Kinetic theory of transport for inhomogeneous electron fluids

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Ohm’s law – the “simplest” experiment...

\[ E = \rho J \quad (V = IR) \]
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...yet \( \rho \) hard to compute in interesting systems:

- gases
- fluids
- stat.mech.
- black holes
- chaos

- QFT

57 years later, we have ‘completed’ this theory.
Introduction to Transport

Transport in Metals

- Ohm’s law – the “simplest” experiment...

\[ E = \rho J \quad (V = IR) \]

- ...yet \( \rho \) hard to compute in interesting systems:

- **textbooks** on canonical \textbf{Boltzmann/kinetic} transport:
  
Ohm’s law – the “simplest” experiment...

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...yet \( \rho \) hard to compute in interesting systems:

gases \( \rightarrow \) transport \( \rightarrow \) QFT

fluids \( \rightarrow \) stat.mech. \( \rightarrow \) black holes \( \rightarrow \) chaos

textbooks on canonical Boltzmann/kinetic transport:


57 years later, we have ‘completed’ this theory
assume **quasiparticles**:

\[
\langle \psi^\dagger(k, \omega) \psi(k, \omega) \rangle \sim \frac{1}{\omega - \epsilon(k) + ic\omega^2 + \cdots}.
\]
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non-equilibrium distribution function:

\[
f(x, p) \sim \text{“particles of momentum } p \text{ at position } x\text{”}
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weak interactions + \( \Delta x \Delta p \gg \hbar \Rightarrow \text{kinetic theory:} \)

\[ \partial_t f + v \cdot \partial_x f + F \cdot \partial_p f = C[f]. \]

free-particle streaming collagen
assume:
  • inversion and time reversal symmetry
  • thermodynamic equilibrium $f_{eq}$ is not unstable
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- thermodynamic equilibrium $f_{eq}$ is not unstable
- static linear response: $f = f_{eq} + \delta f$,

$$\mathbf{v} \cdot \partial_x \delta f + \mathbf{F} \cdot \partial_p \delta f + e\mathbf{E} \cdot \mathbf{v} \frac{\partial f_{eq}}{\partial \epsilon} = \left. \frac{\delta C}{\delta f} \right|_{eq} \cdot \delta f$$

* streaming terms

* source term

* collision operator
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  ▶ inversion and time reversal symmetry
  ▶ thermodynamic equilibrium $f_{eq}$ is not unstable

static linear response: $f = f_{eq} + \delta f,$

\[
\begin{align*}
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\text{streaming terms} & \quad \text{source term} & \quad \text{collision operator}
\end{align*}
\]

BIG linear algebra problem...schematically:

\[
\begin{align*}
L|\Phi\rangle + W|\Phi\rangle = E_i|J_i\rangle, \\ 
\delta f = -\left. \frac{\partial f_{eq}}{\partial \epsilon} \right|_{\Phi} \Phi \\
\Phi \text{ not singular}
\end{align*}
\]
assume:

- inversion and time reversal symmetry
- thermodynamic equilibrium $f_{eq}$ is not unstable

static linear response: $f = f_{eq} + \delta f$,

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streaming terms

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collision operator

BIG linear algebra problem...schematically:

\[
L |\Phi\rangle + W |\Phi\rangle = E_i |J_i\rangle, \quad \delta f = - \frac{\partial f_{eq}}{\partial \epsilon} \Phi
\]

\Phi not singular

choose inner product $\langle J_x | \Phi \rangle = J_x^{avg}$:

\[
\sigma_{xx} = \langle J_x | (W + L)^{-1} | J_x \rangle.
\]
exact solution hard. common approximations:

\[ W = \frac{1}{\tau} \]

homogeneous fluid

relaxation time approximation
exact solution hard. common approximations:

- \( W = \frac{1}{\tau} \).
- \( L \rightarrow 0 \).

homogeneous fluid

relaxation time approximation

however there are conservation laws:

\[
C \left[ f_{eq} \left( \frac{\epsilon(p) - \mu - \delta \mu}{T} \right) \right] = 0
\]
exact solution hard. common approximations:

\[ W \left\langle \frac{1}{\tau} \right\rangle \]

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however there are **conservation laws**:

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C \left[ f_{eq} \left( \frac{\epsilon(p) - \mu - \delta \mu}{T} \right) \right] = 0
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conservation of charge: \( W|\Phi = 1\rangle = 0 \)
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relaxation time approximation

however there are conservation laws:

\[ C \left[ f_{eq} \left( \frac{\epsilon(p) - \mu - \delta \mu}{T} \right) \right] = 0 \]

- conservation of charge: \( W|\Phi = 1\rangle = 0 \)
- and conservation of momentum: if density \( \rho \neq 0 \):

\[ W|\Phi = p_x\rangle = 0 \implies \sigma_{xx} = \langle J_x|W^{-1}|J_x\rangle = \infty \]
Strange Metals in Experiment

Viscous Flows in Constrictions

• at finite $T$, more scattering? $\frac{\partial}{\partial T} W > 0 \implies \frac{\partial}{\partial T} \rho = \frac{\partial}{\partial T} \left( \langle J_x | W^{-1} | J_x \rangle \right)^{-1} > 0$?
Strange Metals in Experiment

Viscous Flows in Constrictions

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- however, $\frac{\partial}{\partial T} \rho < 0$ in graphene constrictions!

[Kumar et al; 1703.06672]
in a Fermi liquid:

\[ \tau_{ee} \sim \frac{\hbar \mu}{(k_B T)^2}, \quad \rho = AT^2 \sim \frac{1}{\tau_{ee}} \ldots \]
Strange Metals in Experiment

Electron-Electron Interaction Limited Resistivity in Fermi Liquids

- In a Fermi liquid:
  \[ \tau_{ee} \sim \frac{\hbar \mu}{(k_B T)^2}, \quad \rho = AT^2 \sim \frac{1}{\tau_{ee}} \ldots \]

- \( A \) depends on thermodynamics (not disorder?):
  [Jacko, Fjaerestad, Powell; 0805.4275]
Strange Metals in Experiment

Linear Resistivity: A Challenge

– in a theory without quasiparticles:

\[ \tau_{ee} \gtrsim \frac{\hbar}{k_B T}. \]
Strange Metals in Experiment

Linear Resistivity: A Challenge

▶ in a theory without quasiparticles:

\[ \tau_{ee} \gtrsim \frac{\hbar}{k_B T}. \]

▶ “Drude” \[ \rho = \frac{m}{ne^2} \frac{1}{\tau_{ee}} \sim \frac{m}{ne^2} \frac{k_B T}{\hbar} : \]

[Bruin, Sakai, Perry, Mackenzie; (2013)]
charge puddles: \( H = H_{\text{clean}} + \int d^d x \ n(x)V_{\text{imp}}(x) \):

\[ F_{\mu}(x) = \mu_0 V_{\text{imp}}(x) \]

for simplicity: neglect umklapp, phonons

\( |\psi_P\rangle = 0 \) – ee collisions conserve momentum

must include streaming terms: \( \lambda_F \neq 0 \)

\( V_{\text{imp}} \) perturbatively small – efficient analytical/numerical algorithm to exactly solve the transport problem:

\[ \rho = V_{\text{imp}}^2 \times \cdots \]

\[ \mu(x) = \mu_0 - V_{\text{imp}}(x) \]
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- \( W|P\rangle = 0 \) – ee collisions conserve momentum
- **must include streaming terms**: \( L \neq 0 \)
- \( V_{\text{imp}} \) perturbatively small – efficient analytical/numerical algorithm to **exactly solve** the transport problem:

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\rho = V_{\text{imp}}^2 \times \cdots
\]
simple argument:

\[
\frac{1}{\tau_{\text{Drude}}} = \frac{v_F}{\xi} \times \Delta \theta^2
\]

\[\text{arrow of time} \Rightarrow \text{entropy production when } W = 0!\]
New Boltzmann Transport

Free Fermions ($L \neq 0$, $W = 0$)

- simple argument:
  \[ \frac{1}{\tau_{\text{Drude}}} = \frac{v_F}{\xi} \times \Delta \theta^2 \]

- formally:
  \[ \rho \sim \frac{\nu(\mu) V_{\text{imp}}^2}{n^2 e^2 v_F \xi} \]
simple argument:

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\frac{1}{\tau_{\text{Drude}}} = \frac{v_F}{\xi} \times \Delta \theta^2
\]

formally:

\[
\rho \sim \frac{\nu(\mu)V_{\text{imp}}^2}{n^2 e^2 v_F \xi}
\]

arrow of time \( \Rightarrow \) entropy production when \( W = 0 \)!
New Boltzmann Transport

Toy Model: Single Fermi Surface

\[ \Phi \approx \sum_{j \in \mathbb{Z}} \Phi_j(x)e^{ij\theta} \]

\[ W \sim v_F \ell_{ee} \sum_{|j| \geq 2} |j\rangle\langle j| \]

\[ \rho/\rho_{res}, k_T \xi \gg 1 \]

\[ k_T \xi \ll 1 \]
low $T$ limit:

$$\Phi \approx \sum_{j \in \mathbb{Z}} \Phi_j(x) e^{ij\theta}$$
New Boltzmann Transport

Toy Model: Single Fermi Surface

- low $T$ limit:

\[ \Phi \approx \sum_{j \in \mathbb{Z}} \Phi_j(x) e^{ij\theta} \]

- conserve charge ($j = 0$), momentum ($j = \pm 1$):

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- analytically compute $\rho$!

interactions always decrease $\rho$:

![Graph showing $\rho/\rho_{\text{res}}$ vs $\xi/\ell_{ee}$ with two curves: $k_{TF}\xi \gg 1$ and $k_{TF}\xi \ll 1$.]
Comparison with Viscous Hydrodynamics

\[ \ell_{ee} \ll \xi \implies \text{viscous hydrodynamic transport} \]

\[ \rho \sim \eta \int \frac{d^2x}{V} \left( \nabla \frac{1}{ne} \right)^2 \]

with \( \eta \sim \ell_{ee} \)
New Boltzmann Transport

Comparison with Viscous Hydrodynamics

- $\ell_{ee} \ll \xi \implies$ viscous hydrodynamic transport

  $$\rho \sim \eta \int \frac{d^2x}{V} \left(\nabla \frac{1}{n_e}\right)^2$$

  with $\eta \sim \ell_{ee}$

- QP random walks – sees $V_{imp}$ slower:

  $$\rho \sim \frac{1}{\tau} \sim \frac{v_F \ell_{ee}}{\xi^2}$$
New Boltzmann Transport

Toy Model: Two Fermi Surfaces

![Diagram showing two Fermi surfaces: one filled and one empty, with labels for k_x and k_y axes.]

- Pockets have different conservation laws:
  - Charge in pocket 1
  - Charge in pocket 2
  - Total momentum

Numerical computation gives:

\[ v_F, 1/v_F, 2 = 0.3 \]
\[ v_F, 1/v_F, 2 = 0.5 \]
\[ v_F, 1/v_F, 2 = 0.7 \]
\[ v_F, 1/v_F, 2 = 1 \]

Similar to Baber scattering, but novel mechanism.
Toy Model: Two Fermi Surfaces

- pockets have different $v_F$
pockets have different $v_F$

- conservation laws:
  - charge in pocket 1
  - charge in pocket 2
  - total momentum
pockets have different $v_F$
conservation laws:
  - charge in pocket 1
  - charge in pocket 2
  - total momentum
To understand the transport properties of the system, it is important to consider the different Fermi pocket structures depicted. Each pocket has its own Fermi velocity, $v_F$, leading to different transport characteristics. For instance, the pockets have different $v_F$ values, leading to similar to Baber scattering, but a novel mechanism.

Numerical computation gives the following relationship: $k_{TF} \xi \ll 1$. The graph illustrates the variation of $\rho/\rho_{res}$ with $\xi/\ell_{ee}$ for different $v_F,1/v_F,2$ ratios: $0.3$, $0.5$, $0.7$, and $1$.
Why Do Interactions Increase the Resistivity?

\[ \nabla \cdot \mathbf{J}_1 = \nabla \cdot \mathbf{J}_2 = 0; \text{ QPs} \]

pushed out of equilibrium:
New Boltzmann Transport

Why Do Interactions Increase the Resistivity?

- $\nabla \cdot J_1 = \nabla \cdot J_2 = 0$; QPs pushed out of equilibrium:

  - all excited QPs relax momentum:

    \[
    \frac{1}{\tau} \sim \frac{\ell_{ee} v_F}{\xi^2} \times \frac{\xi^2}{\ell_{ee}} \sim \frac{v_F}{\ell_{ee}}
    \]

    - diffusion
    - imbalance
we proved a unified transport bound:

$$\rho \leq \frac{T \dot{s}}{J^2} = \frac{\langle \Phi_{\text{odd}} | \tilde{W} | \Phi_{\text{odd}} \rangle}{\langle \Phi_{\text{odd}} | E \rangle^2}, \quad \text{subject to } \nabla \cdot \mathbf{J} [\Phi_{\text{odd}}] = 0$$

conservation of charge, energy, imbalance...

upon integrating out non-conserved even modes:

$$\tilde{W} = W_{\text{odd}} + L^T W_{\text{even}}^{-1} L$$
we proved a unified transport bound:

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hydrodynamic limit:

\[ T \dot{S} \sim \int d^d \mathbf{x} \left[ (\mathbf{J} - n \mathbf{v}) \cdot \Sigma^{-1} \cdot (\mathbf{J} - n \mathbf{v}) + \eta (\nabla \mathbf{v})^2 \right]. \]
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perturbative results from above become non-perturbative bounds on \( \rho \)
Phenomenology: Enhanced Resistivity near Criticality

- sample phase diagram:

$\rho \sim T$

$\rho \sim T^2$

$\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$

Non-Fermi Liquid

SDW

Superconductivity

$T$ → doping
Phenomenology: Enhanced Resistivity near Criticality

- Sample phase diagram:

- Imbalance diffusion + smooth disorder $\Rightarrow$ entire phase diagram?
sample phase diagram:

- imbalance diffusion + smooth disorder $\implies$ entire phase diagram?
- layered materials with chemical doping $\implies$ smoother potentials...
Phenomenology: Enhanced Resistivity near Criticality

- sample phase diagram:

- imbalance diffusion + smooth disorder \(\Rightarrow\) entire phase diagram?

- layered materials with chemical doping \(\Rightarrow\) smoother potentials...

- many ways to get this effect besides two Fermi surfaces...