Odd and even $\mathbb{Z}_2$ spin liquids

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Abstract


These notes derive dual models of (what are now called) $\mathbb{Z}_2$ spin liquids, and show for the first time that $\mathbb{Z}_2$ spin liquids come in 2 varieties: these are now labeled ‘odd’ and ‘even’ $\mathbb{Z}_2$ spin liquids. For spin quantum antiferromagnets, odd (even) spin liquids appear for half-integer (even-integer) spin per unit cell. The even case corresponds ultimately to the $\mathbb{Z}_2$ gauge theory studied by Wegner (and later in the toric-code model by Kitaev), and this can have a trivial confining phase with no broken symmetry. The odd case is new, and was considered here for the first time. In this case, the $\mathbb{Z}_2$ gauge theory has a background gauge charge, and the confining phase is shown here to break lattice symmetry by the appearance of valence bond solid order. This ensures consistency with the Lieb-Schultz-Mattis-Hastings-Oshikawa theorems.

In the deconfined phase, with topological order, both the even and odd $\mathbb{Z}_2$ spin liquids have 4-fold degeneracy on the torus. However, these notes show that vison (or ‘$m$’ particle) excitations are at least doubly degenerate for the odd $\mathbb{Z}_2$ spin liquid.
Duality Mappings for Dimer Models

with on-site bond

Allowed moves

\[
\begin{align*}
K & \rightarrow K + (2a, \phi - 2\pi) \\
K & \rightarrow K + (2b, \phi) \\
K & \rightarrow K + (2c, \phi)
\end{align*}
\]

and

\[
\begin{align*}
& \rightarrow \\
& \rightarrow \\
& \rightarrow \\
& \rightarrow
\end{align*}
\]

Dynamical variables

\[
\hat{L}_{i+1} = e^{i\phi R_i} \hat{L}_{i+1} \\
\text{number of dimers on a bond.}
\]

\[
N_i = e^{i\phi R_i} x \text{ (number of bonds on site).}
\]

Constraint

\[
\Delta_i L_i + 2N = (-1)^{i} n_b
\]

Conjugate variables \(a_i, \phi\)

\[
[a_i, L] = i \text{ and } [\phi, N] = i
\]
Hamiltonian

\[ H = \frac{\hbar}{2} \sum_i \left( \frac{\nabla_i^2}{2} \right) + \frac{\hbar^2}{2} \sum_i \nabla_i^2 + \frac{\hbar^2}{2} \cos \left( \epsilon_i \Delta_i a_j \right) \]

\[ + \frac{\hbar^2}{2} \cos \left( \Delta_i \phi - 2\alpha_i \right). \]

We approximate as via by Villain form in

The e^{-\beta H} whence we will get terms of the form

\[ \frac{k^2}{2K_3} + ik \epsilon_{ij} \Delta_i a_j \]

\[ + \frac{P_i^2}{2K_4} + iP_i (\Delta_i \phi - 2\alpha_i) \]

where \( k, P_i \) are integers "dual" to \( \epsilon_{ij} \Delta_i a_j \) and \( \Delta_i \phi - 2\alpha_i \) respectively.

The integral over \( \phi \) and \( \alpha_i \) can be performed

after introducing discrete time shifts.

This yields

Then the constraints take the form

\[ \Delta_i b_j = 0 \]
\[
Z = \mathcal{T}_{N, L_i, P_i, k} e^{-H}
\]

\[
H(N, L_i, P_i, k) = \frac{e^2}{2} (L_i^2 + k^2) + \frac{g}{2} (N^2 + P_i^2)
\]

where \( N, L_i, P_i, k \) are integers subject to the constraints

\[
\Delta_i L_i + 2N = (-1)^i n_b
\]

\[
\Delta_t N = \Delta_i P_i
\]

\[
\Delta_t L_i = \epsilon_{ij} \Delta_j k = -2 P_i
\]

The couplings \( e^2 \) and \( g \) have been chosen for convenience and anticipate further mappings.

We now introduce the three-vectors

\[
\vec{b}_\mu = (P_i, -N) \quad \text{and} \quad \vec{c}_\mu = (\epsilon_{ij} L_j, -k)
\]

Then the constraints take the form

\[
\Delta_\mu \vec{b}_\mu = 0 \quad (1)
\]
and

$$\epsilon_{\mu \nu} \Delta^\nu \xi^\mu + 2b_\mu = \epsilon_{\mu \nu} \Delta^\nu \xi^\mu \quad (2)$$

where $\xi^\mu$ is $t$-independent and has the following values

Thus

$$\epsilon_{\mu \nu} \Delta^\nu \xi^\mu = (-1)^i \xi_0 \eta_0 \eta_1$$

The most general form of (1) and (2) has the form

$$b_\mu = \epsilon_{\mu \nu} \Delta^\nu \eta_\nu$$

$$\xi_\mu = \Delta_\mu m - 2\eta_\mu + \xi^\mu$$

where $\eta_\mu, m$ are arbitrary vectors.
\[ Z = \Phi \text{Tr} m \eta \ e^{-H} \] (\star)

\[ H = \frac{e^2}{2} (\Delta m - 2 \eta + S) + \frac{g}{2} (\eta \nabla \Delta \eta) \]

(\star) is the most useful form \( H \).

We now perform some manipulations to find other equivalent forms:

1. Split \( S \mu \) into its curl-free and divergence-free parts

\[ S \mu = \frac{\eta \nu}{4} \Delta \eta + \frac{\eta \nu}{4} \nabla \nabla \Delta \eta \]

Where \( S = 0, 1, 2, 3 \) on 4 and lettings as before

and

\[ B \mu = \frac{\delta^\mu_\nu (-1)^i}{2} \] [Staggered magnetic field in \( i \) direction]

Thus

\[ H = \frac{e^2}{2} (\Delta m - 2 \eta + \frac{\eta \nu}{4} \Delta S)^2 \]

\[ + \frac{g}{2} (\eta \nabla \Delta \eta + \frac{\eta \nu e^2}{2g} B_\mu)^2 + \text{constant} \]
(2) Rewrite $H$ in the form

$$H = \frac{e^2}{2} \sum_{\nu} B_{\nu} \left( d_{\nu} + \frac{n_{\nu}}{4} \Delta_{\nu} \right)^2$$

$$+ \frac{g}{2} \left( e_{\nu} - \frac{n_{\nu} e^2}{2g} B_{\nu} \right)^2$$

where $d_{\nu}, e_{\nu}$ are integers subject to constraints

$$\epsilon_{\nu} \Delta_{\nu} d_{\nu} + 2e_{\nu} = 0.$$ 

$$\Delta_{\nu} e_{\nu} = 0$$

Enforce these constraints be Lagrange multipliers.

$$Z = \sum_{d_{\nu}} \sum_{e_{\nu}} \int d\phi d\theta A_{\mu} e^{-H'}$$

$$H' = \frac{e^2}{2} \left( d_{\nu} + \frac{n_{\nu}}{4} \Delta_{\nu} \right)^2 + \frac{g}{2} \left( e_{\nu} - \frac{n_{\nu} e^2}{2g} B_{\nu} \right)^2$$

$$+ i \Phi \Delta_{\nu} e_{\nu} + i A_{\nu} \left( \epsilon_{\nu} \Delta_{\nu} d_{\nu} + 2e_{\nu} \right)$$

The sum over $e_{\nu}, d_{\nu}$ can be performed exactly and we obtain

$$Z = \sum_{A_{\mu}} \sum_{\epsilon_{\nu}} \int d\phi d\theta A_{\mu} e^{-H''}$$

$$H'' = \frac{1}{2e^2} \left( \epsilon_{\mu} \Delta_{\mu} A_{\nu} - 2\pi g_{\mu} \right)^2 + \frac{1}{2g} \left( \Delta_{\mu} \Phi - 2A_{\mu} - 2\pi P_{\mu} \right)^2$$

$$+ \frac{i\Phi}{2\epsilon} \Delta_{\mu} g_{\nu} + \frac{ie^2}{2g} B_{\nu} \left( \Delta_{\mu} \Phi - 2A_{\mu} - 2\pi P_{\mu} \right)$$
So $\Delta u B_u = 0$ we may write

$$B_u = \epsilon_{\nu u} \Delta_\nu A^\nu$$

We have the lowest energy state and that:

$$H'''' = \frac{1}{2e^2} (\epsilon_{\nu u} \Delta_\nu A^\nu - 2\pi p^\nu)^2 + i \frac{n_\nu}{2} \Delta_\nu \eta_\nu$$

$$+ \frac{1}{2g} (\Delta_\mu \phi - 2A_\mu - 2\pi p_\mu)^2$$

$$+ i \frac{n_\nu}{2g} \Delta_\mu \epsilon_{\mu \nu \kappa} \Delta_\kappa (A_\kappa - 2\pi p_\kappa)$$

We now return to the action (*)&

$$\frac{e^2}{2} (\Delta_\mu m - 2n_\mu + S_\mu)^2 + \frac{g}{2} (\epsilon_{\nu u} \Delta_\nu \eta_\nu)^2$$

Recall

$$L_i = \epsilon_{ij} c_j$$

$$L_i = \epsilon_{ij} (\Delta_j m - 2n_j + S_j)$$

This allows

$$N = -b_\nu = -\epsilon_{\nu u} \Delta_\nu \eta_\nu.$$
their dimers coverings.

For large $e^2$ and $\eta = 1$ and $g$ arbitrary (1)

the lowest energy states and their

associated dimer coverings are

(1) $m = 0$, $\eta_m = 0$

\[ E = \left( \frac{1}{2} \right) E(G) + \frac{1}{2} \rho \frac{\beta}{\nu} \]

(2) $\eta_m = 0$

\[ m = \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \]

and so on.

Since $E_{\nu m} \Delta m = 0$ for all these states

the value of $g$ is not relevant at the saddle point.
Fully frustrating d=2 Ising model.

Unfrustrated model.

\[ \hat{T}(g) = \exp \left( 2K^* (T_3 \cos^2 g + T_1 \sin^2 g) \right) \exp \left( -2K T_3 \right) \]

\[ \Theta = 2K^* \sqrt{\cos^2 g + \sin^2 g} = 2K^* \]

\[ \vec{a} = (\sin g, 0, \cos g) \]

\[ \hat{T} = \begin{bmatrix} \cosh 2K^* + \sinh 2K^* (T_3 \cos^2 g + T_1 \sin^2 g) \\ \cosh 2K^* - \sinh 2K \end{bmatrix} \]

\[ = \begin{bmatrix} \cosh 2K^* + \sinh 2K^* \cos g & \sinh 2K^* \sin g \\ \sinh 2K^* \sin g & \cosh 2K^* - \sinh 2K^* \cos g \end{bmatrix} \]

\[ \times R \left[ a + d = 2 \left( a^* - s a^* s \cos g \right) \right] \]