Review of Holographic (and other) computations of Entanglement Entropy, and its role in gravity
Entanglement Entropy

• what is entanglement entropy?
  general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
• in QFT, typically introduce a (smooth) boundary or entangling surface $\Sigma$ which divides the space into two separate regions
• integrate out degrees of freedom in “outside” region
• remaining dof are described by a density matrix $\rho_A$

→ calculate von Neumann entropy: $S_{EE} = -Tr[\rho_A \log \rho_A]$
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$

  \[
  S_{EE} = -Tr [\rho_A \log \rho_A]
  \]

- result is UV divergent!

- must regulate calculation: $\delta = \text{short-distance cut-off}$

  \[
  S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots
  \]

- careful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{A_\Sigma}{\delta^{d-2}} + \cdots$
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$

\[ S_{EE} = -Tr[\rho_A \log \rho_A] \]

(t = constant)

- must regulate calculation: $\delta = \text{short-distance cut-off}$

\[ S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots \]

$d = \text{spacetime dimension}$

- leading coefficients sensitive to details of regulator, eg, $\delta \rightarrow 2\delta$

- find universal information characterizing underlying QFT in subleading terms, eg,

\[ S = \cdots + c_d \log (R/\delta) + \cdots \]
General comments on **Entanglement Entropy**:

• nonlocal quantity which is (at best) very difficult to measure
  → no accepted experimental procedure

• in **condensed matter theory**: diagnostic to characterize quantum critical points, quantum spin fluids or topological phases

• in **quantum information theory**: useful measure of quantum entanglement (a computational resource)
General comments on **Entanglement Entropy**:

Where did “Entanglement Entropy” come from? \(\rightarrow\) black holes

- **Sorkin ’84**: looking for origin of black hole entropy
  \[ S_{BH} = A / 4G_N \]

- recall that leading term obeys “area law”:
  \[ S = c_0 \frac{A_S}{\delta^{d-2}} + \cdots \]
  \(\rightarrow\) suggestive of Bekenstein-Hawking formula if \(\delta \simeq \ell_P\)

  (Sorkin ’84; Bombelli, Koul, Lee & Sorkin; Srednicki; Frolov & Novikov; . . .)

- **problem?**: leading singularity not universal; regulator dependent
- active topic in 90’s but story left unresolved \((\text{return to this later})\)

- recently considered in **AdS/CFT correspondence**
  (Ryu & Takayanagi ’06)
**AdS/CFT Correspondence:**

**Bulk:** gravity with negative $\Lambda$ in $d+1$ dimensions

**Boundary:** quantum field theory in $d$ dimensions

"holography"

anti-de Sitter space

conformal field theory

radius $\leftrightarrow$ energy

energy
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{\partial V = \Sigma} \frac{A_V}{4G_N} = \infty!! \]

- "UV divergence" because area integral extends to \( r = \infty \)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{\partial V = \Sigma} \frac{A_V}{4G_N} \]

- “UV divergence” because area integral extends to \( r = \infty \)
- finite result by stopping radial integral at large radius: \( r = R_0 \)
  
\[ \text{short-distance cut-off in boundary theory:} \quad \delta = \frac{L^2}{R_0} \]

(Ryu & Takayanagi `06)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext}_{\partial V = \Sigma} \frac{A_V}{4G_N} \approx \frac{L^{d-1}}{G_N} \frac{A_\Sigma}{\delta^{d-2}} + \cdots \]

central charge (counts dof) \( (L/\ell_{\text{Planck}})^{d-1} \)

"Area Law"
Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)

cut-off in boundary CFT:
\[ \delta = L^2 / R_0 \]

general expression (as desired):
\[ S(A) \simeq c_0 (R/\delta)^{d-2} + c_1 (R/\delta)^{d-4} + \cdots \]

\[ + c_{d-2} \log(R/\delta) + \cdots \quad \text{(d even)} \]

\[ + c_{d-2} + \cdots \quad \text{(d odd)} \]

universal contributions
Holographic Entanglement Entropy:

\[ S(A) = \text{ext} \frac{A_V}{\partial V = \Sigma 4G_N} \]  

**conjecture**

Extensive consistency tests:

1) leading contribution yields “area law”

\[ S \approx \frac{L^{d-1}}{G_N} \frac{A_\Sigma}{\delta^{d-2}} + \cdots \]

2) recover known results of Calabrese & Cardy for \( d=2 \) CFT

\[ S = \frac{c}{3} \log \left( \frac{C}{\pi \delta \sin \frac{\pi \ell}{C}} \right) \]

(\text{also result for thermal ensemble})

\( C = \text{circumference} \)

(Ryu & Takayanagi ’06)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext} \frac{A_V}{4G_N} \]

conjecture

Extensive consistency tests:

3) \( S(A) = S(\bar{A}) \) in a pure state

\( A \) and \( \bar{A} \) both yield same bulk surface \( V \)
Holographic Entanglement Entropy:

\[ S(A) = \text{ext} \frac{A_V}{4G_N} \]

**conjecture**

Extensive consistency tests:

3) \( S(A) = S(\bar{A}) \) in a pure state

\( A \) and \( \bar{A} \) both yield same bulk surface \( V \)

**cf:** thermal ensemble \( \neq \) pure state

horizon in bulk \( S(A) \neq S(\bar{A}) \)

AdS/CFT Dictionary: thermal bath \( \leftrightarrow \) black hole

(Ryu & Takayanagi `06)
Holographic Entanglement Entropy:

\[ S(A) \ = \ \text{ext} \frac{A_V}{4G_N} \]

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double

(Ryu & Takayanagi `06)

(Headrick)
Holographic Entanglement Entropy:

\[
S(A) = \text{ext} \frac{A_V}{\partial V = \sum 4G_N}
\]

conjecture

Extensive consistency tests:

5) sub-additivity: \( S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \)

[ “all” other inequalities: Hayden, Headrick & Maloney]
Holographic Entanglement Entropy: 

\[ S(A) = \text{ext} \left( \frac{A_V}{4G_N} \right) \]

conjecture

Extensive consistency tests:

4) Entropy of eternal black hole = 
   entanglement entropy of boundary CFT & thermofield double

5) sub-additivity: 
   \[ S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \]
   (Headrick & Takayanagi)
   (Hayden, Headrick & Maloney)

for more general holographic framework, expect

\[ S'(A) = \text{ext} \left[ S_{\text{horizon}} \right] + \cdots \]

includes \( \alpha' \) corrections

g_s corrections
for more general holographic framework, expect

\[
S'(A) = \text{ext} \left[ S_{\text{horizon}} \right] + \cdots
\]

\[
\partial V = \Sigma
\]

(deBoer, Kulaxizi & Parnachev)

(Hung, Myers & Smolkin)

some progress with classical higher curvature gravity:

• note \( S_{\text{horizon}} \) is not unique! and \( S_{\text{Wald}} \) is wrong choice!

• correct choice understood for “Lovelock theories”

• test with universal term for \( d=4 \) CFT:

\[
S_{\text{uni}} = \log(R/\delta) \frac{1}{2\pi} \int_{\Sigma} d^2x \sqrt{h} \left[ c \left( C^{ijkl} \hat{g}_{ik} \hat{g}_{jl} - K^i_b K^j_a + \frac{1}{2} K^i_a K^j_b \right) - a R \right]
\]

(Solodukhin)
Topics currently trending in Holographic $S_{EE}$:

• suggest C/F-theorems  
  (Myers & Sinha; Liu & Mezei; . . .)

• diagnostic of holo-fermi surfaces  
  (Ogawa, Takayanagi & Ugajin; Huijse, Sachdev & Swingle; . . .)

• probe of holo-quantum quenches  
  (Abajo-Arrastia, Aparicio, Lopez; Balasubramanian et al; . . .)

• probe of holo-RG flows  
  (Albash & Johnson; Myers & Singh; . . .)

• probe of holo-insulator/superconductor transition  
  (Albash & Johnson; Cai, He, Li & Li . . .)

• probe of holo-fractionalized states  
  (Albash, Johnson & Macdonald; Hartnoll & Huijse, . . .)

• probe of holo-QFT in curved/deSitter space  
  (Maldacena & Pimental)

• probe of holo-logarithmic CFT’s  
  (Grumiller, Riedler, Rosseel & Zojer)

  . . . .
Lessons for bulk gravity theory:

????
AdS/CFT Dictionary:

**Boundary**: thermal plasma  \(\rightarrow\)  **Bulk**: black hole

Temperature

Energy

Entropy
Lessons for bulk gravity theory:

\[(\text{entanglement entropy})_{\text{boundary}} = (\text{entropy associated with extremal surface})_{\text{bulk}}\]

What Entropy?/Entropy of what?

not black hole! not horizon!
not causal domain!

(eg, Hubney, Rangamani & Takayanagi)

What are the rules?
Proposal: Geometric Entropy

- In a theory of quantum gravity, for any sufficiently large region with a smooth boundary in a smooth background (e.g., flat space), the leading contribution to entanglement entropy takes the form:

\[ S_{EE} = \frac{A_\Sigma}{4G_N} \quad + \quad \ldots \]

- In QG, (short range) quantum entanglement corresponding to area law is a signature of macroscopic spacetime geometry.
Lessons for bulk gravity theory:

\[(\text{entanglement entropy})_{\text{boundary}} = (\text{entropy associated with extremal surface})_{\text{bulk}}\]

- R&T construction assigns entropy \(S_{BH} = A/4G_N\) to bulk regions with “unconventional” boundaries:
  - not black hole! not horizon!
  - not causal domain!

- indicates \(S_{BH}\) applies more broadly!

- what about extremization?
  - needed to make match above

- other surfaces might be expected to give other entropic measures of entanglement in boundary theory

(Hubeny & Rangamani; Balasubramanian, McDermott & van Raamsdonk)
Proposal: Geometric Entropy

- in a theory of quantum gravity, for any sufficiently large region with a smooth boundary in a smooth background (e.g., flat space), there is an entanglement entropy which takes the form:

\[ S_{EE} = \frac{A_\Sigma}{4G_N} + \cdots \]

- evidence comes from several directions:
  1. holographic \( S_{EE} \) in AdS/CFT correspondence
  2. QFT renormalization of \( G_N \)
  3. Randall-Sundrum 2 model
  4. spin-foam approach to quantum gravity
  5. Jacobson’s “thermal origin” of gravity
Where did “Entanglement Entropy” come from? → black holes

• **Sorkin ’84**: looking for origin of black hole entropy
  \[ S_{BH} = \mathcal{A}/4G_N \]

• recall that leading term obeys “area law”: \[ S = c_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]
  suggestive of BH formula if \( \delta \simeq \ell_P \)
  (Sorkin ’84; Bombelli, Koul, Lee & Sorkin; Srednicki; Frolov & Novikov)

• **problem?**: leading singularity not universal; regulator dependent

• **resolution**: this singularity represents contribution of “low energy” d.o.f. which actually renormalizes “bare” area term \( S_0 = \mathcal{A}/4G_0 \)
  (Susskind & Uglum)

\[
\delta \left( \frac{1}{G} \right) = \frac{4c_0}{\delta^{d-2}} \quad \text{and} \quad S = c_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots
\]

both coefficients are regulator dependent but for a given regulator should match!
BH Entropy \sim \text{Entanglement Entropy}

\[ \delta \left( \frac{1}{G} \right) = \frac{4c_0}{\delta^{d-2}} \]

\[ S = c_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]

both coefficients are regulator dependent but for a given regulator should match!

• “a beautiful killed by ugly facts”?? 😞

• seems matching is not always working??

(Demers, Lafrance & Myers; Kabat, Strassler, Frolov, Solodukhin, Fursaev, . . .)

• all numerical factors seem to be resolved! 😊

(Cooperman & Luty)

matching of area term works for any QFT (s=0, 1/2, 1, 3/2)
to all orders in perturbation theory for any Killing horizon

• technical difficulties for spin-2 graviton

♥ • results apply for Rindler horizon in flat space

(some technical/conceptual issues may remain! Jacobson & Satz)
BH Entropy \sim\text{Entanglement Entropy} \\
(Susskind \& Uglum)
\[ \delta \left( \frac{1}{G} \right) = \frac{4c_0}{\delta^{d-2}} \]
\[ S = c_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots \]

both coefficients are regulator dependent 
but for a given regulator should match!

• “a beautiful killed by ugly facts”??

• result “unsatisfying”:
\[ S_{BH} = S_0 + S_{EE} = \left( \frac{A}{4G_0} \right) + \frac{A}{4} \delta \left( \frac{1}{G} \right) + \cdots \]

where did bare term come from?

• formally “off-shell” method is precisely calculation of \( S_{EE} \)
(Susskind \& Uglum; Callan \& Wilczek; Myers \& Sinha: extends to \( S_{wald} \))

• challenge: understand microscopic d.o.f. of quantum gravity

\[ \text{AdS/CFT: eternal black hole (or any Killing horizon)} \]
(Maldacena; van Raamsdonk et al; Casini, Huerta \& Myers)
• return to considerations of $S_{EE}$ of general region in some QFT
• first step in calculation of $S_{EE}$ is to determine $\rho_A$

• $\rho_A$ encodes standard correlators, eg, if global vacuum:

$$\text{Tr}(\rho_A \phi(x)\phi(y)) = \langle 0|\phi(x)\phi(y)|0\rangle$$

• by causality, $\rho_A$ describes physics throughout causal domain $\mathcal{D}$
Entanglement Hamiltonian:

- hermitian and positive semi-definite, hence $\rho_A = e^{-H}$
  
  $H$ = modular or **entanglement Hamiltonian**

- formally can consider evolution by $U(s) = \rho_A^{is} = e^{-iHs}$

- unfortunately $H$ is **nonlocal** and flow is nonlocal/not geometric

\[
H = \int d^{d-1}x \gamma_1^{\mu\nu}(x) T_{\mu\nu} + \int d^{d-1}x \int d^{d-1}y \gamma_2^{\mu\nu;\rho\sigma}(x, y) T_{\mu\nu} T_{\rho\sigma} + \cdots
\]
Entanglement Hamiltonian:

- hermitian and positive semi-definite, hence $\rho_A = e^{-H}$

$H$ = modular or entanglement Hamiltonian

- $H$ explicitly known in only a few instances
- most famous example: Rindler wedge

$H = \text{boost generator}$

$$= 2\pi K = -2\pi \int_{A(x>0)} d^{d-2}y \, dx \left[ x \, T_{00} \right]$$
Entanglement Hamiltonian:

Can this formalism provide insight for “area law” term in $S_{EE}$?

$L_{geo}L_{geom} \gg \delta \gg \delta$

- zoom in on infinitesimal patch of $\partial \mathcal{D}$
  - region looks like flat space
  - $\partial \mathcal{D}$ looks like Rindler horizon
- assume Hadamard-like state correlators have standard UV sing’s
- UV part of $\rho_A$ same as in flat space
- $H$ must have Rindler term: $H = 2\pi K + \cdots$
Entanglement Hamiltonian:

Can this formalism provide insight for “area law” term in $S_{EE}$?

for each infinitesimal patch:

- UV part of $\rho_A$ must be same as in flat space
- $H$ must have Rindler term:
  \[ H = 2\pi K + \cdots \]
  Rindler $H$ yields area law; hence $\delta S_{EE} = c_0 \delta A_\Sigma/\delta^{d-2} + \cdots$
  hence $S_{EE}$ must contain divergent area law contribution!
- invoke Cooperman & Luty: area law divergence matches precisely renormalization of $1/G$
  \[ S_{EE} = \frac{A}{4} \delta \left( \frac{1}{G} \right) + \cdots \]
  for any large region of smooth geometry!!
Entanglement Hamiltonian:

Can this formalism provide insight for “area law” term in $S_{EE}$?

- Cooperman & Luty: area law divergence matches precisely renormalization of $1/G$:

$$S_{EE} = \frac{A}{4} \delta\left(\frac{1}{G}\right) + \cdots$$

for any large region of smooth geometry!!

- we can formulate a formal geometric argument analogous to the “off-shell” method for black holes (or Rindler horizons)

- **challenge**: understand microscopic d.o.f. of quantum gravity
Proposal: Geometric Entropy

• in a theory of quantum gravity, for any sufficiently large region with a smooth boundary in a smooth background (e.g., flat space), there is an entanglement entropy which takes the form:

\[ S_{EE} = \frac{A_\Sigma}{4G_N} + \cdots \]

• evidence comes from several directions:

1. holographic \( S_{EE} \) in AdS/CFT correspondence
2. QFT renormalization of \( G_N \)
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MERA & Holography:

- **Multi-scale Entanglement Renormalization Ansatz**: approximate and efficient approach to build ground-state wavefunction of critical lattice models
  
  - close connection between MERA and AdS/CFT

(Vidal; . . .)

(Swingle)
MERA & Holography:

- **Multi-scale Entanglement Renormalization Ansatz**: approximate and efficient approach to build ground-state wavefunction of critical lattice models (Vidal; ...)
- close connection between MERA and AdS/CFT (Swingle)

- short-hand:

  \[ \begin{align*}
  & \begin{array}{c}
  \times \\
  \hline
  \times
  \end{array} = \begin{array}{c}
  \text{rectangle}
  \end{array} \\
  \begin{array}{c}
  \times \\
  \hline
  \bullet
  \end{array} = \begin{array}{c}
  \text{triangle}
  \end{array}
  \end{align*} \]

- question: where is the entanglement entropy?
- introduce conjugate circuit
- trace over exterior bonds
• introduce conjugate circuit
• trace over exterior bonds
• $2^k (= L_{\text{eff}})$ bonds $\rightarrow$ 1 bond in $k$ layers
• each layer contributes $c/3$ to $S_{\text{EE}}$

$S_{\text{EE}} = c/3 \log(L_{\text{eff}})$
$S_{EE}$ and $S_{Ren}$ are unchanged by
\[ \hat{\rho} = U^\dagger \rho U \]
• $S_{EE}$ and $S_{Ren}$ are unchanged by
\[ \hat{\rho} = U^\dagger \rho U \]
• only narrow band of circuit contributes to $S_{EE}$ and $S_{Ren}$
• only narrow band of circuit contributes to $S_{EE}$ and $S_{Ren}$
**MERA & Holography:**

- **Multi-scale Entanglement Renormalization Ansatz:** approximate and efficient approach to build ground-state wavefunction of critical lattice models (Vidal; ...)

- Close connection between MERA and AdS/CFT (Swingle)

- Question: where is the entanglement entropy? Only narrow band of circuit contributes to $S_{EE}$ and $S_{Ren}$

- RT prescription: $S_{EE}$ depends on geometry near extremal surface

- Common theme:
  - Long-range entanglement $\rightarrow$ short-range entanglement

- MERA might provide insight for questions:
  - $\rightarrow$ nature of extremization procedure?
  - $\rightarrow$ information in non-extremal surfaces?
  - $\rightarrow$ calculating Renyi entropies?
Conclusions:

• holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

Dialogue:

• potential to learn lessons about issues in boundary theory
• potential to learn lessons about issues in quantum gravity in bulk

Lots to explore!