Logarithmic (and worse) violations of the area law for entanglement entropy in holography, tensor networks and lattice models

Liza Huijse – Harvard University
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S. Sachdev, S. Hartnoll, B. Swingle
Outline

- Compressible phases
- Fractionalization
- Hyperscaling violation geometries
- Study entropy properties and compare with FS
- Open questions
- Ideas involving yet another emergent dimension
- Toy-models with emergent dimensions and many low energy modes
Compressible phase

- Spatial $d>1$, translational invariance, unbroken $U(1)$ symmetry
- Globally conserved $U(1)$ charge, $Q$
- Expectation value, $\langle Q \rangle$, varies smoothly as a function of chemical potential $\mu$
  \[ d\langle Q \rangle/d\mu \neq 0 \]
- Gapless (but CFT is not compressible in $d>1$)
- Landau Fermi liquid, key feature is Fermi surface
- FS can be ‘hidden’ when fermions couple to gauge field
Bose metal

- Example of non-Fermi liquid
- Compressible phase of bosons
- Boson **fractionalizes** into two fermions, $f_1$ and $f_2$, that each form FS.
- Fermions necessarily couple to emergent gauge field, so FS is ‘hidden’.

\[ b = f_1 f_2 \]
Gauge invariance
\[ f_1(x) \rightarrow f_1(x)e^{i\theta(x)} \]
\[ f_2(x) \rightarrow f_2(x)e^{-i\theta(x)} \]

Motrunich, Fisher, PRB 75 235116 (2007);
LH, Sachdev, PRD 86, 026001 (2011)
Problem with non-Fermi liquids

- Effective field theory: fermions coupled to gauge field
- Strongly coupled theory
- Large N-expansion is not controlled...

- More generally, compressible phases that are not FL are seen in experiments, but are hard to find in CMT. In contracts, in holography compressible phases arise naturally. What can we learn by comparing?

AdS/CFT duality

**Bulk**
- Dimensions: $d+2$
- AdS spacetime

**Boundary**
- Dimensions: $d+1$
- CFT

Finite charge density

**Bulk**
- Dimensions: \(d+2\)
- AdS spacetime
- Emanating electric flux

**Boundary**
- Dimensions: \(d+1\)
- CFT
- Finite chemical potential

\(\varepsilon_r = Q\)

\(\mu\)
Electric field sources

- Compressible phase: bulk theory with emanating electric flux
- Two types of charge sources for electric flux:
  - Charged fermionic particles in the bulk
  - Charge hidden behind horizon
- The two cases correspond to different phases
- Phase with charge hidden behind horizon is dual to a fractionalized phase

Hartnoll, LH, CQG 29, 194001 (2012)
Charged fluid

- Effective field theory in the bulk including fermions explicitly:

\[ \mathcal{L} = \frac{1}{2\kappa^2} \left( R - 2(\nabla \Phi)^2 - \frac{V(\Phi)}{L^2} \right) - \frac{Z(\Phi)}{4e^2} F_{\mu \nu} F^{\mu \nu} + p(\mu_{\text{loc}}) \]

- Pressure of the fluid: \(-p = \rho - \mu_{\text{loc}} \sigma\)

- Fluid description allows us to include backreaction

Hartnoll, Tavanfar, PRD 83, 046003 (2011)
Method

- Dilaton potential and coupling:
  \[ V(\Phi) = -6 \cosh\left(\frac{2\Phi}{\sqrt{3}}\right) \quad Z(\Phi) = e^{\frac{2\Phi}{\sqrt{3}}} \]

- UV solution (\(\Phi \to 0\)): AdS\(_4\)

- Multiple IR solutions:
  - Fractionalized (\(\Phi \sim \log r \to +\infty\)),
  - Cohesive (\(\Phi \sim -\log r \to -\infty\)),
  - Lifshitz (\(\Phi \to \text{const}\))

- We obtain phase diagram as a function of coupling \(\phi_0\) to relevant operator dual to dilaton

Hartnoll, LH, CQG 29, 194001 (2012)
Phase diagram

$z$ is dynamic critical exponent

Hartnoll, LH, CQG 29, 194001 (2012)
Charge fractionalization

\( \frac{(Q - Q_{\text{fluid}})}{Q} \)

Ratio of charge behind horizon and total charge

<table>
<thead>
<tr>
<th>Cohesive</th>
<th>Partially fractionalized</th>
<th>Fully fractionalized</th>
</tr>
</thead>
<tbody>
<tr>
<td>all charge carried by fermions in bulk</td>
<td>charge partially hidden behind horizon rest carried by fermions in bulk</td>
<td>all charge hidden behind horizon</td>
</tr>
</tbody>
</table>

Hartnoll, LH, CQG 29, 194001 (2012)
1\textsuperscript{st} order vs 2\textsuperscript{nd} order transition

- Lifshitz solution characterizes phase transition
- It depends on fluid parameters $m$ and $\beta$
- Scaling dimension of relevant operator can be complex: indicates \textbf{first order} transition

Hartnoll, LH, CQG 29, 194001 (2012)
1\textsuperscript{st} order vs 2\textsuperscript{nd} order transition

\begin{align*}
\{\hat{m}, \hat{\beta}\} &= \{0.5, 10\} \\
M &\text{ complex}
\end{align*}

\begin{align*}
\{\hat{m}, \hat{\beta}\} &= \{0.1, 20\} \\
M &\text{ real}
\end{align*}

\begin{align*}
\text{Free energy (dimless units)} &\quad \text{Fraction of charge behind horizon (dimless units)} \\
\text{relevant coupling } \phi_0 &\quad \text{relevant coupling } \phi_0
\end{align*}
Order parameter

- Charge behind the horizon seems to behave as order parameter for phase transition to fractionalized phase
- This was made more precise by Hartnoll, Radicevic
  - Flux through minimal surface
  - What is FT analog?
Fractionalized phases

Hyperscaling violation geometries

NB d refers to spatial dimension of boundary
Fractionalized solution

General solution for fractionalized phase:

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

Scale transformation: \( x_i \rightarrow \zeta x_i, \quad t \rightarrow \zeta^z t, \quad ds \rightarrow \zeta^{\theta/d} ds \)

\( z \) dynamical critical exponent

\( \theta \) hyperscaling violation exponent

(CFT has \( z=1, \theta=0 \rightarrow \text{AdS} \))

Ogawa, Takayanagi, Ugajin, arXiv:1111.1023;
LH, Sachdev, Swingle, PRB 85, 035121 (2012)
Fractionalized phases

- Large number of holographic theories where charge is hidden behind a horizon parametrized by $\theta$ and $z$
- Is there a ‘hidden’ Fermi surface?
- Probes: thermal entropy and entanglement entropy
- Thermal entropy of fractionalized holographic theories:
  \[ S \sim T^{(d-\theta)/z} \]
- For a system with a Fermi surface we have $S \sim T^{1/z}$, this suggests $\theta=d-1$

LH, Sachdev, Swingle, PRB 85, 035121 (2012)
State of system: $|\Psi\rangle$, density matrix: $\rho = |\Psi\rangle \langle \Psi|$

$\rho_A = \text{Tr}_B \rho$ = density matrix of region A

Entanglement entropy:

$$S_A = -\text{Tr}(\rho_A \log \rho_A)$$
Boundary law for gapped systems:

\[ S_A \sim \Sigma \]

\( \Sigma \) is boundary of region A
Systems with FS have logarithmic violation of the boundary law for entanglement entropy:

\[ S_A \sim k_F \Sigma \log(k_F \Sigma) \]

\( \Sigma \) is boundary of region A,
\( k_F \) is Fermi wavevector

Gioev, Klich, PRL 96, 100503 (2006);
Swingle, PRL 105, 050502 (2010);
Zhang, Grover, Vishwanath, PRL 107, 067202 (2011)
Holographic entanglement entropy

Area of minimal surface = entanglement entropy

For holographic entanglement entropy in fractionalized phase we find ($\Sigma$ is boundary of entangling region $A$):

$$S_A = \begin{cases} 
\Sigma & \theta < d - 1 \\
\Sigma \ln \Sigma & \theta = d - 1 \\
\Sigma^{\theta/(d-1)} & \theta > d - 1
\end{cases}$$

logarithmic violation of boundary law for $\theta = d - 1$, as expected for FS!

Ogawa, Takayanagi, Ugajin, arXiv:1111.1023;
LH, Sachdev, Swingle, PRB 85, 035121 (2012);
Dong, Harrison, Kachru, Torroba, Wang, ArXiv:1201.1905
Moreover entanglement entropy has right scaling with charge:

For $\theta = d - 1$ we find ($d = 2$)

$$S_A \sim \sqrt{Q} \Sigma \log(\sqrt{Q} \Sigma)$$

where $Q$ is charge behind the horizon

Comparison with $S_A \sim k_F \Sigma \log(k_F \Sigma)$ suggests: $k_F \sim \sqrt{Q}$

This matches expectation from Luttinger theorem: $Q \sim k_F^2$

Ogawa, Takayanagi, Ugajin, arXiv:1111.1023;
LH, Sachdev, Swingle, PRB 85, 035121 (2012)
Including fluid

- Including fluid we find (d=2)
  \[ S_A \sim \sqrt{Q - Q_{\text{fluid}}} P \log[\sqrt{Q - Q_{\text{fluid}}} P] \]
  for \( \theta = d-1 \)

- Still consistent with FS interpretation of hidden charge:
  \[ k_F \sim \sqrt{Q - Q_{\text{fluid}}} \]

- But how should Ryu-Takayanagi formula be generalized to include fluid contribution to entanglement?

LH, Sachdev, Swingle, PRB 85, 035121 (2012)
Fermi surface continued

- Thermal entropy and entanglement entropy consistent with FS at $\theta=d-1$
- BUT no spectral weight at finite momentum in current-current correlator
- Spectral weight not exponentially suppressed for $z \to \infty$ and $\theta/z$ fixed
- Probe for Friedel oscillation by studying monopole events

Hartnoll, Shaghoulian, ArXiv:1203.4236
Silverstein, Polchinski, ArXiv:1203.1015
Iqbal, Faulkner, ArXiv:1207.4208
Sachdev, ArXiv:1209.1637
Yet another emergent dimension?

- For $d=2$ (spacetime dimension $d+2=4$) the fractionalized solution with $\theta=d-1$ can be obtained from a 5 dimensional pure gravity theory by compactifying one dimension on a sphere: $\text{AdS}_4 \times S^1$
- The size of the sphere grows as we probe lower energy scales
- A growing emergent direction is also seen in the branching MERA

Goutéraux, Smolic, Smolic, Skenderis, Taylor, JHEP (2012)
Evenbly, Vidal, ArXiv:1210.1895
Kaluza-Klein reduction

- Pure gravity theory on AdS$_5$
- Non-diagonal KK reduction on AdS$_4 \times S^1$

\[ ds^2_{(5)} = e^{2\alpha\phi} ds^2_{(4)} + e^{-4\alpha\phi} (dy + A)^2 \quad A = A_M dx^M \quad \alpha = -\frac{1}{2\sqrt{3}} \]

- Einstein-Maxwell-dilaton theory with $\theta = d-1$ and $z = 3$

\[ \mathcal{L} = \frac{1}{2\kappa^2} \left( R - 2(\nabla \Phi)^2 - \frac{V(\Phi)}{L^2} \right) - \frac{Z(\Phi)}{4\epsilon^2} F_{\mu\nu} F^{\mu\nu} \]

- Charge $Q$ is related to size of compactified dimension

\[ e^{-2\alpha\phi} \sim Qr^2 \]
MERA and AdS

- MERA is tensor network constructed to simulate GS of CFT (it can capture entanglement of such a state)

Entanglement entropy

= Number of links on optimal surface crossing minimal number of links

Geometry interpretation of network gives AdS

Vidal, PRL 99, 220405 (2007)
Swingle, ArXiv:0905.1317
Fig. from Evenbly, Vidal, ArXiv:1210.1895
Entanglement captured by MERA in scheme where k sites merge to 1 at each step:

\[ S_E \sim \text{# of links cut} \sim \sum_{n=0}^{\log_k(L/\varepsilon)} \left( \frac{L}{k^n \varepsilon} \right)^{d-1} \]

In $d=1$ this leads to $\log(L/\varepsilon)$

Boundary law for $d>1$

Swingle, ArXiv:0905.1317;
Swingle, ArXiv:1209.3304
Branching MERA

- Branching MERA can capture entanglement of FS
- At each step $k$ sites merge to 1 and $b$ branches are created

Evenbly, Vidal, ArXiv:1210.1895
Swingle, ArXiv:1209.3304
Branching MERA

- Entanglement captured by branching MERA in scheme where $k$ site merge to 1 at each step and $b$ branches are created:

$$S_E \sim \# \text{ of links cut} \sim \sum_{n=0}^{\log_k \left( \frac{L}{\epsilon} \right)} \left( \frac{L}{k^n \epsilon} \right)^{d-1} b^n$$

- For $b = k^{d-1}$ this gives log violation of area law
- It follows that a dimension emerges that grows effectively as a $d-1$ dimensional object

Swingle, ArXiv:1209.3304
Branching MERA

- Branching MERA can capture entanglement of FS
- It has growing emergent direction to account for growing number of gapless modes on FS

Evenbly, Vidal, ArXiv:1210.1895
Swingle, ArXiv:1209.3304
If one naively geometrizes the branching MERA in d=2 one gets AdS$_4 \times S^1$

It this connected to uplifted solution of hyperscaling violation geometry with $\theta = d-1$?

Does it only tell us about the entanglement, or also about the FS?

Idea: compute current-current correlator in uplifted theory. Are there non-trivial corrections?

What about thermal entropy?

If this picture is correct, what are the higher d uplifted solutions?

Also, can the $\theta > d-1$ geometries be uplifted?

Cf. fractal FS with Swingle

If this picture is too naive, the entanglement can be due to low energy modes at zero momentum (or not at a FS), however, the number of these modes has to grow as one approaches the IR!

In this picture $\theta=d-1$ does not seem special

Simplest scenario: large ground state degeneracy

There are supersymmetric lattice models that can be tuned to have a GS degeneracy $L^a$ with $0<\alpha<d$ (full translation invariance only for $\alpha=0$ and $\alpha=d$)

Entanglement entropy shows scaling in entire range between area law and volume law of certain states in ground space

In which state does Ryu-Takayanagi compute $S_E$?

Growing an extra dimension

- There are 0+1 dimensional supersymmetric matrix models that, in the large N limit, behave as 1+1 dimensional spin chains.
- Bosons and fermions are matrices:

\[
[a_{ij}, a^\dagger_{kl}] = \delta_{il}\delta_{jk} ; \quad \{f_{ij}f^\dagger_{kl}\} = \delta_{il}\delta_{jk} ; \quad i, j, k, l = 1, \ldots N ,
\]

Veneziano-Wosiek model:

\[
H = \{Q^\dagger, Q\}
\]

\[
Q = \text{Tr}[f a^\dagger (1 + ga^\dagger)], \quad Q^\dagger = \text{Tr}[f^\dagger (1 + ga)a],
\]

Veneziano, Wosiek, hep-th/0609210
In large N limit the Hamiltonian only couples single-trace states to leading order (coupling to multi-trace states is suppressed by factors of $1/N$).

Then in strong coupling limit $B = \sum_i m_i$ and $F = \sum_i n_i$ are conserved:

$$|m_i, n_i\rangle = \frac{1}{\mathcal{N}_n} \text{Tr}[a^{m_1} (f^\dagger)^{n_1} a^{m_2} (f^\dagger)^{n_2} \ldots (f^\dagger)^{n_k}]|0\rangle \ ; \ m_i, n_i > 0$$

Map $m$ bosons to $m$ spin ups and $n$ fermions to $n$ spin downs.

Strong coupling Hamiltonian in 0+1d maps to Hamiltonian for 1+1d XXZ Heisenberg spin chain.

Veneziano, Wosiek, hep-th/0609210
Length of spin chain is \( L = F + B = \# \) operators in the trace

To probe IR physics in spin chain we need to send \( L \) to infinity

The continuum theory is Luttinger liquid

Fermions in large-\( N \) matrix model build up a single FS? Cf Iqbal, Faulkner

NB supersymmetric lattice model in 1+1d also maps to XXZ spin chain
Another toy model

- Supersymmetric lattice model on triangular ladder

- Ground state entropy per site

\[ S_{GS} = \frac{\ln Z_{GS}}{2L} \approx 0.21 \]
Frustration and QC in 1d

- Zero energy dark states: particle resonating on rung
- Mapping to chain by excluding dark states
- Charge ordered phase at $\frac{1}{4}$ filling
- Quantum critical phase at $\frac{1}{6}$ filling
  “superposition” of SCFTs (c=1)
- Phase separation in between
- Near $\frac{1}{6}$ filling: do we have growing number of low energy modes?

[LH - Mehta - Moran - Schoutens - Vala ’11]
Summary

- Using entropy as guiding principle we studied compressible fractionalized phases in holography.
- Hyperscaling violation geometry with $\theta = d - 1$ is consistent with Fermi surface associated to charge behind horizon.
- Open questions: absence of spectral density at finite momentum, contribution to entanglement entropy of fluid, theories with other values of $\theta$, issues with stability, …
- Emergent compactified dimension in branching MERA and for $\theta = d - 1$ geometry.
- Toy models: matrix models, fractal FS, supersymmetric lattice models, …
Thank you