Quantum Phase Transitions

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Chapter 8: *Exercises*

The problems below refer to the ϕ^4 field theory, defined by the partition function ($\alpha = 1 \ldots N$).

$$
\begin{aligned}
Z &= \int \mathcal{D}\phi_\alpha(x) \exp\left(- \int d^D x \mathcal{L} \right) \\
\mathcal{L} &= \frac{1}{2}\left[(\nabla \phi_\alpha)^2 + r\phi_\alpha^2 \right] + \frac{u}{24}(\phi_\alpha^2)^2 \qquad (1)
\end{aligned}
$$

1. In the paramagnetic phase, rotational invariance implies that we can write for the susceptibility, $\chi(q)\delta_{\alpha\beta} = \langle \phi_\alpha(q)\phi_\alpha(-q)\rangle$, where q is a D-dimensional spacetime momentum. Also, Dyson's equation has the form $\chi^{-1}(q) = q^2 + r - \Sigma(q)$. Obtain the perturbative expansion for $\Sigma(q)$ to order u^2. Leave the result in the form of integrals over momenta.

2. Another useful identity in the theory of Gaussian integrals is

$$
\prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{\pi}} \exp\left(-\frac{1}{2}\sum_{i,j} x_i M_{ij} x_j \right) = (\det M)^{-1/2} = \exp\left(-\frac{1}{2} \mathrm{Tr} \ln M \right) \qquad (2)
$$

where M is a real, symmetric, positive-definite matrix (*i.e.* all eigenvalues are positive). This identity can be easily established by changing variables of integration to a basis in which M is diagonal. We will use this identity to compute the free energy density F, defined by $Z = \exp(-VF)$ where V is the volume of spacetime. In the paramagnetic phase, $r > 0$, the perturbative expansion for F takes the form $F = C_1 + C_2 u + \mathcal{O}(u^2) + \ldots$, while in the magnetically ordered phase, $r < 0$, it takes the form $F = C_3/u + C_4 + \mathcal{O}(u)$. Obtain expressions for C_{1-4}. Assume we have normalized the $\mathcal{D}\phi_\alpha$ in Z to absorb the factor of $1/\sqrt{\pi}$ in (2).

3. This is adapted from Problem (6.5a-c) in Plischke and Bergersen to the notation we are using. You may follow their approach if you wish. We consider the consequences of anisotropy in the O(N) symmetry of $\mathcal{L}$. In some applications to classical ferromagnets and quantum antiferromagnets (which correspond to the case $N = 3$), spin-orbit interactions may introduce a weak anisotropy in which the $r\phi_\alpha^2$ term in $\mathcal{L}$ is replaced

by

$$r_s \sum_{\alpha<N} \phi_\alpha^2 + r_n \phi_N^2, \tag{3}$$

while the quartic term is replaced by

$$\frac{u_1}{24} \sum_{\alpha,\beta<N} \phi_\alpha^2 \phi_\beta^2 + \frac{u_2}{12} \sum_{\alpha<N} \phi_\alpha^2 \phi_N^2 + \frac{u_3}{24} \phi_N^4. \tag{4}$$

Clearly, the original problem with full O(N) symmetry is the case $r_s = r_n$ and $u_1 = u_2 = u_3$. The model with $r_s = \infty$, $u_1 = u_2 = 0$ is the field theory of the Ising model, while the model with O($N-1$) symmetry is $r_n = \infty$, $u_2 = u_3 = 0$.

(a) Show that the one-loop RG flow equations for this model are:

$$\begin{aligned}
\frac{dr_s}{d\ell} &= 2r_s + \frac{(N+1)}{6(1+r_s)} K u_1 + \frac{1}{6(1+r_n)} K u_2 \\
\frac{dr_n}{d\ell} &= 2r_n + \frac{(N-1)}{6(1+r_s)} K u_2 + \frac{1}{2(1+r_n)} K u_3 \\
\frac{du_1}{d\ell} &= \epsilon u_1 - \frac{(N+7)}{6(1+r_s)^2} K u_1^2 - \frac{1}{6(1+r_n)^2} K u_2^2 \\
\frac{du_2}{d\ell} &= \epsilon u_2 - \frac{2}{3(1+r_s)(1+r_n)} K u_2^2 - \frac{(N+1)}{6(1+r_s)^2} K u_1 u_2 - \frac{1}{2(1+r_n)^2} K u_2 u_3 \\
\frac{du_3}{d\ell} &= \epsilon u_3 - \frac{3}{2(1+r_n)^2} K u_3^2 - \frac{(N-1)}{6(1+r_s)^2} K u_2^2,
\end{aligned} \tag{5}$$

where K is the phase space factor discussed in class.

(b) Show that these equations reduce to the expected equations in the limits corresponding to the models with O(N), Ising, and O($N-1$) symmetry just noted.

(c) Consider the fixed point of the flow equations with O(N) symmetry: $r_s = r_n = r^*$, and $u_1 = u_2 = u_3 = u^*$. Show that, to leading order in ϵ, this fixed point has *two* relevant eigenvalues $2-(N+2)\epsilon/(N+8)$ and $2-2\epsilon/(N+8)$ (see Plischke and Bergersen for some calculational hints).