Entanglement Entropy at 2D “Quantum Lifshitz” Critical Point

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Backgrounds

Entanglement:

- a useful characterization of quantum states

(Gapped) topological phases:

- universal “topological entanglement entropy”
- \( S_E = \alpha L - \gamma + \ldots \)

(Gapped) symmetry-protected topological phases:

- degeneracy in entanglement spectrum

Gapless critical phases

- F-theorem, CFT, AdS/CFT
- (but relatively less understood except in 1+1 D)
RVB states

Originally introduced as a candidate groundstate for quantum antiferromagnets (Anderson 1973～)

4 spins on a square: Groundstate is exactly

\[ \frac{1}{\sqrt{2}} ( \begin{align*} \text{singlet pair} \\ \text{a.k.a. “valence bond”} \end{align*} ) \]

\[ = \frac{1}{\sqrt{2}} ( | \uparrow\downarrow \rangle - | \downarrow\uparrow \rangle ) \]

\[ \sum \text{ all the valence-bond patterns} \]
Quantum Dimer Model

Model defined in terms of valence bonds (=dimers) not equivalent to the spin model, but still nontrivial……..

\[ \mathcal{H} = \sum_{\text{Plaquette}} \left[ -J \left( \left| \begin{array}{c} \uparrow \downarrow \\ \downarrow \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| + \text{H.c.} \right) + V \left( \left| \begin{array}{c} \uparrow \downarrow \\ \downarrow \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| + \left| \begin{array}{c} \uparrow \uparrow \\ \down\down \end{array} \right\rangle \left\langle \begin{array}{c} \up\up \\ \down\down \end{array} \right| \right) \right] \]

On the square lattice, RVB state appears as a critical (Rokhsar-Kivelson) point – all the phases are “valence bond crystals” (with SSB)

Rokhsar-Kivelson point

~0.2 ? 1
Quantum ↔ Classical

\[ Z = \sum_c e^{-E(c)} \]

\[ |\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\frac{1}{2} E(c)} |c\rangle \]

\[ |\psi(c)|^2 = |\langle c|\Psi\rangle|^2 = \frac{1}{Z} e^{-E(c)} \]

(Groundstate) wavefunction of 2+1 D quantum system ↔ classical stat mech in 2 D
Classical Dimer Model

Transfer matrix can be diagonalized exactly in terms of free fermions (in 1+1D)

Continuum limit at the critical point:
\[ c = 1 \] Conformal Field Theory (in 1+1D)
Quantum Lifshitz Field Theory

Ardonne-Fendley-Fradkin 2004

\[ S = \int d^3 x \left[ \frac{1}{2} (\partial_t \phi)^2 - \frac{\kappa^2}{2} (\nabla^2 \phi)^2 \right] \]

\[ H = \int d^2 x \ Q^\dagger(\vec{x}) Q(\vec{x}) \]

\[ Q(x) = \frac{1}{\sqrt{2}} \left( \frac{\delta}{\delta \phi} + \kappa \nabla^2 \phi \right) \]

Groundstate wavefunction

\[ \Psi_0[\phi] = \frac{1}{\sqrt{Z}} e^{-\frac{\kappa}{2} \int d^2 x (\nabla \phi(x))^2} \]

\[ |\psi(c)|^2 = |\langle c|\Psi\rangle|^2 = \frac{1}{Z} e^{-E(c)} \]
Entanglement Entropy?

= classical (Shannon) entropy in the corresponding classical stat mech model

CFT calculation (Hsu-Mulligan-Fradkin-Kim 2009)
Lattice calculation (Stephan-Furukawa-Misguich-Pasquier 2009)

“area (perimeter) law” + universal constant (same form as in gapped topological phases)

\[
S_E = \log \left( \sqrt{2gR} \right) - \frac{1}{2}.
\]

\[
\mathcal{L} = \frac{g}{4\pi} (\partial_\mu \phi)^2.
\]

\[
\phi \sim \phi + 2\pi R
\]
Lattice calculation (cylinder)

Stephan-Furukawa-Misguich-Pasquier 2009
Numerics and constant in EE

\[ S_{\text{topo}}^{\text{KP}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}, \]

Kitaev-Preskill / Levin-Wen proposed geometrical schemes to cancel out the leading non-universal area-law term

\[ S_E \sim \alpha \mathcal{L} - \gamma \]

But in practical numerical calculations, “naive” fitting of data on cylinder often works better (Furukawa-Misguich 2007) ↔ curvature expansion (Grover-Turner-Vishwanath 2011)
Replica approach

\[ S_E = -\text{Tr} \left( \rho_A \log \rho_A \right) \]

\[ S_E = -\frac{\partial \text{Tr} \rho_A^n}{\partial n} \bigg|_{n=1} \]

\[ \rho_A = \text{Tr}_B |\Psi_0\rangle \langle \Psi_0| \]

replica fields

\[ \text{Tr} \rho_A^n = \frac{Z_P}{Z_F} \]

\[ \phi_1, \phi_2, \ldots \phi_n \]

partition function without any restriction at the boundary

\[ Z_F = (z_F)^n \]

partition function with \( \phi_1 = \phi_2 = \ldots = \phi_n \) at the boundary \( \Gamma \) between regions A and B
Resolution of the b.c.?

\[ \varphi_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_j \]

\[ \varphi_1 = \frac{1}{\sqrt{2}} (\phi_1 - \phi_2) \]

... \[ \varphi_{n-1} = ... \]

"center-of-mass" boson \( \varphi_0 \) is free at the boundary \( \Gamma \) all the other \( \varphi_j \)'s obey the Dirichlet b.c. on \( \Gamma \)

\[ \varphi_j = 0 \quad (j = 1, \ldots, n - 1) \]

\[ Z_P = (z_D)^{n-1} z_F \]
Resolution of the b.c.?

\[ Z_P = (z_D)^{n-1} z_F \]
\[ Z_F = (z_F)^n \]

\[ \text{Tr} \rho_A^n = \frac{Z_P}{Z_F} = \left( \frac{z_D}{z_F} \right)^{n-1} \]

nonuniversal, extensive part of the boundary free energy

\[ \leftrightarrow \text{“area-law” term in EE} \]

universal boundary entropy (Affleck-Ludwig)

\[ \leftrightarrow \text{universal constant part in EE} \]

But the result

\[ S_E = \log \left( \sqrt{2gR} \right) \]

does not agree w/ lattice calculation?
Pitfall

Lagrangian of the free boson theory is invariant under the "change of basis"

\[
\varphi_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_j
\]

\[
\varphi_1 = \frac{1}{\sqrt{2}} (\phi_1 - \phi_2)
\]

\[
\ldots
\]

\[
\varphi_{n-1} = \ldots
\]

.... but the compactification is NOT invariant!

\[
\phi_j \sim \phi_j + 2\pi R \
\]

\[
\varphi_j \sim \varphi_j + 2\pi R
\]
Simple Example

- c=1 free boson on torus
  - periodic b.c.
  - folding

- c=2 free boson on cylinder w/ boundary condition
  - \( \phi_1 = \phi_2 \) at the ends
Resolution of b.c.?

\[ \Phi_0 = \frac{\phi_1 + \phi_2}{\sqrt{2}}, \]
\[ \Phi_1 = \frac{\phi_1 - \phi_2}{\sqrt{2}}, \]

\( \Phi_0 \) remains free (Neumann b.c.) at the ends
\( \Phi_1 = 0 \) (Dirichlet b.c.) at the ends

\[ Z_{\text{simple}}(q) = z_{DD}(R, q)z_{NN}(R, q) \]

But this does not agree with the torus partition function (obtained without folding)
Compactification Lattice

correct partition fn. is obtained by proper treatment of the lattice

cf.) Wong-Affleck 1994

\[
\Phi_0 \sim \Phi_0 + 2\pi n_0 \frac{R}{\sqrt{2}},
\]

\[
\Phi_1 \sim \Phi_1 + 2\pi n_1 \frac{R}{\sqrt{2}},
\]

\[n_0 \equiv n_1 \mod 2.\]

\[
\Theta_0 \sim \Theta_0 + 2\pi m_0 \frac{1}{\sqrt{2gR}},
\]

\[
\Theta_1 \sim \Theta_1 + 2\pi m_1 \frac{1}{\sqrt{2gR}},
\]

\[m_0 \equiv m_1 \mod 2.\]
Boundary state

conformally invariant b.c. $\leftrightarrow$ “boundary state”
Boundary states of free bosons

conformal invariance ⇒ $(L_m - \bar{L}_{-m}) |B\rangle = 0$

boundary state is a superposition of “Ishibashi states”

Virasoro generators

$$L_m = \frac{1}{2} \sum_l : \bar{\alpha}^L_{m-l} \alpha^L_l :$$

$$\bar{\alpha}^L_n = \begin{cases} -i \sqrt{n} \bar{\alpha}^L_n & (n > 0) \\ \frac{1}{\sqrt{2}} \left( \sqrt{g} \bar{R} + \frac{1}{\sqrt{g}} \bar{K} \right) & (n = 0) \\ i \sqrt{n} (\bar{\alpha}^L_{-n})^\dagger & (n < 0) \end{cases}$$
Boundary states of free bosons

CMT applications: Chamon-MO-Affleck 2005, etc.

sufficient condition for the conformal inv.

solution (Ishibashi states)

$$|(\vec{R}, \vec{K})\rangle = \exp \left( -\sum_{n=1}^{\infty} (\vec{a}_n^L)^\dagger \mathcal{R} (\vec{a}_n^R)^\dagger \right) |(\vec{R}, \vec{K})\rangle$$

$$|\vec{R}, \vec{K}\rangle : \text{vacuum w/ winding numbers}$$

$$\vec{\phi} \sim \vec{\phi} + 2\pi \vec{R} \quad \vec{\theta} \sim \vec{\theta} + \frac{2\pi}{g} \vec{K}$$

$$\left( \sqrt{g} \vec{R} + \frac{1}{\sqrt{g}} \vec{K} \right) = \mathcal{R} \left( -\sqrt{g} \vec{R} + \frac{1}{\sqrt{g}} \vec{K} \right)$$

Dirichlet: $\mathcal{R} = 1$

Neumann: $\mathcal{R} = -1$
Boundary state of free bosons

cylinder amplitude

modular transf. to open string channel

\[ Z_{AB}(\tilde{q}) = \langle A | e^{-L\hat{H}_P} | B \rangle \]

\[ Z_{AB}(q) = \sum_h N^h_{AB} \chi^\text{Vir}_h(q) \]

To satisfy the Cardy condition, physical boundary state is given by an infinite superposition of Ishibashi states

non-negative integer (Cardy condition)
Back to Entanglement Entropy

Dirichlet b.c.

Dirichlet b.c.

$\beta$

$A$

$B$

$\Gamma$

$L$

$L$

$L$
Boundary condition

\[ \phi_1 = \phi_2 = \ldots = \phi_{N-1} = \phi_N \]

(roughly speaking) Neumann on and Dirichlet on other l.c.'s

\[ \Phi_0 \equiv \frac{1}{\sqrt{\mathcal{N}}} \sum_{j=1}^{\mathcal{N}} \phi_j \]

\[ \mathcal{R} = 1 - 2\vec{d}\vec{d}^T \]

\[ \vec{d} = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} \]
Boundary state for “replica” bc

\[ |\mathcal{P}(\vec{\theta}_0, \vec{\phi}_0)\rangle = g_{\mathcal{P}} \sum_{\vec{R}=n_0 R d, \vec{K} \in \Xi_{D\mathcal{P}}} e^{-i(\vec{\theta}_0 \cdot \vec{R} + \vec{\phi}_0 \cdot \vec{K})} |(\vec{R}, \vec{K})\rangle \]

\[ \Xi_{D\mathcal{P}} = \{ \vec{K} \in \Lambda^*/\sqrt{g} | \vec{d} \cdot \vec{K} = 0 \} \]

boundary entropy (Affleck-Ludwig)

\[ g_{\mathcal{P}} = \left( \frac{g}{2} \right)^{\mathcal{N}/4} R^{\mathcal{N}/2} v_0(\Xi_{D\mathcal{P}}) \]

\[ v_0(\Xi_{D\mathcal{P}}) \] unit cell volume of the lattice
Geometry of the lattice

\[ v_0(\frac{\Lambda^*}{\sqrt{g}}) = \frac{1}{\sqrt{gNR}} v_0(\Xi_{DP}) \]

\[ v_0(\Xi_{DP}) = \sqrt{2g}^{-\frac{(N-1)}{2}} R^{-\frac{(N-1)}{2}} \frac{N}{2} \]

\[ g_{\mathcal{P}} = (\sqrt{2gR})^{-(n-1)} \sqrt{n} \]

\[ S_E = \log \left( \sqrt{2gR} - \frac{1}{2} \right) \]
Lessons

EE at quantum Lifshitz critical point in 2+1D
= Classical (Shannon) entropy in 2D
= Affleck-Ludwig boundary entropy of 1+1D CFT
EE alone does not quite distinguish it from gapped topological phases (except for the sign in some cases)

“Changing the basis” trick works for free bosons, but care should be taken on compactification

Replica trick works for free bosons?
(cf. phase transition at $n=n_c$ ,
replica-free formulation Stephan-Misguich-Pasquier 2011)
Open problems

Interacting CFT: “changing the basis” does not work
(Lagrangian density is not invariant)

e.g. Ising model
quantum $\text{EE} = \text{classical (Shannon) entropy}$
still holds if

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_c e^{-\frac{1}{2} E(c)} |c\rangle$$
Numerical Result for Ising

Stephan-Furukawa-Misguich-Pasquier 2009

Their conjecture:

\[ S_E = \begin{cases} 
0 & (n > 1) \\
-0.4387(1) & (n = 1) \\
- \log 2 & (n < 1) 
\end{cases} \]

how to prove/disprove?
Take the same approach?

Need to solve 2n layers of Ising model coupled only at the boundary

But this is as difficult as 2n layers of Ising model coupled also in the boundary ($c=n/2$)
- I don’t know how to solve in general

cf.) $2n=2$ layers $\leftrightarrow$ defect line in the Ising model
    $\leftrightarrow c=1$ CFT ($\mathbb{Z}_2$ orbifold of free boson)

M.O.-Affleck 1996