ entangled (quantum)
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{entangled (quantum)}$
- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{more entangled}$
Quantum entanglements through examples

• $|\uparrow\rangle \otimes |\downarrow\rangle$ = direct-product state $\rightarrow$ unentangled (classical)
• $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$ $\rightarrow$ entangled (quantum)
• $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$
  $= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle$ $\rightarrow$ unentangled
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle$ = direct-product state $\rightarrow$ unentangled (classical)
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- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$ $= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle$ $\rightarrow$ unentangled
- $|\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\rightarrow$ unentangled
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{entangled (quantum)}$
- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle = (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled}$
- $|\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \cdots \rightarrow \text{unentangled}$
- $(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \cdots \rightarrow \text{short-range entangled (SRE) entangled}$

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How to make long range entanglements (topo. orders)

To make topological order, we need to sum over different product states, but we should not sum over everything.

- Sum over a subset of the particle configurations, by first join the particles into strings, then sum over the loop states

\[ |\Phi_{\text{loop}}\rangle = \sum_{\text{all loop conf.}} \left| \right\rangle \]

which is not a direct-product state and not a local deformation of direct-product states

→ non-trivial **topological orders** (long-range entanglements)
Topological orders through pictures

- **Global dance:**
  
  All spins/particles dance following a local dancing “rules”
  
  → The spins/particles dance collectively
  
  → a global dancing pattern.

FQH state

String liquid (spin liquid)
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of string-net liquid:
  - Up-spins from closed strings with no ends.
  - Strings can fluctuate and re-connect freely.

$\Phi_{\text{str}}(\text{string}) = \Phi_{\text{str}}(\text{string})$, $\Phi_{\text{str}}(\text{string}) = \Phi_{\text{str}}(\text{string})$

$\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{global pattern}) = 1$
Local dancing rule → global dancing pattern

- Local dancing rules of string-net liquid:
  - Up-spins from closed strings with no ends.
  - Strings can fluctuate and re-connect freely.

\[ \Phi_{\text{str}} (\text{closed string}) = \Phi_{\text{str}} (\text{closed string}), \quad \Phi_{\text{str}} (\text{fluctuating string}) = \Phi_{\text{str}} (\text{re-connected string}) \]

→ Global dancing pattern \( \Phi_{\text{str}} (\text{fluctuating string}) = 1 \)

- Local dancing rules of another string liquid:
  1. Dance while holding hands (no open ends)
  2. \( \Phi_{\text{str}} (\text{closed string}) = \Phi_{\text{str}} (\text{closed string}), \quad \Phi_{\text{str}} (\text{fluctuating string}) = -\Phi_{\text{str}} (\text{re-connected string}) \)

→ Global dancing pattern \( \Phi_{\text{str}} (\text{fluctuating string}) = (-)^{\# \text{ of loops}} \)

- Two string-net condensations → two topological orders

Levin-Wen 05

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Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings are point-like excitations, which can carry spin-1/2 and Fermi statistics?

Fidkowski-Freedman-Nayak-Walker-Wang 06
Emergence of fractional spin/statistics

• Why electron carry spin-1/2 and Fermi statistics?
• Ends of strings are point-like excitations, which can carry spin-1/2 and Fermi statistics?

Fidkowski-Freedman-Nayak-Walker-Wang 06

\[ \Phi_{\text{str}} \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) = 1 \]

String liquid \[ \Phi_{\text{str}} \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) = \Phi_{\text{str}} \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) \]

360° rotation: \[ \bullet \rightarrow \bigcirc \] and \[ \bigcirc = \bigcirc \rightarrow \bullet : R_{360°} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \bullet + \bigcirc \] has a spin 0 mod 1. \[ \bullet - \bigcirc \] has a spin 1/2 mod 1.
Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings are point-like excitations, which can carry spin-1/2 and Fermi statistics?

Fidkowski-Freedman-Nayak-Walker-Wang 06

- $\Phi_{\text{str}} = 1$ string liquid $\Phi_{\text{str}} = \Phi_{\text{str}}$

360° rotation: $\bullet \rightarrow \oplus$ and $\ominus \rightarrow \oplus$: $R_{360°} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\bullet + \ominus$ has a spin 0 mod 1. $\bullet - \ominus$ has a spin 1/2 mod 1.

- $\Phi_{\text{str}} = (-)^{\# \text{ of loops}}$ string liquid $\Phi_{\text{str}} = -\Phi_{\text{str}}$

360° rotation: $\bullet \rightarrow \ominus$ and $\oplus = -\bullet \rightarrow -\bullet$: $R_{360°} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\bullet + i\ominus$ has a spin $-1/4$ mod 1. $\bullet - i\ominus$ has a spin $1/4$ mod 1.
Spin-statistics theorem

- $(a) \rightarrow (b) = \text{exchange two string-ends.}$
- $(d) \rightarrow (e) = 360^\circ \text{ rotation of a string-end.}$
- Amplitude $(a) = \text{Amplitude (e)}$
- Exchange two string-ends plus a $360^\circ$ rotation of one of the string-end generate no phase.

$\rightarrow \textbf{Spin-statistics theorem}$
New math for entanglements $\rightarrow$ Tensor category theory

**Levin-Wen 05**

Generalize the local dancing rule

$$\Phi \left( \begin{array}{c} \text{gray box} \\ \text{gray box} \end{array} \right) = \Phi \left( \begin{array}{c} \text{gray box} \\ \text{gray box} \end{array} \right)$$

to

$$\Phi \left( \begin{array}{c} \text{gray box} \quad i \\ \text{gray box} \quad j \\ \text{gray box} \quad k \end{array} \right) = \sum_{n=0}^{N} F_{ijm}^{kmn} \Phi \left( \begin{array}{c} \text{gray box} \quad i \\ \text{gray box} \quad j \\ \text{gray box} \quad k \end{array} \right)$$

which must satisfy

$$\sum_{n=0}^{N} F_{kpn}^{mlq} F_{mns}^{jip} F_{lnr}^{jsn} = F_{qkr}^{jip} F_{mls}^{riq}$$

The theory about the solutions $=$ tensor category theory

$\rightarrow$ classify 2D gapped phases with no symmetry (topological order)
$\rightarrow$ produce non-Abelian statistics in 2D and fermions in any dim.
Long range entanglements (closed oriented strings) → emergence of electromagnetic waves (photons)

- Wave in superfluid state $|\Phi_{SF}\rangle = \sum_{\text{all position conf.}} |\rangle$:
  - density fluctuations:
  - Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
  - → Longitudinal wave

- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}} |\rangle$:
  - String density $E(x)$ fluctuations → waves in string condensed state.
  - Strings have no ends → $\partial \cdot E = 0$ → only two transverse modes.
  - Equation of motion for string density → Maxwell equation:
    $\dot{E} - \partial \times B = \dot{B} + \partial \times E = \partial \cdot B = \partial \cdot E = 0$. ($E$ electric field)
Long range entanglements (string nets) \rightarrow \text{Emergence of Yang-Mills theory (gluons)}

- If string has different types and can branch\[\rightarrow \text{string-net liquid} \rightarrow \text{Yang-Mills theory}\]
- Different ways that strings join \rightarrow \text{different gauge groups}

\[\text{Closed strings} \rightarrow \text{Maxwell gauge theory}\]
\[\text{String-nets} \rightarrow \text{Yang-Mills gauge theory}\]

A picture of our vacuum
A string-net theory of light and electrons
A unification of gauge interaction and Fermi statistics

- Previous understanding of gauge interaction:
  - Connection of a fiber bundle. Weyl 20
  - Geometric phases of quantum spins.
  - “Glue” to glue partons into physical spins (Dynamically generated gauge field)
    D'Adda & Di Vecchia & Lüscher 78, Witten 79, Baskaran & Anderson 88
  - Collective modes of long-range entanglement (of strings)
    Foerster & Nielsen & Ninomiya 80, Kitaev 97, Wen 02
    Senthil & Motrunich 02, Levin-Wen 05

- Previous understanding of emergent fermions:
  - Bound states of charge and flux (works only in 2+1D).
    Leinaas & Myrheim 77, Wilczek 82
  - Bound states of charge and monople (only for $U(1)$)
    Goldhaber 82, Wilczek 82
  - End of strings (for any dim. and any gauge groups)
    Levin-Wen 03
Can long-range entangled qubits produce everything?

At moment, the fundamental properties in our universe can be summarized by the following short list:

1. Locality.
2. Identical particles.
5. Tiny masses of fermions ($\sim 10^{-20}$ Planck mass).
6. Chiral fermions.
7. Lorentz invariance.
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**Long-range entangled qubits can produce (1–6)**

$\rightarrow$ (an extended) standard model.
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**Long-range entangled qubits can produce (1–6)** → (an extended) standard model.

**An experimental prediction:** All composite fermions must carry non-trivial gauge charge. There are additional discrete gauge theories beyond $U(1) \times SU(2) \times SU(3)$. There are new cosmic strings ($Z_2$ gauge flux) at high energies.
What is long-range entanglement

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  → all systems belong to one trivial phase

For short-range entangled (SRE) states:

- There are SRE states → one phase

\[ |\text{LRE}\rangle \neq |\text{SRE}\rangle \]

Local unitary transformation

- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different patterns of long-range entanglements
defined by the LU trans. = different topological orders

→ A category theory of topological order
Levin-Wen 05, Chen-Gu-Wen 2010
What is long-range entanglement

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break → all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are long range entangled (LRE) states
  - There are short range entangled (SRE) states

\[ |\text{LRE}\rangle \neq |\text{product state}\rangle = |\text{SRE}\rangle \]

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What is long-range entanglement

For gapped systems with no symmetry:

• According to Landau theory, no symmetry to break → all systems belong to one trivial phase

• Thinking about entanglement: Chen-Gu-Wen 2010
  - There are long range entangled (LRE) states → many phases
  - There are short range entangled (SRE) states → one phase

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• All SRE states belong to the same trivial phase

• LRE states can belong to many different phases
  = different patterns of long-range entanglements defined by the LU trans.
  = different topological orders

→ A category theory of topological order Levin-Wen 05, Chen-Gu-Wen 2010
Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ one phase

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Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

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- there are SRE symmetric states $\rightarrow$ many different phases

We may call them symmetry protected trivial (SPT) phase

---

$g_2$

| SY−SRE 1 | SB−SRE 1 |
| SY−SRE 2 | SB−SRE 2 |

$g_1$

| SY−LRE 1 | SB−LRE 1 |
| SY−LRE 2 | SB−LRE 2 |

$g_1$

| SY−LRE 1 | SB−LRE 1 |
| SY−LRE 2 | SB−LRE 2 |

| SY−SRE 1 | SB−SRE 1 |
| SY−SRE 2 | SB−SRE 2 |

phase transition
preserve symmetry
no symmetry

SPT 1 SPT 2

topological orders ( ??? )
symmetry breaking (group theory)
SPT phases ( ??? )

---

- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm.
- Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06 and 3D Moore-Balents 07; Fu-Kane-Mele 07

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Short-range entanglements w/ symmetry → SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):
- there are **LRE symmetric states** → many different phases
- there are **SRE symmetric states** → many different phases

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- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm.  
  Haldane 83
Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ many different phases

We may call them **symmetry protected trivial (SPT)** phase
or **symmetry protected topological (SPT)** phase

- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
- Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06
  and 3D Moore-Balents 07; Fu-Kane-Mele 07

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Interacting bosonic SPT phase: A group-cohomology theory

Chen-Liu-Wen 11, Chen-Gu-Liu-Wen 11

For any symmetry group $G$ and in any dimensions $d$

Two key observations:

- **Short-range-entangled states have a simple canonical form:**

  After we treat each block as an effective site.

  - Each effective site has several independent degrees of freedoms entangled with its neighbors.
  - The combined degrees of freedoms on a site form a rep. of $G$

- Each degree of freedoms on the effective site may not form a rep.
Non-trivial short-range entangled states w/ symmetry

- **Haldane phase w/ \( SO(3) \) symm.**: spin-1/2 is not a rep. of \( SO(3) \)

\[
\begin{array}{c}
\text{spin-0} & \text{spin-1} & \text{spin-1/2 \times spin-1/2} \\
\bullet & + & \\
\text{one site} & \text{one site} & \text{spin-1/2 \times spin-1/2}
\end{array}
\]

\[
\begin{array}{c}
\uparrow \uparrow \\
\text{spin-1/2 \times spin-1/2} \\
\text{one site}
\end{array}
\]

- Physical states on each site: \( (\text{spin-1/2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle \)
- The ground state wave function: \( |\Psi_{\text{CZ}}\rangle = \otimes \text{all squares} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle) \)
- The on-site \( Z_2 \) symmetry: (acting on each site \( |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle \)):
  \[ U_{\text{CZ}} = U_{\text{CZ}} U_X, \]
  \[ U_X = X_1 X_2 X_3 X_4, \]
  \[ U_{\text{CZ}} = \text{CZ}_{12} \text{CZ}_{23} \text{CZ}_{34} \text{CZ}_{41} \]
- \( Z_2 \) symm. Hamiltonian \( H = \sum \Box H_p, H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}, \)
  \[ X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle \langle \downarrow\down\down\down| + |\down\down\down\down\rangle \langle \up\up\up\up|, \]
  \[ P = |\up\up\rangle \langle \up\up| + |\down\down\rangle \langle \down\down|. \]
Non-trivial short-range entangled states w/ symmetry

- **Haldane phase w/ $SO(3)$ symm.:** spin-1/2 is not a rep. of $SO(3)$
  
  \[
  \begin{align*}
  \text{spin-0} & \quad \text{spin-1} \\
  \text{spin-1/2 x spin-1/2} & \\
  \text{one site} & \quad \text{one site}
  \end{align*}
  \]

  \[
  \begin{align*}
  \text{spin-1/2} & \quad \text{one site} \\
  \text{spin-1/2} & \quad \text{one site}
  \end{align*}
  \]

- **2D SPT phase w/ $Z_2$ symm.:**

  Chen-Liu-Wen 2011

  - Physical states on each site:
    
    \[(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle\]

  - The ground state wave function:
    
    \[|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)\]
Non-trivial short-range entangled states w/ symmetry

- **Haldane phase w/ SO(3) symm.**: spin-1/2 is not a rep. of SO(3)

  ![Diagram of spin-0 and spin-1 combined with spin-1/2 tensor product]

- **2D SPT phase w/ Z₂ symm.**:

  **Chen-Liu-Wen 2011**

  - Physical states on each site:
    \[(\text{spin-1/2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle\]
  
  - The ground state wave function:
    \[|\psi_{\text{CZX}}\rangle = \otimes_{\text{all squares}} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)\]

  - The on-site Z₂ symmetry: (acting on each site \(|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle\)):

    \[U_{\text{CZX}} = U_{\text{CZ}} U_{X}, \quad U_{X} = X_1 X_2 X_3 X_4, \quad U_{\text{CZ}} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}\]

    \[CZ : |\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle \rightarrow |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle \rightarrow |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle \rightarrow -|\downarrow\downarrow\rangle\]
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- **Haldane phase w/ \( SO(3) \) symm.:** spin-1/2 is not a rep. of \( SO(3) \)

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  - Physical states on each site:
    
    \[
    (\text{spin-1/2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle
    \]
  - The ground state wave function:
    
    \[
    |\Psi_{CZX}\rangle = \otimes_{\text{all squares}} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)
    \]
  - The on-site \( Z_2 \) symmetry: (acting on each site \( |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle \)):
    
    \[
    U_{CZX} = U_{CZ} U_X, \quad U_X = X_1 X_2 X_3 X_4, \quad U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}
    \]
    \[
    CZ : |\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle \rightarrow |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle \rightarrow |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle \rightarrow -|\downarrow\downarrow\rangle
    \]
  - \( Z_2 \) symm. Hamiltonian \( H = \sum \Box H_p, \quad H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}, \)
    \[
    X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|, \quad P = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|.
    \]

Xiao-Gang Wen, Perimeter Institute/MIT, Feb, 2013
Edge excitations for the 2D $\mathbb{Z}_2$ SPT state

- **Bulk Hamiltonian** $H = \sum \Box H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij}$, $P_{kl}$, $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$, $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

- **Edge excitations**: *gapless or break the $\mathbb{Z}_2$ symmetry, robust against any perturbations that do not break the $\mathbb{Z}_2$ symmetry.*

- **Edge effective spin** $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$.

- **Edge effective $\mathbb{Z}_2$ symmetry**: $\exp \left( \sum_i \frac{1}{4} (\tilde{Z}_i \tilde{Z}_{i+1} - 1) \right) \prod_i \tilde{X}_i$
  *which cannot be written as* $U_{\mathbb{Z}_2} = \prod_i O_i$, *such as* $U_{\mathbb{Z}_2} = \prod \tilde{X}_i$. *Not an on-site symmetry!*

- **Edge effective Hamiltonian** ($c = 1$ gapless if the $\mathbb{Z}_2$ is not broken)
  
  $H_{\text{edge}} = -J \sum \tilde{Z}_i \tilde{Z}_{i+1} + B_x \sum [\tilde{X}_i + \tilde{Z}_{i-1} \tilde{X}_i \tilde{Z}_{i+1}]$
  
  $+ B_y \sum [\tilde{Y}_i - \tilde{Z}_{i-1} \tilde{Y}_i \tilde{Z}_{i+1}]$
SPT states for any symmetry in any dimensions

- Generic SPT state: \((|↑↑↑↑⟩ + |↓↓↓↓⟩ \rightarrow \sum_{g \in G} |gggg⟩)\)

\[ |\text{SPT state}⟩ = \bigotimes_{\text{all squares}} \sum_{g \in G} |ggg⟩ \]

- Generic twisted symmetry transformations

\[ |g_1, g_2, g_3, g_4⟩ \rightarrow \eta(g_1, g_2, g_3, g_4)|gg_1, gg_2, gg_3, gg_4⟩ \]

where the twisting phase factor \(\eta(g_1, g_2, g_3, g_4)\) correspond to cocycles in \(\mathcal{H}^{d+1}[G, U_T(1)]\).
### Group cohomology $H^{d+1}[G, U_T(1)] \rightarrow$ bosonic SPT phases

<table>
<thead>
<tr>
<th>Symmetry $G$</th>
<th>$d = 0$</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1) \times Z_2^T$ (top. ins.)</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$ (0)</td>
<td>$\mathbb{Z}_2$ (0)</td>
<td>$\mathbb{Z}_2$^2 (0)</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T \times$ trans</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2^3$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2^8$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$ (spin sys.)</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$\mathbb{Z}_2$^3</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T \times$ trans</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^9$</td>
</tr>
<tr>
<td>$Z_2^T$ (top. SC)</td>
<td>0</td>
<td>$\mathbb{Z}_2$ (0)</td>
<td>0 (0)</td>
<td>$\mathbb{Z}_2$ (0)</td>
</tr>
<tr>
<td>$Z_2^T \times$ trans</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
</tr>
<tr>
<td>$U(1)$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1) \times$ trans</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
</tr>
<tr>
<td>$Z_n$</td>
<td>$\mathbb{Z}_n$</td>
<td>0</td>
<td>$\mathbb{Z}_n$</td>
<td>0</td>
</tr>
<tr>
<td>$Z_n \times$ trans</td>
<td>$\mathbb{Z}_n$</td>
<td>$\mathbb{Z}_n$</td>
<td>$\mathbb{Z}_n^2$</td>
<td>$\mathbb{Z}_n^4$</td>
</tr>
<tr>
<td>$D_{2h} = Z_2 \times Z_2 \times Z_2^T$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^9$</td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>$SO(3) \times Z_2^T$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^3$</td>
</tr>
</tbody>
</table>

Table of $H^{d+1}[G, U_T(1)]$

- **“$Z_2^T$”**: time reversal, 
- **“trans”**: translation, 
- others: on-site symm.

0 $\rightarrow$ only trivial phase.

($\mathbb{Z}_2$) $\rightarrow$ free fermion result

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Xiao-Gang Wen, Perimeter Institute/MIT, Feb, 2013

The world of long-range entanglement – from new states of q
A classification of gauge anomalies

- SPT states in \( d + 1 \) space-time dimensions are classified by \( \mathcal{H}^{d+1}[G, U(1)] \).
- The non-on-site symmetries in \( d \) space-time dimensions are classified by \( \mathcal{H}^{d+1}[G, U(1)] \).
- The non-on-site symmetries = the gauge anomalies.
- The gauge anomalies in \( d \) space-time dimensions are classified by \( \mathcal{H}^{d+1}[G, U(1)] \).

The free part of \( \mathcal{H}^{d+1}[G, U(1)] \):
\[ \rightarrow \text{Adler-Bell-Jackiw chiral-anomalies} \]

The torsion part of \( \mathcal{H}^{d+1}[G, U(1)] \):
\[ \rightarrow \text{new global anomalies (?)} \]
Many (non-exact) examples of SPT states

- **$U(1)$** bosonic SPT state in 2+1D: Lu Vishwanath 12, Senthil Levin 12, Chen Wen 12
  \[
  \mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I_\mu \partial_\mu a^J_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu \partial_\mu a^I_\lambda \epsilon^{\mu\nu\lambda}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 2k \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
  \]
  Even-integer quantize Hall conductance ($\sigma_{xy} = (2k - 2)/2\pi$).

- **$U^f(1)$** fermionic SPT state in 2+1D: Lu Vishwanath 12, Wen 12
  \[
  K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 2k_1 + 1 \\ 2k_2 + 1 \end{pmatrix}
  \]
  Hall conductance quantized as 8 times integers.

- **$SU(2)$** bosonic SPT state in 2+1D ($\lambda \to \infty$):
  \[
  S = \int_M \frac{1}{\lambda} |\partial g|^2 - i \frac{\theta}{24\pi^2} \int_M \text{Tr}(g^{-1} dg)^3, \quad \theta = 2\pi k, \quad g \in SU(2)
  \]
  $SU(2)$ symmetry acts as $g(x) \to hg(x), \quad h \in SU(2)$
  Quantized spin-Hall conductance.
Entanglement is everything

The world of long-range entanglement – from new states of quantum matter to a unification of gauge interactions and Fermi statistics.