Quantum Phase Transitions

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Chapter 2: *Exercises*

1. Consider the infinite classical Ising chain with first and second neighbor exchange ($K_1 > 0$, $K_2 > 0$):

$$H = -\sum_i \left(K_1 \sigma_i^z \sigma_{i+1}^z + K_2 \sigma_i^z \sigma_{i+2}^z \right) \tag{1}$$

   (a) Let Δ be the energy cost to create a domain wall between a state with all spins up and a state with all spins down. Find the value of Δ.

   (b) Write down the partition function for H as a transfer matrix product. The transfer matrix will be 4×4 and "transfers" the spin configuration by 2 sites. Alternatively, think of it in terms of a model of "superspins" with 4 states, with each superspin representing the states of a pair of nearest neighbor Ising spins.

   (c) Determine the correlation length, ξ, of H in the limit of large K_1, K_2: show that $\xi = (a/2)e^{\Delta}$.

   (d) We will now show that the relationship $\xi = (a/2)e^{\Delta}$ holds quite generally. First argue that for large Δ, the density of domain walls, ρ, is $\rho = (1/a)e^{-\Delta}$. So we need to establish that $\xi = 1/(2\rho)$. Assume that the positions of the domain walls are statistically uncorrelated from each other. Consider a long chain of length $L \gg \xi$ with $M = \rho L$ domain walls in it. The probability that any given domain wall is between positions 0 and $x > 0$ is $q = x/L$. Now use the statistical independence of the domain wall positions to argue that

$$\langle \sigma_0^z \sigma_x^z \rangle = \sum_{j=0}^{M} (-1)^j q^j (1-q)^{M-j} \frac{M!}{j!(M-j)!} \tag{2}$$

   Evaluate the above in the limit $M, L \to \infty$, $\rho = M/L$ fixed, to establish the desired result.

2. *The Poisson summation formula.* Consider the function

$$f(x) = \sum_{m=-\infty}^{\infty} \delta(x - m) \tag{3}$$

This is clearly a periodic function of x with period 1. Restrict the function to the fundamental domain $|x| < 1/2$. We can write $f(x)$ in a Fourier series expansion

$$f(x) = \sum_{\omega_n} e^{i\omega_n x} F(\omega_n) \tag{4}$$

where $\omega_n = 2\pi n$, and

$$F(\omega_n) = \int_{-1/2}^{1/2} dx f(x) e^{-i\omega_n x}. \tag{5}$$

In this manner, establish the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} \delta(x - m) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x}. \tag{6}$$

Apply this formula to (2.53) by writing

$$A(y) = \int_{-\infty}^{\infty} dx f(x) e^{-\pi x^2 y}. \tag{7}$$

Now prove (2.65).