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Conformal Field Theories in 2+1 Dimensions: Entanglement Entropy and F-Theorem


In 2+1 dimensional CFT, the Entanglement Entropy (EE) across a circle of radius R is

\[ S_E = \frac{R}{\varepsilon} \]

Carmi, Hsu, Myers

F = \frac{\log 12}{\sqrt{3}}

F generalizes to CFT

Euclidean Path Integral of the CFT on a round 3-sphere (after subtracting a term found in gapped theories).

UV CFT \Ndash; F-theorem

For RG flows connecting different CFT_s, F\text{UV} > F\text{IR}

IR CFT

For a free massless real scalar,

\[ F = \frac{\log 2}{8} - \frac{3\sqrt{3}}{16\pi^2} \approx 0.0638 \]

Add a mass so that the theory becomes infrared at long distances

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 \] relevant operator

UV CFT
The "Renormalized Entanglement Entropy"  

$$F = R \frac{ds}{dR} - S(R)$$

where $S(R)$ is the EE of the disk of radius $R$ in non-conformal Lorentz invariant theory. The UV cut-off dependence cancels!

Here $F$ is a function of $g = m^2 r^2$.

Calculate $F(g)$ numerically using radial discretization and a finite number of angular momentum modes.

$F(\nu) \approx 0.063\nu$;  \hspace{1cm} $F(g \to \infty) \sim \frac{1}{g} + ...$

$F(g)$ smoothly interpolates between $F_{\nu=0}$ and $F_{g=0}$.

In general $F' = R S''(R)$.

$S''(R) < 0$ (proven by Carini and Herada using the strong subadditivity and Lorentz inv. of EE).

However, numerical results show that $F'(\nu) < 0$, i.e. $F$ is not stationary at the UV CFT.

Does this mean that RG flow in $d+1$ is not a gradient flow? Or is there a better interpolating function that is monotone and stationary?
How do we calculate $F$ in strongly coupled CFT's?

**Method 1:** AdS/CFT

If a large $N$ CFT is dual to M-theory on $AdS_4 \times Y^7$ with $N$ units of flux, then

$$F \sim \frac{N^{3/2}}{\text{Vol}(T^2)}$$

**Method 2:** If the CFT has $N=2$ supersymmetry, we can use "localization" + F-maximization to calculate divergences of chiral operators and $F$ exactly. (Kapustin, Wilczeck, Yaakov, Tikhov)

When both methods are applicable, they agree $\implies$ tests of AdS/CFT using localization.

**Example of Method 2:** $N=2$ SUSY QED with

If superfields $\Phi^i$ of charge $+1$ and $\Phi^i \bar{\Phi}^i$,

Each superfield = complex scalar + Dirac fermion

Exact path integral

$$Z = \int \mathcal{D}[\Phi, \bar{\Phi}] e^{\frac{1}{g^2} \sum \left[ \lambda(1 - A^2 + \Delta) + \lambda(1 - A + i\Delta) \right]}$$

$\Delta = \text{spacing} = \text{R-charge of fields } \Phi, \bar{\Phi}$

$$\lambda(z) = -\log(1 - e^{-2\pi i z}) + \frac{1}{2} \left( \gamma^2 + \frac{1}{4} \log^2 (e^{2\pi i}) - \frac{\pi^2}{4} \right)$$
Maximize $F$ and to final $R$-charge $\Delta$

This gives (for large $N_f$)

$$
\Delta = \frac{1}{2} - \frac{1}{N_f} + O\left(\frac{1}{N_f^2}\right)
$$

$$
F = N_f \log 2 + \frac{1}{2} \log \left(\frac{N_f}{2}\right) + \frac{1}{N_f} \left(-\frac{1}{4} + \frac{2}{N_f^2}\right) + O\left(\frac{1}{N_f^2}\right)
$$

More precise results found numerically $\frac{1}{N_f}$ expansion good down to $N_f = 1$

$\Delta (N_f = 1) = \frac{1}{3}$ due to a certain "mirror symmetry".

Even for $N_f = 1$ the theory is unitary $\Rightarrow$

$\Rightarrow$ no lower boundary of the "conformal window".

"Chiral operators" $\Phi, \bar{\Phi}$ have dimension $2\Delta$

which is above the unitarity bound $\frac{1}{2}$ for scalar operators.

Another $N=2$ model $U(1)$ theory coupled to $2N_f$ superfields $\Phi$ of the

same charge $+$

$$
Z = \int d\lambda e^{2\lambda} e^{(1-\Delta)\lambda}
$$

Note, $Z = -\log|Z|$ is independent of $\Delta$

Can calculate numerically or develop $\frac{1}{N_f}$ expansion

$$
F = N_f \log 2 + \frac{1}{2} \log \left(\frac{N_f}{2}\right) + \frac{1}{N_f} \left(-\frac{1}{4} + \frac{10}{3N_f^2}\right) + O\left(\frac{1}{N_f^2}\right)
$$
\[ Z(N_f, N_c) = \frac{1}{N_c!} \prod_{j=1}^{N_c} \left( \frac{N_c}{\pi} \right)^{N_f} e^{N_f(1-\Delta+j)} e^{-(1-\Delta-j)} + e^{-(1-\Delta+j)} \right] \]
\[ \times \prod_{\alpha < \beta} \left( 2 \sinh(\Delta_j - \Delta_k) \right)^2 \]

For \( \Delta = \frac{1}{2} \), \( e(1-\Delta+j) e(1-\Delta-j) = -\log(\cos(\alpha \Delta)) \)

For general \( \Delta \), it behaves as \( -2\pi(1-\Delta)/\Delta \) at large \( \Delta \)

The large \( \Delta \) behavior of integrand: \( e^{2\pi(1-\Delta)/(\Delta - 1 - 4\Delta - 4)} \)

Integral converges only if \( N_f (1-\Delta) - N_c + 1 > 0 \)

The dimension of the "monopole" operators is \( \Delta_v = N_f (1-\Delta) - N_c + 1 \), \( \Delta_v \geq \frac{1}{2} \) by unitarity

For \( N_c = N_f = 2 \), \( \Delta = \frac{1}{4} \) and \( \Delta_v = \frac{1}{2} \)

Both mesons and monopoles must be light at the bound.

For \( N_c > 2 \), find theories with \( N_f > N_c \) where \( \Delta_v < \frac{1}{2} \Rightarrow F\)-maximization stops making sense! What to do?

Answer: monopole fields decouple and become free

Consider Veneziano limit: \( N_c \to \infty \), \( x = N_f/N_c \) fixed

Now, convergence and unitarity bounds some:\n\[ x(1-\Delta) > 1 \Rightarrow -\frac{1}{x} > \Delta > \frac{1}{x} \Rightarrow x > \frac{4}{3} \]
Numerically, find $x_c = 1.45$ where monopoles hit the bound. $x_c$ is the “crack in the conf. window.” Below $x_c$, use stringy dual $U(N_f - N_c)$ theory with $N_f$ flavors $q, \bar{q}$,

\[ W = q_i N_i^j \bar{q}_j + V_+ V_+ V_+ V_+ \]

but set $R_V = \frac{1}{2}$ (they are decoupled free fields).

This provides a sensible definition for the theory with $N_c < N_f < x_c N_c$.

For $N_f < N_c$, SUSY is broken.

At $N_f = N_c$, $R_+ = R_+ R_+ R_+ \frac{1}{2}$ and the monopoles become free.

As $x$ is decreased, the monopoles $V_+$ hit the constantly bound and decouple before monopoles do.

A clue for non-SUSY conformal window m34? Non-SUSY theory with $N_f$ quarks.

For $m = 0$, this is a CFT for $N_f > N_c$.

What is the physics at the lower edge of conf. window? XSB? Monopoles?