Quantum Phase Transitions

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Chapter 5: *Exercises*

1. As we will discuss later in the course, a superconducting quantum dot separated from a bulk superconductor by a Josephson tunnel barrier can be modeled as a $O(2)$ quantum rotor coupled to an external field

$$H = \frac{g}{2}\hat{L}^2 - h\hat{n}_x \tag{1}$$

where g is a measure of the Coulomb gap of the dot, and h is the Josephson coupling. Determine the first two terms in the series for the ground state energy in limit of small and large g.

2. Derive the result (5.6) for the dispersion of the triplet quasiparticle excitation in the large g limit of a $O(3)$ quantum rotor model.

3. Provide the missing steps leading to the last equation in (5.15). For this you simply have to find the normal modes of the "spin-wave" Hamiltonian discussed in class, and then quantize them. You may find the discussion in Section 3-1-1 of Itzykson and Zuber helpful.

4. Compute the value of

$$F_\alpha(\theta) = \exp\left(i\theta n_\beta \hat{S}_\beta\right)\hat{S}_\alpha \exp\left(-i\theta n_\gamma \hat{S}_\gamma\right) \tag{2}$$

where $\hat{S}_\alpha$ are quantum spin operators of angular momentum S $[\hat{S}_\alpha\hat{S}_\alpha = S(S+1)]$, n_α is an arbitrary vector of unit length, and θ is an angle of rotation. First show that all the $d^n F_\alpha/d\theta^n$ can be written solely in terms of the commutators of $\hat{S}_\alpha$ at $\theta = 0$. Hence argue that $F_\alpha(\theta)$ can be written as

$$F_\alpha(\theta) = f_{\alpha\beta}(\theta)\hat{S}_\beta \tag{3}$$

where the functions $f_{\alpha\beta}(\theta)$ are independent of the value of S. Finally, determine the $f_{\alpha\beta}(\theta)$ by explicitly evaluating everything using the Pauli matrix representation valid for $S = 1/2$.